сообщения
объединенного
института
ядерных
исследований
дубна

E11-83-692

A.Krastev, J.Nedialkov

A NUMERICAL SOLUTION OF THE EQUATION OF THE COMPUTERIZED TOMOGRAPHY AND ITS APPLICATION IN ASTROPHYSICS

1983

During the last years the method of the computerized tomography was applied for the solution of many problems. In order to cope the specific features on condition of a large variety of such problems, it is advisable to develope different methods for their mathematical treatment. Following this line we offer here a specific procedure for the numerical solution of the equations of the computerized tomography based on the A.Cormak's formulae [1]. These formulae, although seldom used, in some cases may have advantages, at least because of their transparency.

We shall have always in view a medium with a central symmetry. The method which we offer is such that the generalization for the nonsymmetric case can be carried out automatically.

Our method can be applied for the solution of various problems. But to be more specific we shall expose it in connection with problems arising when studying the structure of the Earth and other celestial bodies by neutrons [2].

When the Earth is subjected to neutrino beam irradiation in one or more directions, we can estimate its density profile provided we know the degree of attenuation. Until now this problem could be solved in two ways:

1) the elimination method;

2) the neutrino tomography method.

The elimination method has been proposed in [3,4] respectively in 1973 and 1974; and its modifications, in [5,6] in 1983. The underlying idea is as follows: If we have several rivalling models for the density distribution obtained on the basis of seismic and other geophysical data for each one of these models, one can calculate which would be the attenuation of the neutrino beam if the Earth had a density distribution corresponding to these models. The thus calculated attenuations are then compared with the experimentally measured attenuations. If there is a discrepancy between the calculated and the measured values, then the corresponding model has to be eliminated. In [4] and [5] the comparison must be made for a great number of directions and the probability of a misinterpretation becomes smaller but, on the other hand, the measurement time is longer. In the paper of Volkova and Zatsepin it was proposed to do the measurement only in one direction, but the probability of a misinterpretation is higher. In (6) the measurement has to be made in a small number of points. Thus the probability of a misinterpretation becomes smaller, while the measurement time becomes shorter.

The second method is the neutrino tomography method [2]. In [2] some problems of neutrino detection and the optimal energy of neutrino experiments for investigation of stars have been considered. The idea of this method is to use the methods of computerized tomography worked out for the purpose of X-ray diagnostic in medicine for an exploration of the Earth's density profile with neutrino experiments. This means that the attenuation of neutrino beams which pass through the Earth in different directions have to be measured. Having this information with the formulas from the computerized tomography, the density distribution can be calculated in a model-independent way. The investigation of different aspects of this method [7,8,9] showed that if neutrino beams are produced from TeV accelerators and detected with sufficiently large detectors, for example, DUMAND, from the opposite side of the Earth the experiment for the reconstruction of the density profile will last years or decades.

In this work we shall show that with TeV proton-beam accelerators, a number of celestial bodies can also be explored: for example, the planets of the Solar system, the Sun itself and some other stars. The exposure time should be roughly the same as for the Earth. This is achieved by amending the method described in [10] and applying it to other celestial bodies. This method concerns the numerical solution of a specific equation of the computerized tomography. Although only a short time has passed since [2] appeared, now the perspective for the exploration of celestial bodies by means of computer tomography methods have become more realistic. We can say that there are intensive investigations under way of different methods for neutrino exploration of the Earth. Meanwhile, a proposal has been put forward to accelerate elementary particles with the help of laser beams [11,12]. The compactness of such devices will probably make possible to launch them with satellites in orbit around planets and the Sun. If the present rate of advance remains the same, future systems for a generation of powerful multi-TeV beams might be expected. Thus the size of neutrino detectors will decrease sufficiently to enable their transportation on satellites.

On the other hand, there is a great advance in the space technology in last years. These considerations bring us to the conclusion that it is possible to build in future satellite systems for neutrino exploration of planets, natural satellites and the Sun.

When the experiment is carried out on Earth, the neutrino source and the detector are situated on its surface. The detector measures the attenuation of neutrino beams which pass through the Earth's interior. This attenuation is proportional to the dimensionless parameter I, which is connected with the density distribution as follows:

$$I(t) = \int_t^1 \frac{2r\rho(r)\,dr}{\sqrt{r^2 - t^2}}. \qquad (1)$$

Here $\rho(r)$ is the density distribution, r is the distance from the Earth's centre and t is the distance between the beam and the center of the Earth. All parameters in this equation are dimensionless. The unit for length is the Earth's radius; and the unit for density, the mean density of the Earth.

In neutrino astrophysics the detector and the source of neutrino are situated on a joint circular orbit so that the neutrino beam remains all the time at constant distance t from the centre of the celestial body. When source and detector are circling around, the chord defined by the part of the beam, which transpierce the body, is circling too. In this way the averaged value of $I(t)$ for a certain t is measured. But as celestial bodies can be regarded with good approximation to be spherically symmetrical, the measured value will be approximately the same as if the source and the detector were situated on the surface of the celestial body. Changing appropriately the mutual disposition of source and detector on the orbit we are changing t and thus the obtaining of values of $I(t)$ for different values of t becomes feasible. With the same experimental complex one can explore also the cases with deviation from central symmetry. This can be done simply by changing the organization of the recording of experimental data.

For the central symmetric case, which will be studied thoroughly later on, the mathematical problem is: having the values $I(t_1), I(t_2), \ldots$ to reconstruct the density profile of the celestial body solving the integral equation (1)*. The problem of solving this equation is an ill-posed one and could be treated with some methods developed to cope with such problems. From our point of view it is convenient to use the inversion formula

$$\rho(r) = \frac{d}{dr}\left(-\frac{r}{\pi}\int_r^1 \frac{I(t)\,dt}{t\sqrt{t^2 - r^2}}\right). \qquad (2)$$

We use this formula because of the possibility it gives to study more easily the three sources of incorrectness. The first

---

*Note added in proof: Equation (1) is an Abel equation which could be regarded also as a special case of the equations of computerized tomography but in what follows we shall study it simply as Abel equation. But in the more general case mentioned above, when there is no central symmetry, the mathematical problem is more complex. For its solution one could use methods of computerized tomography.

one is thé numerical differentiation of a function, which is known only in discrete points with certain errors. The second one is connected with the stochastic treatment of the problem. The third source of errors is a boundary effect in a sense. So, when the neutrino beam approaches the Earth's surface the projected mass density tends to zero and therefore the useful information together with the quotient signal/noise tends to zero.

Further on we proceed with the estimation of the duration and accuracy of such experiments. This will be made on the basis of the so-called Roche-model [13] for the density distribution in celestial bodies.

In what follows we describe the numerical procedure used to solve eq.(1).

We first divide the interval [0, 1] into N equally large subintervals. In every subinterval we approximate I(t) from (2) with a second order polynom, integrate explicitly and sum over all subintervals. For the integral in the brackets in formula (2) we obtain:

$$J\left(\frac{k}{N}\right) = \sum_{m=k}^{N-1} S(k, m), \qquad k = 0,1,\dots,N, \text{ where} \tag{3}$$

$$S(k, m) = \lambda(k, m)\, I_{m-1} + \mu(k, m)\, I_m + \nu(k, m)\, I_{m+1}. \tag{4}$$

Here I_m stands for $I(t = \frac{m}{N})$ and λ, μ, and ν are explicitly known coefficients, which are obtained after the integration in each subinterval. Having $J(\frac{k}{N})$ in all points one can find $G(\frac{k}{N})$ using the standard discrete approximation of the first derivative:

$$G\left(\frac{k}{N}\right) = \frac{1}{2\pi}\left[(k-1)\, J\left(\frac{k-1}{N}\right) - (k+1)\, J\left(\frac{k+1}{N}\right)\right]. \tag{5}$$

Now having only a few I_m and knowing their values with certain errors, which are assumed to have their origin in the statistical character of every physical measurement, one can calculate the corresponding values of $G(\frac{k}{N})$. We want to investigate also how the error with which $G(\frac{k}{N})$ is calculated depends on the error in the input data and on the number of points in which the function I(t) is supposed to be known. To do this we have made some computer experiments and have calculated the expected resulting error as a function of the input error and the number of points in two mutually independent ways. The discrepancy between the values thus obtained was small enough to reassure us in the correctness of our result. Next we explain how the reconstruction error ϵ_R depends on the input error and the number of subintervals in which the interval [0,1] has been divided.

The number of subintervals is important because it equals the number of measurements which have to be carried out.

We distinguish between various kinds of errors. ϵ_R can be regarded as consisting of two kinds of errors - ϵ_{RD} and ϵ_{RP}. ϵ_{RD} - the deterministic error is due to the approximation of integrals and derivatives with their discrete analogues, while ϵ_{RP} - the probabilistic error is due to the probabilistic character of the input data. Besides the reconstruction error, which characterizes the accuracy of the whole solution, we also use the reconstruction errors $\epsilon_R^{(k)}$ for the reconstruction error in different points $k = 0,1,\dots,N$. Analogously are defined the errors $\epsilon_{RD}^{(k)}$ and $\epsilon_{RP}^{(k)}$. We calculate ϵ_R by the formula

$$\epsilon_R = \sqrt{\sum_{k=2}^{N-2} [\epsilon_R^{(k)}]^2/(N-4)} \text{ and respectively } \epsilon_{RD} = \sqrt{\sum_{k=2}^{N-2} [\epsilon_{RD}^{(k)}]^2/(N-4)}.$$

The reason for the omission of points on both ends of the interval is that there the errors are too great and have to be treated by other methods. ϵ_R, ϵ_{RD} and $\epsilon_{RD}^{(k)}$ are deterministic numbers. But the input errors in the different points $k = 0, 1,\dots,N$, which will be denoted by $\epsilon_I^{(k)} \cdot \eta_k$, η_k being centered random numbers with dispersion 1, so that $\epsilon_I^{(k)}$ is the standard deviation of the random number $\epsilon_I^{(k)} \cdot \eta_k$. We choose $\epsilon_I^{(k)}$, $k = 0,1,\dots$ equal to each other and denote them by ϵ_I. One can find the deterministic error calculating (3), (4), and (5) and using the exact values of I(t), i.e., putting $\epsilon = 0$. Proceeding in this way and changing N from 5 to 60, we have obtained curve 1 on Fig.1. The curve ϵ_{RD} falls with increasing the number N until N becomes greater than 40 when computer errors, due to working with limited accuracy, begin to play a role.

One can find the probabilistic error as follows: Let $I(t) = 0$. Then clearly $G(r)$ should also be identically equal to zero. Now, one puts instead of zero on the place of I(t) in formula (2) random numbers $\epsilon_I^{(k)} \cdot \eta_k$, $k = 0,1,\dots,N$. Then one calculates the resulting values of $G(r)$. After calculating the resulting errors in different points $k = 0, 1,\dots,N$ one obtains random numbers which are denoted by $\epsilon_{RP}^{(k)} \eta_k$. We define the total probabilistic error by the number:

$$\epsilon_{RP} = \sqrt{\sum_{k=2}^{N-2} [\epsilon_{RP}^{(k)}]^2/(N-4)}, \quad \epsilon_{RP}^{(k)} \text{ depends on } \lambda, \mu, \nu \text{ and } \epsilon_I.$$

The calculated values of ϵ_{RP} are shown on Fig.1. On the same figure the curve ϵ_R is also shown. It depends on ϵ_{RD} and ϵ_{RP} as follows $\epsilon_R = \sqrt{\epsilon_{RD}^2 + \epsilon_{RP}^2}$.

The second way to calculate the probabilistic error consists in putting in (3), (4), (5) instead of I(k) the disturbed data $I(k) + \epsilon_I \eta_k$, $k = 1,2,\dots,N$. The resulting solution is given in the form of random numbers in $N-4$ points, which we denote $G(k)\,\eta_k$. We run the whole procedure M times, calculate the root mean square of each $G_k\eta_k$ and denote the last by $H(k, m)$. Now the total error $\epsilon_R(M)$ is calculated as follows:

$$\epsilon_R(M) = \sqrt{\sum_{k=2}^{N-2} H^2(k, M)/(N-4)}\,, \quad \lim_{M\to\infty} \epsilon_R(M) = \epsilon_R\,,$$

M was chosen 20, 30 and 40. $\epsilon_{RP}(40)$ is shown on Fig.2. The curve for $\epsilon_I = 1\%$ with sufficient accuracy coincides with ϵ_{RP} on Fig.1.

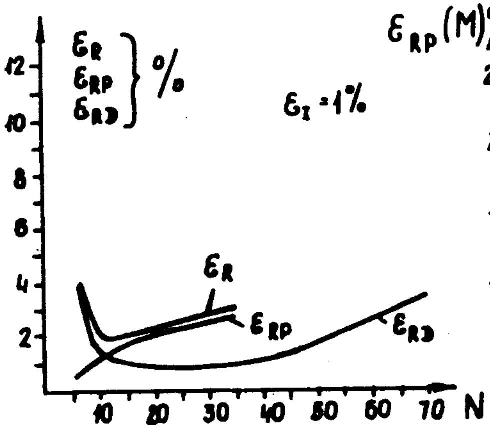

Fig.1

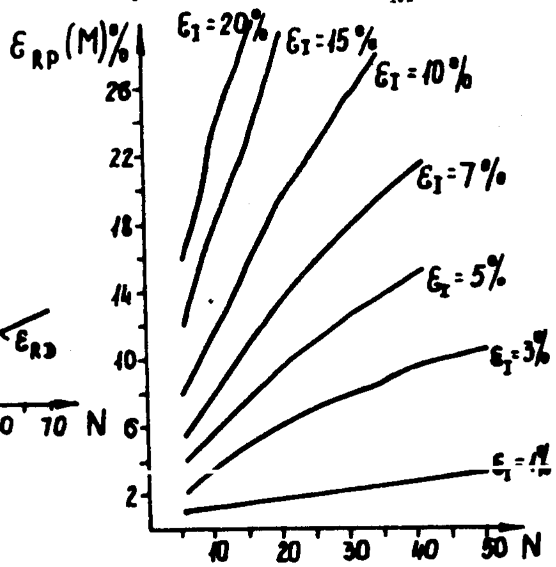

Fig.2

We now proceed with the application of the above-described algorythm to study the possibilities for exploration of the structure of celestial bodies by neutrino.

It is well known that the so-called Roche-model represents the simplest two-parameter model for the density distribution of the Earth, which satisfies simple and basic physical requirements - it satisfies the Laplace equation for the hydrostatic equilibrium state of the Earth and the requirement $\frac{dG}{dr}=0$ at $r = 0$ [18]. In this model the density distribution is represented by a second order polynom

$$G(r) = \alpha + \beta r^2. \tag{6}$$

The two parameters α and β can be determined by replacing (6) in the formulas for the total mass m of a planet and its momentum of inertia i, for example, which are known for the planets of the Solar system from astronomical observations.

Proceeding in this way and using the known data for m and i, one can easily check that the planets of the Solar system can be roughly classified in three groups.

The first group consists of Merkurius only. Its density distribution is constant.

For the third group of planets the Roche formula gives relatively small negative values for the density for a small region near the surface. It means that in this case it is advisable to put approximately α equal $-\beta = 2{,}5$. So that $G(1) = 0$. In this group come planets like Jupiter, Saturn and Uranus.

The second group is intermediate between the first and the third. It consists of Mars, Venus and the Earth. Here we choose α and β to be those determined for the Earth on the basis of the standard model for the Earth's density distribution.

The stars can be considered as members of the third group.

As the problem is ill-posed, one has to choose $N = N_{opt}$, where N_{opt} is that point of the curve $\epsilon_R(N)$, where it attains its minimum (Fig.3)*.

The curves showing the dependence of N_{opt} and of the coefficient $C_{IR} = \frac{\epsilon_I}{\epsilon_R}$ on ϵ_R are shown on Fig.4 and Fig.5*, respectively.

From the figures one can see that for the second group the reconstruction error ϵ_R is about two times greater than ϵ_I

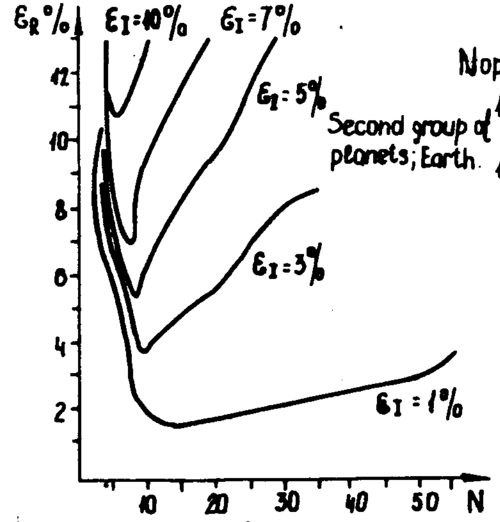

Fig.3

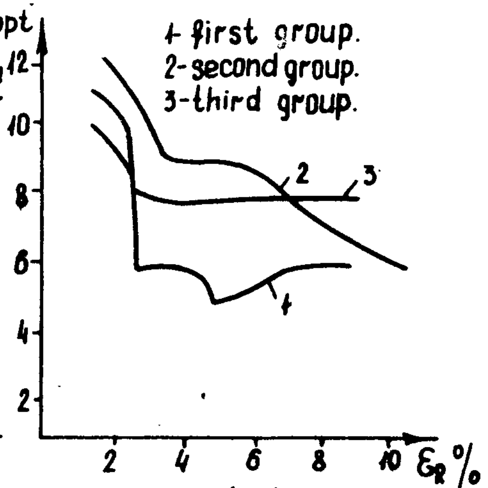

Fig.4

*The results from the calculations with N = 5 and 6 must be regarded as rough estimates.

for small ϵ_R and approximately equal to ϵ_I for greater values of ϵ_R. Another important conclusion which one can draw is that even by the most conservative prognosis where the experimental complex should have characteristics roughly the same as the present facilities, the duration of particular measurements will not exceed few years. This is motivated in the following way. For the Earth the duration of an experiment for a given value of t can be of the order of parts of the year [7]. As the number of experiments is $N = N_{opt} \approx 5 \div 10$, (see Fig.4) the total exposure time will be of the order of few years. But as the curves and some other parameters for all three groups of planets are similar, the conclusion is true for all of them.

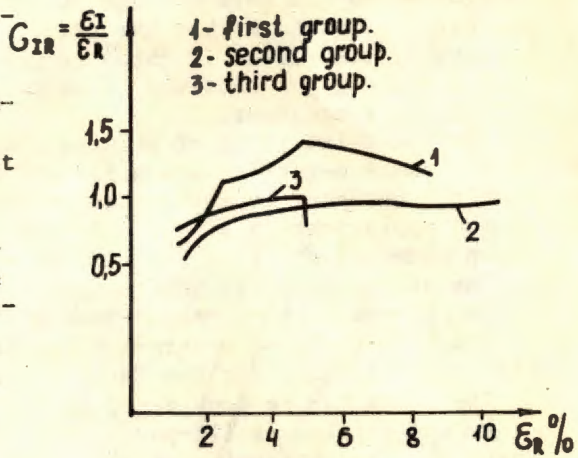

Fig.5

REFERENCES

1. Cormack A.M. J.Appl.Phys., 1963, 34 (9), p.2722.
2. Nedyalkov I.P. JINR, 18-81-189, Dubna, 1981.
3. Placci A., Zavattini E. A CERN Note Geneva-9, October, 1973.
4. Volkova L.V., Zatsepin G.T. Izv.Akad.Nauk SSSR, ser.fiz., 1974, 38, p.1060.
5. De Rujula A. et al. Preprint HUTP-83/A019, Harvard University, 1983.
6. Nedyalkov I.P. Compt.rend.Acad.Bulg.Sci., 1983, 36, No.5.
7. Id. ibid., 1981, 34, No.4, p.493.
8. Id. Proc. of the Conf. Neutrino'82. Balatonfüred, Hungary, 1982, Part I, p.300.
9. Nedyalkov I.P. JINR, P2-81-645, Dubna, 1981.
10. Krastev P., Nedialkov I., Nikolova L. University Annual Technical Physics. Sofia, 1982, vol.XIX, book 2, p.171.
11. CERN Courier No.4, Geneva, 1982.
12. CERN Courier No.11, Geneva, 1982.
13. Bullen K.E. The Earth's Density. Chapman and Hall, London, 1975.

Received by Publishing Department
on October 4,1983.

Крастев П., Недялков И. E11-83-692

Об одном численном решении уравнения компьютерной томографии и его применении в астрофизике

Предлагается метод численного решения уравнений компьютерной томографии. Метод основан на использовании формул А.Кормака [1] и разработан для сред с центральной симметрией. Обобщение на несимметрический случай не связано с дополнительными трудностями. При помощи этого метода в работе изучены возможности исследования распределения плотности во внутренней области Земли, других планет и Солнца путем их просвечивания нейтринными пучками.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1983

Krastev P., Nedyalkov J. E11-83-692

A Numerical Solution of the Equation of the Computerized Tomography and its Application in Astrophysics

A numerical method for solving the equation of the computerized tomography is proposed in this paper. The method, which is based on Cormack's formulae [1], is worked out for material media with central symmetry. The generalization for the nonsymmetric case does not lead to complications. The method is applied to study the possibilities for the investigation the interior of the Earth, the other planets and the Sun by means of neutrino experiments.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1983