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FAST RELAXATION METHOD
FOR SOLVING THE DIFFERENCE PROBLEM
FOR THE POISSON EQUATION
ON THE SEQUENCE OF GRIDS

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1. INTRODUCTION

Let us consider the difference problem

$$Au = f,$$ (1)

in the region $G \in R^n$, where A is a nonlinear operator, which provides a single solution u of the problem (1). Suppose also that the following difference approximation of the equation (1) is given

$$A_h u_h = f_h ,$$ (2)

where u_h and f_h are defined on the grids Ω_h and to solve the equation (2) the iterative process

$$\frac{u_h^{n+1} - u_h^n}{\tau} = -\Psi(u_h^n); \quad \Psi(u_h) = 0,$$ (3)

is constructed which converges to the solution u_h with a rate

$$\|u_h^n - u_h\| \le [q(h)]^n \|u_h^0 - u_h\|, \quad q < 1.$$ (4)

If q depends on h, then, as a rule,

$$q(h) \to 1 \qquad h \to 0; \qquad \text{as } q(h_1) < q(h_2), \quad h_2 < h_1 .$$ (5)

When constructing iterative processes on the sequence of grids Ω_{h_i}, $i = 1, 2, \ldots p$, $h_1 > h_2 > \ldots > h_p$, we solved the problem (2) by the method (3) on the grid Ω_{h_1} with accuracy ϵ_1

$$\|u_{h_1}^k - u_{h_1}\| \le \epsilon_1 .$$

Then the grid function $u_{h_1}^k$ is interpolated on the grid Ω_{h_2} and is used there as an initial approximation in (3) for decreasing the error to the value $\epsilon_2 < \epsilon_1$ and so on. In papers [1,2,3] it is shown that such a process gives a considerable economy as compared with the solution of the problem (2) with accuracy ϵ_p by the method (3) without using auxiliary grids Ω_{h_i}, $i = 1, 2, \ldots p$.

Here only general characteristics of the algorithms are used, such as the approximation error and the iterations convergence rate. If

$$\epsilon_i = O(h_i^a), \quad a > 0,$$

where h_i^a is the error of approximate solutions u_{hi}, then the use of a sequence of grids usually decreases the computational work $O(\ln \epsilon_p^{-1})$ times as $h_p \to 0$.

2. METHOD FOR ACCELERATING THE CONVERGENCE RATE

In this paper we suggest the method for accelerating the convergence of iterations on the sequence of grids, which is practically much more efficient, then the above-mentioned one. In this method one has to use additional information concerning the regularity of the decomposition of the error of the solution u_h into the degrees of the discretization step h:

$$u_h = u(\mathbf{x}) + \sum_{i=1}^m c_i(\mathbf{x}) h^{a_i} + o(h^{a_m}), \quad \mathbf{x} \in \Omega_h, \quad a_1 < a_2 < \ldots,$$ (6)

where the functions $c_i(\mathbf{x})$ are independent of h. The existence conditions for the decomposition (6) for nonlinear operators A were obtained in [4] and we will not consider them here.

The main idea of the proposed method is the following. Suppose we obtained solutions u_{h_i} on the grids Ω_{h_i}, $i = 1, \ldots, p$, $p = m+1$, with equal accuracy $\epsilon = o(h_{p+1}^{a_m})$ for the steps h_i different in pairs. Then for calculating the solutions $u_{h_{p+1}}$, $h_{p+1} < h_p$ we take as an initial approximation the combination

$$u_{h_{p+1}}^0 = \sum_{i=1}^p \gamma_i u_{h_i}, \quad \mathbf{x} \in \Omega_{h_{p+1}}.$$ (7)

where u_{h_i} are interpolated on the grid $\Omega_{h_{p+1}}$ with accuracy $o(h_{p+1}^{a_m})$. and the coefficients γ_i are defined from the set of equations

$$\begin{aligned}
\gamma_1 + \gamma_2 + \ldots + \gamma_p &= 1 \\
\gamma_1 h_1^{a_1} + \gamma_2 h_2^{a_1} + \ldots + \gamma_p h_p^{a_1} &= h_{p+1}^{a_1} \\
\gamma_1 h_1^{a_m} + \gamma_2 h_2^{a_m} + \ldots + \gamma_p h_p^{a_m} &= h_{p+1}^{a_m}
\end{aligned}$$ (8)

assuming that the solution of (8) exists. Thus, using solutions on the grids Ω_{h_i}, $i \le p$ we perform the extrapolation to the exact solution $u_{h_{p+1}}$ of the difference problem on the grid $\Omega_{h_{p+1}}$, since from (6) and (8) it follows

$$u_{h_{p+1}} = \sum_{i=1}^p \gamma_i u_{h_i} + o(h_{p+1}^{a_m}), \quad \mathbf{x} \in \Omega_{h_{p+1}}.$$ (9)

Note, that if $a_i = i$ or $a_i = 2i$, which usually takes place, then the determinant of the system (8) is of the Wandermond type and differs from zero. Estimates of the coefficients γ_i are given in [5].

Though the initial approximation $u_{h_{p+1}}^0$ from (7) is close to $u_{h_{p+1}}$ as $h_p \to 0$, however it doesn't provide fast convergence of iterations (3) because of (5). Here it is necessary to use additional information about the nature of the interpolation error.

3. THE DIFFERENCE PROBLEM FOR THE POISSON EQUATION

Consider, for example, Dirichlet's problem for the Poisson equation. For accelerating the convergence we will use only two auxiliary grids.

Let us consider the Poisson equation (1) in the stepped region $G \in R \times R$ with the boundary Γ, where $A = \Delta$, $u_\Gamma = \phi(\xi)$, which may be replaced by the difference problem

$$\Delta_h u_h = f, \quad u_{h,\Gamma} = \phi(\xi), \quad \xi \in \Gamma, \tag{10}$$

defined in the grid Ω_h, correlated with the boundary Γ and having the same step h for both variables. The operator Δ_h is defined by the usual five-point difference scheme.

Suppose the solution $u(x,y) \in c^6(\bar{G})$ of the problem (1) is such, that the function $c_1(x,y)$, defined by the equation

$$\Delta c_1 = -\frac{1}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right), \quad c_{1,\Gamma} = 0$$

satisfies the relation $c_1 \in c^4(\bar{G})$. Then, according to [6] we have

$$u_h = u(x,y) + c_1(x,y) h^2 + O(h^4), \quad (x,y) \in \Omega_h. \tag{11}$$

Writing the decomposition for the steps $\frac{h}{2}$ and $\frac{h}{4}$ and using the system (8) we easily find

$$u_h = \frac{5}{4} u_{\frac{h}{2}} - \frac{1}{4} u_h \quad . \quad (x,y) \in \Omega_h. \tag{12}$$

For interpolating (12) on the grid $\Omega_{\frac{h}{4}}$ the operator Q is used, which is of the order $o(h^4)$ for the exact solutions $u(x,y)$ and of the order $o(h^2)$ for the functions $v \in C^4(\bar{G})$. Then, according to (11) it has accuracy $O(h^4)$ on the difference solution u_h. If

$$\Omega_h = \{x_{i_1} = 2i_1 h, \quad y_{j_1} = 2j_1 h; \quad i_1 \in I, \quad j_1 \in J\},$$

then the operator Q is defined by

$$Qu_{i,j} = \begin{cases} u_{i_1 j_1} ; \; i=2i_1, \; j=2j_1 \\ \frac{1}{4}(u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}) - h^2 f_{i,j} \quad \begin{matrix} i=2i_1 \pm 1 \\ j=2j_1 \pm 1 \end{matrix} \\ \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}) - \frac{h^2}{2} f_{i,j} \quad \begin{matrix} i=2i_1, j=2j_1 \pm 1 \\ i=2i_1 \pm 1, i=2j_1 \end{matrix} \end{cases} \tag{13}$$

which is twice applied to u_h and once to $u_{\frac{h}{2}}$: $Q(u_h)$ is defined on $\Omega_{\frac{h}{2}}$; $Q^2(u_h)$ and $Q(u_{\frac{h}{2}})$ are defined on $\Omega_{\frac{h}{2}}$.

It should be noticed, that in calculations on the square grids besides the step discretization one can use the turn of the grid to the angle $\frac{\pi}{4}$ as suggested in [7]. If we denote the solution on such a grid by u_r, where for the given h we have $r = \sqrt{2}h$, then the extrapolation formulae will take form

$$u_{\frac{h}{2}} = \frac{3}{4} u_h + \frac{1}{4} u_r + \frac{h^2}{16} f + o(r^4); \quad (x,y) \in \Omega_h,$$

$$u_{r/2} = \frac{1}{2} u_h + \frac{1}{2} u_r + \frac{h^2}{16} f + o(r^4); (x,y) \in \Omega_h.$$

In this case solutions u_r and u_h are defined on the same grid, therefore, only one interpolation by the formula (13) is needed.

4. NUMERICAL EXPERIMENTS

Efficiency of the extrapolation (12) using the solutions on two grids is investigated for the single-linked and multi-linked regions, at different values of the residue ϵ_{p+1}, and besides, the dependence of the number of iterations on the smoothness of the solution $u(x,y)$ is considered.

The problem $\Delta u = 0$, $u_\Gamma = \exp(\pi y)\sin \pi x|_\Gamma$ is considered in the square region $G = \{0 \leq x \leq 1, \; 0 \leq y \leq 1\}$, whence $u = \exp(\pi y \sin(\pi x))$. For solving the problem in the square grid Ω_h of the dimension 129x129, $h = 1/128$ four additional grids Ω_{2h}, Ω_{4h}, Ω_{8h}, Ω_{16h} were used. Calculations on each of these grids were carried out with the same accuracy ϵ. The criteria for the end of the process on each grid is taken from the relation

$$\max_{(x,y) \in \Omega_h} |u_h^k - u_h^{k-1}| < \epsilon, \tag{14}$$

where k is the number of iteration. As a basic ralaxation scheme we used either the Successive-over-Relaxation method (SOR), or Seidel's method (S). The use of the latter is caused by the fact, that Seidel's method well suppresses the high-frequency error component, which arises in the interpolation process. This component contains the eigenfunctions of the operator corresponding to large eigenvalues. The smooth part can be well restored using the relation (12). Therefore Seidel's method being used on the last grids, essentially accelerates the process. Calculations were carried out in the following way. The solution u_{16h} was transformed by the operator Q (13) into the initial approximation for u_{8h}; then using

Table 1

Table 1

ε	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	21	24	28	2	I	3.7	2.00
10^{-5}	24	32	47	6	2	7.0	1.43
10^{-6}	27	37	64	44	5	20.6	0.67
10^{-7}	30	44	67	278	33	107	0.15
	SOR	SOR	S	S	S		

Table 2

ε	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	21	24	9	2	I	3.5	2.62
10^{-5}	24	32	53	7	2	8.6	1.50
10^{-6}	27	37	218	48	5	32.	0.43
10^{-7}	30	44	45	302	33	113.	0.1
	SOR	SOR	S	S	S		

Table 3

ε	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	21	24	28	7	I	4.9	1.55
10^{-5}	24	32	47	46	7	22.0	0.5
10^{-6}	27	37	64	87	80	106.	0.13
	SOR	SOR	SOR	SOR	SOR		

Table 4

ε	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	64	87	8	2	I	3.6	2.00
10^{-5}	77	146	49	7	2	9.3	1.1
10^{-6}	91	206	214	48	5	33.9	0.40
10^{-7}	106	265	455	303	32	140.	0.1
	S	S	S	S	S		

Table 5

ε	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	21	24	37	31	I	11.5	0.73
10^{-5}	24	32	48	76	71	91.3	0.12
10^{-6}	27	37	65	100	156	185.	0.07
10^{-7}	30	44	74	129	212	249.	0.06
	SOR	SOR	SOR	SOR	SOR		

Table 6

ε	K(16h)	K(8h)	K(4h)	K(2h)	KΣ	γ	γ
10^{-6}	27	52	101	197	376	376	0.04
	SOR	SOR	SOR	SOR	SOR		

u_{16h} and u_{8h} we constructed by the formula (12) the initial approximation for u_{4h} and so on, up to the last grid Ω_h. Tables 1-4 show the main characteristics of iterations at various combinations of SOR and S methods on the sequence of five grids. k(h) denotes the number of iterations on the grid Ω_h before the condition (14) is fulfilled. kΣ designates the whole number of iterations calculated by the formula

$$k\Sigma = \sum_{i=0}^{4} k(2^i h) 4^{-i} + 1$$

which takes into account the interpolation. The number γ indicates the asymptotic efficiency of the iteration process:

$$\exp(-\gamma k\Sigma) = \epsilon \cdot \epsilon_0^{-1},$$

where ϵ_0 is the initial error.

Note, that the highest efficiency of the relaxation method proposed by R.P.Fedorenko for the analogous problem is $\gamma = 0.42$ [8] which doesn't depend on the dimension of the problem. The method of fictitious variables devised by E.S.Nikolaev and I.E.Kaporin [9] requires for decreasing the error 10^{-4} times (according to [10]) on the average six iterations for the single-linked regions and nine iterations for the couple-linked regions. The corresponding values of γ are approximately $1.1 \geq \gamma \geq 1.0$.

Table 5 shows the results of calculations for the same problem when the solution on the grid Ω_{h_k} was used only as an initial approximation for the calculation of $u_{h_{k+1}}$. For all grid the SOR method is used. For the comparison the results

of calculations for $\epsilon = 10^{-6}$ by SOR method with the initial approximation $u_{i,j} = 0$ on each grid are given in table 6. In tables 7-8 the results for the multi-linked region "B" are presented. Table 7 corresponds to the region G_1, which is obtained from the initial square by excluding the rectangles Π_1 and Π_2.

$$\Pi_1 = \{0.25 \leq x \leq 0.75; \quad 0.25 \leq y \leq 0.375\}$$

$$\Pi_2 = \{0.25 \leq x \leq 0.75; \quad 0.625 \leq y \leq 0.75\}$$

and table 8 is constructed for the rectangles

$$\Pi_1 = \{0.375 \leq x \leq 0.625; \quad 0.25 \leq y \leq 0.375\}$$

$$\Pi_2 = \{0.375 \leq x \leq 0.625; \quad 0.625 \leq y \leq 0.75\}.$$

Sizes of the rectangles Π_1 and Π_2 are correlated with the initial grid Ω_{16h}. It is seen, that the iterations convergence rate practically doesn't differ from that one for the rectangular region and is even greater for $\epsilon = 10^{-7}$. In table 9 the dependence of the convergence rate on the smoothness of solution

Table 7

ϵ	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	21	18	27	2	I	4.43	1.70
10^{-5}	23	25	39	6	2	6.42	1.54
10^{-6}	26	33	55	58	6	24.5	0.54
10^{-7}	29	42	72	168	35	82.2	0.19
	SOR	SOR	SOR	S	S		

Table 8

ϵ	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
10^{-4}	20	19	26	2	I	3.7	1.94
10^{-5}	23	26	43	6	2	6.7	1.40
10^{-6}	26	34	58	49	6	22.5	0.59
10^{-7}	28	39	71	164	34	79.0	0.20
	SOR	SOR	SOR	S	S		

Table 9

P	K(16h)	K(8h)	K(4h)	K(2h)	K(h)	KΣ	γ
I	17	22	26	2	I	3.53	2.54
2	17	29	44	6	2	8.13	1.27
3	18	31	58	64	4	24.1	0.46
4	17	32	61	72	7	29.3	0.38
5	17	34	65	67	42	63.4	0.18
6	16	35	69	60	65	84.9	0.14
7	I	36	73	54	72	90.6	0.13
8	19	37	72	49	73	90.4	0.13
9	18	39	72	44	70	86.1	0.13
10	19	39	73	41	67	84.3	0.13
20	15	39	91	97	41	72.6	0.17
	SOR	SOR	SOR	S	S		

$u(x,y)$ is shown. All calculations were performed up to $\epsilon = 10^{-5}$. The problem

$$\Delta u = -2p^2 \pi^2 \sin \pi p x \cdot \sin \pi p y$$

$$u_\Gamma = 0$$

with the exact solution $u = \sin \pi p x \cdot \sin \pi p y$ is solved in the region G. Calculations for $p = 1 \div 10$ and $p = 20$ are presented. It is seen that beginning with $p = 7$ the quantity γ does not decrease and even increases for $p = 20$. In all cases $\min \gamma \geq 0.13$. This can be explained by the fact that the smooth component of solution is extrapolated exactly by (12) and the high-frequency error component can be suppressed easily by Seidel's method.

5. CONCLUSIONS

These numerical results allow us to expect that when the error of approximate solution exhibits a regular behaviour according to (6), then the suggested method of accelerating the convergence in the case of elliptic equations gives an algorithm, which is optimal in viewpoint of the order of the number of arithmetic operations and the required storage exceeds the corresponding storage for the finest grid less than 1.5 times.

It should be noticed, that the quantity γ can be controlled during calculations and when its value becomes as small as $\gamma_{min} \geq \gamma$, one can easily change to the relaxation method[8], which requires analogous organization of calculation on the sequence of grids.

Note also, that in tables 1-4 and 7-9 the number of iterations on the last grid is always less, than the maximum number of iterations on each of the rest four grids, which means, that for the given ϵ the number of iterations does not increase with increasing h. For $\epsilon = o(h^2) = 10^{-4}$ the number of iterations decreases monotonously from one grid to another, that is, the summary computational work required is estimated by the quantity $o(h^{-2})$.

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Айрян Э.А., Жидков Е.П., Хоромский Б.Н.　　　　　E11-82-492
Быстрый итерационный метод на последовательности сеток решения разностной задачи для уравнения Пуассона

Предложен метод ускорения сходимости итерационных процессов на последовательности сеток, учитывающий разложение разностного решения по степеням шага дискретизации. Приближенные решения с нескольких вспомогательных сеток экстраполируются к точному решению разностной задачи на более мелкой метке. В случае разностной задачи для уравнения Пуассона погрешность такой экстраполяции на последней сетке быстро подавляется простыми итерациями в силу известных свойств гладкости собственных функций разностного оператора. Приведены результаты численных экспериментов, иллюстрирующие высокую эффективность метода для решения указанной задачи.

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Ayrjan E.A., Zhidkov E.P., Khoromsky B.N.　　　　E11-82-492
Fast Relaxation Method for Solving the Difference Problem for the Poisson Equation on the Sequence of Grids

A method for accelerating the convergence of iterative processes on the sequence of grids is proposed, which makes use of the decomposition of difference solution into the degrees of discretization step. Approximate solutions from a number of auxiliary grids are extrapolated to the exact solution on the finest grid. In the case of a difference problem for the Poisson equation the error of such extrapolation on the last grid is quickly suppossed by simple iterations due to same properties of smoothness of the difference operator's eigenfunctions. The results of numerical experiments are presented which illustrate the high efficiency of the proposed method for solution of the given problem.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.