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VARIANCE ANALYSIS OF THE MONTE-CARLO PERTURBATION SOURCE METHOD IN INHOMOGENEOUS LINEAR PARTICLE TRANSPORT PROBLEMS. DISCUSSION

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1. Introduction

In the first paper on the given topic we have formulated the Monte-Carlo perturbation source method (PSM) in inhomogeneous linear particle transport problems on the basis of FREDHOLM integral equations for the particle fields /1/. In that framework the formulae for the second moment of the difference event point estimator were derived. This was accomplished by an adequate extension of the adjoint integral method used by Coveyou et al. for representing the variances of the event point estimator in analog and biased solutions of ordinary particle transport problems /2/. In the present paper we analyse the general structure of the variance in the PSM, point out the variance peculiarities of this method, discuss the dependences on certain transport games and generation procedures, and draw conclusions with respect to its improvement. Only to complete the paper we preface the discussion of chapter 3 once more by the mathematical formulation and a brief outline of the PSM in chapter 2.

2. Mathematical Formulation and Outline of the Perturbation Source Method

The physical problem considered here is the following. Let us have an arrangement consisting of a constant outer particle source, a nonmultiplying material system and a detector. In the original state of the system ("zero" state) the detector gives a certain counting rate λ_0. After changing the system ("one" state) and waiting for the new equilibrium distribution the detector shows now the counting rate λ_1. We are interested in the effect shown by the detector in consequence of the system modification, i.e., in the difference λ of the counting rates λ_i

in both system states i=0,1

$$\lambda \equiv \lambda_1 - \lambda_0 . \tag{1}$$

The PSM seems to be a powerful MC method which at least in certain cases allows one to estimate the effect λ with an acceptable statistical reliability. In order to analyse its general variance behaviour we have formulated this method in a convenient mathematical model. We describe the particle distributions in both system states by event densities $E_i(x) \geq 0$ which are to be the solutions of the FREDHOLM integral equations

$$E_i(x) = S_0(x) + \int K_i(x' \to x) E_i(x') dx' , \qquad i = 0,1 \tag{2}$$

and give the counting rates

$$\lambda_i = \int D(x) E_i(x) dx , \qquad i = 0,1. \tag{3}$$

In equations (2) $S_0(x)$ represents a nonnegative source distribution which may be assumed to be normalized. The kernels $K_i(x \to y) \geq 0$ describe the transitions of a particle from an event point x to the next point in dy near y. They are completely determined by the system states. In equation (3) $D(x) \geq 0$ is the detector function describing the localization and sensibility of the detector in counting the events.

For the further explanation it is useful to introduce the so-called value functions $W_i(x) \geq 0$ of both system states which are to be the solutions of

$$W_i(x) = D(x) + \int K_i(x \to x') W_i(x') dx' , \qquad i = 0,1. \tag{4}$$

In the PSM the same effect λ is calculated as the difference of two other counting rates λ_j^* , which are the counting rates of the same dedector but of the events of two types (j=0,1) of new particles

$$\lambda = \lambda_1^* - \lambda_0^* , \tag{5}$$

where

$$\lambda_j^* = \int D(x) E_j^*(x)\,dx \quad , \qquad j = 0,1 \tag{6}$$

and $E_j^*(x) \geq 0$ are the event densities of the new so-called perturbation particles, or shorter " perturbatons ",

$$E_j^*(x) = P_j(x) + \int K_1(x' \to x) E_j^*(x')\,dx' \;, \quad j = 0,1. \tag{7}$$

We note that both types of perturbatons live in the same system "one" and thus have the same value function $W_1(x)$, but are emitted by different sources $P_j(x) \geq 0$. The sources $P_j(x)$ themselves are generated by the original particles in the system state "zero" (basic particles)

$$P_j(x) = \int P_j(x' \to x) E_0(x')\,dx' \;, \qquad j = 0,1, \tag{8}$$

where the generation kernels $P_j(x \to y)$ are defined as the nonnegative, nonmultiplying remainders of the difference of the transition kernels K_i

$$K_1(x \to y) - K_0(x \to y) = P_1(x \to y) - P_0(x \to y). \tag{9}$$

The $P_j(x)$ give the so-called perturbation source

$$P(x) = P_1(x) - P_0(x). \tag{10}$$

From equations (2) and (5) through (8) follows the general outline of the PSM:

1) According to $[S_0, K_0]$ simulate the history of a basic particle.
2) According to equations (8) generate both types of perturbatons during the random walk of the basic particle.
3) Simulate the histories of all generated perturbatons in the "one" state of the system and sum up estimators η_j^* of the λ_j^* during the lifetimes of all generated "j" perturbatons.

4) The estimator (per basic particle) of the effect λ is given by

$$\eta^* = \eta_1^* - \eta_0^* . \tag{11}$$

The variance of the difference estimator (11) is composed of the variances $Var(\eta_j^*)$ of the estimators η_j^* and of their covariance $Cov(\eta_0^*, \eta_1^*)$

$$Var(\eta^*) = \sum_{j=0}^{1} Var(\eta_j^*) - 2\,Cov(\eta_0^*, \eta_1^*). \tag{12}$$

Comparing the PSM with other methods which base on estimations of the λ_j we may generally establish that all hopes in the PSM rest on its promising features for introducing positive correlation between its subtracting estimators η_j^*.

3. Variance Analysis of the PSM Event Point Estimator

The event point estimator of the PSM is given by equation (11) with event point estimators η_j^* (j=0,1). Denoting with x_l (l=0,1...L) the event points of a basic particle and with x_{nj}(n=0,1...N) the event points of a generated "j" perturbaton the η_j^* may be represented as

$$\eta_j^* = \sum_{l=0}^{L} \sum_{n=0}^{N} w_j(x_0, x_1, \ldots, x_l; x_{0j}, x_{1j}, \ldots, x_{nj}) D(x_{nj}) , \quad j = 0, 1. \tag{13}$$

Here $w_j(x_0,\ldots,x_l;x_{0j},x_{1j},\ldots,x_{nj}) \geq 0$ is to be the statistical weight of a "j" perturbaton at its event point x_{nj} which was generated in a basic event at x_l and after that has passed the event points $x_{0j}, x_{1j}, \ldots, x_{nj}$. As contribution function the estimators contain the detector function D(x). We point out that the estimators η_j^* are summed up over all perturbatons of the type "j" which were generated during the lifetime of a basic particle.

To study the variance of the PSM event point estimator

$$Var(\eta^*) = M[\eta^{*2}] - \lambda^2 \tag{14}$$

we have to derive expressions for its second moment $M[\eta^{*2}]$. That was accomplished in /1/ with the help of the following properly defined random variables+: ,

$\bar{\xi}_0(x)$ is to be a random variable whose value is, for each possible basic particle of unit weight experiencing an event in dx near x, the total contribution to the estimate of λ , present and future, resulting from the particle during its further random walk in the system K_0 (including the present event) by generating perturbatons which then contribute to the estimate.

$\bar{\xi}_j(x)$ (j=0,1) is to be a random variable whose value is the contribution to the estimate of λ_j^* , made by a "j" perturbaton which is possibly generated in result of an event of a basic particle with unit weight in dx near x.

$\bar{\xi}_1(x)$ is to be a random variable whose value is, for each possible perturbaton of unit weight experiencing an event in dx near x, the total contribution, present and future, to the estimate (of λ_0^* or λ_1^*) of its perturbaton type.

Furthermore, let us define the random variable $\bar{\xi}(x)$ according to

$$\bar{\xi}(x) = \bar{\xi}_1(x) - \bar{\xi}_0(x) . \tag{15}$$

The specified random variables are so defined that their expected values (over all particle histories in question) are given by

$$M[\bar{\xi}_1(x)] = W_1(x) , \tag{16}$$

$$M[\bar{\xi}_0(x)] = W(x) \tag{17}$$

with W(x) as solution of

$$W(x) = M[\bar{\xi}(x)] + \int K_0(x \to x') W(x') dx' , \tag{18}$$

+Furtheron dashed entities announce that they are related to biasing schemes.

where

$$M[\bar{\mathfrak{Z}}(x)] = M[\bar{\mathfrak{Z}}_1(x)] - M[\bar{\mathfrak{Z}}_0(x)] \tag{19}$$

and

$$M[\bar{\mathfrak{Z}}_j(x)] = \int P_j(x \to x') W_1(x')\, dx' \quad , \quad j = 0, 1. \tag{20}$$

According to the definition of $\bar{\mathfrak{Z}}_0(x)$ the mean value of the estimator is given by

$$M[\eta^*] = \int S_0(x) M[\bar{\mathfrak{Z}}_0(x)]\, dx. \tag{21}$$

Starting from this equation it is easy to show that η^* is an unbiased estimator of λ , i.e.,

$$M[\eta^*] = \lambda. \tag{22}$$

The estimator variance (14) depends on the types of tranaport games carried out with the basic particlee and perturbatons and on the procedure used for the generation of the perturbatons. Only for clearness we have restricted our derivation in /1/ to the important class of survival-biasing transport games, the extension to the general biasing /2/ is straightforward. As generation procedures we considered three different biasing schems but all base on the event points of a basic particle.

For the second moment $M[\eta^{*2}]$ we have found the closed expression

$$M[\eta^{*2}] = \int M[\bar{\mathfrak{Z}}(x)]\big(2W(x) - M[\bar{\mathfrak{Z}}(x)]\big)\bar{F}_0(x)\, dx + \int Var(\bar{\mathfrak{Z}}(x))\bar{F}_0(x)\, dx. \tag{23}$$

where $F_0(x) \geq 0$ is the solution of

$$\bar{F}_0(x) = S_0(x) + \int \frac{\sigma_0(x')}{\bar{\sigma}_0(x')} K_0(x' \to x) \bar{F}_0(x')\, dx' \tag{24}$$

and $Var(\mathfrak{Z}(x))$ the variance function of $\mathfrak{Z}(x)$

$$Var(\bar{\mathfrak{Z}}(x)) = M[\bar{\mathfrak{Z}}^2(x)] - M^2[\bar{\mathfrak{Z}}(x)]. \tag{25}$$

In equation (24) and in further equations we use the denotations $\sigma_i(x)$

for the normalization functions of the transition kernels $K_i(x\to y)$, respectively

$$G_i(x) = \int K_i(x\to x')dx' \quad , \qquad i=0,1. \tag{26}$$

The $\bar{G}_i(x)$ are the biasing survival probabilities.

The variance function $Var(\bar{\xi}(x))$ is determined by the generation procedure and by the transport game used for the perturbatons. It is composed of the variance functions of the $\bar{\xi}_j(x)$ and of their covariance function

$$Var(\bar{\xi}(x)) = \sum_{j=0}^{1} Var(\bar{\xi}_j(x)) - 2\, Cov(\bar{\xi}_0(x), \bar{\xi}_1(x)) \, . \tag{27}$$

where

$$Cov(\bar{\xi}_0(x), \bar{\xi}_1(x)) \equiv M[\bar{\xi}_0(x)\bar{\xi}_1(x)] - M[\bar{\xi}_0(x)]M[\bar{\xi}_1(x)] \, . \tag{28}$$

In the <u>generation procedure (A)</u> at an event point x of a basic particle the "j" perturbatons are statistically independent generated and transfered to their first event points according to the generation probabilities $\bar{p}_j(x)$ and the transfere functions $\bar{p}_j(x\to y)$, respectively. Then,

$$M[\bar{\xi}_j^2(x)] = \frac{1}{\bar{p}_j(x)} \int \frac{P_j^2(x\to x')}{\bar{p}_j(x\to x')} M[\bar{\xi}_1^2(x')]dx' \quad , \qquad j=0,1 \tag{29}$$

and

$$M[\bar{\xi}_0(x)\bar{\xi}_1(x)] = \iint P_0(x\to x')P_1(x\to x'')M[\bar{\xi}_1(x')\bar{\xi}_1(x'')]dx'dx'' . \tag{30}$$

In the <u>generation procedure (B)</u> the "j" perturbatons are generated in pairs using a pair generation probability $\bar{p}(x)$ but transfered to their first event points by statistically independent selecting from the distribution functions $\bar{p}_j(x\to y)$. We get

$$M[\bar{\xi}_j^2(x)] = \frac{1}{\bar{p}(x)} \int \frac{P_j^2(x\to x')}{\bar{p}_j(x\to x')} M[\bar{\xi}_1^2(x')]dx' \, , \qquad j=0,1 \tag{31}$$

and

$$M[\bar{\xi}_0(x)\bar{\xi}_1(x)] = \frac{1}{\bar{p}(x)} \iint P_0(x\to x')P_1(x\to x'')M[\bar{\xi}_1(x')\bar{\xi}_1(x'')]dx'dx'' . \tag{32}$$

$$\lambda = \int M[\bar{\xi}(x)] E_o(x)\, dx \qquad (37)$$

what makes clear that λ may be estimated directly in the basic game. In that way the PSM procedure degenerated to the solution of an ordinary linear particle transport problem in the system $[S_0, K_0]$ with $M[\bar{\xi}(x)]$ as the contribution function in an event point estimator η_a^*. Using the results from /2/ for such Monte-Carlo solutions we get

$$M[\eta_a^{*2}] = \int M[\bar{\xi}(x)] \left(2W(x) - M[\bar{\xi}(x)]\right) \bar{F}_o(x)\, dx \qquad (38)$$

what is just the first part of $M[\eta^{*2}]$ in equation (23). In comparison with the true PSM procedure this solution realizes the estimation of λ using the expected value $M[\bar{\xi}(x)]$ instead of the random variable $\bar{\xi}(x)$, i.e.,

$$\bar{\xi}(x) \longrightarrow M[\bar{\xi}(x)]. \qquad (39)$$

Taking into consideration (39) equation (38) may also be derived from balance (II) in /1/.

b) <u>$W_1(x)$ is assumed to be known</u>

Using equations (8),(10),(19) and (20) λ from equation (37) may be rewritten as functional of the perturbation source

$$\lambda = \int W_1(x)\, P(x)\, dx. \qquad (40)$$

This relation shows that, having $W_1(x)$, λ may be estimated by only simulating the perturbation source. Practically the PSM procedure then terminates at the first event points of the perturbatons and $W_1(x)$ must be used as the contribution function in the event point estimators $\eta_{j,b}^*$ for the difference estimator of this variant. That is, this state of the PSM includes the generation process of the perturbatons (up to their first event point) but not their transport game. In comparison with a) the occurrence of the generation process in the Monte-Carlo solution results in an additional variance contribution

$$Var(\eta_b^*) = Var(\eta_a^*) + \int Var_b(\bar{\xi}(x))\, \bar{F}_o(x)\, dx. \qquad (41)$$

In the <u>generation procedure (C)</u> only one perturbaton is generated which now directly represents the difference of both perturbation source terms. In extension of the procedure (B) here we use not only the same generation probability $\bar{p}(x)$ but also a common transfer function $\bar{p}(x \to y)$ for both generation kernels. We find

$$M[\bar{\xi}_j^2(x)] = \frac{1}{\bar{p}(x)} \int \frac{P_j^2(x \to x')}{\bar{p}(x \to x')} M[\bar{\xi}_1^2(x')]dx' \quad , \quad j = 0,1 \tag{33}$$

and

$$M[\bar{\xi}_0(x)\bar{\xi}_1(x)] = \frac{1}{\bar{p}(x)} \int \frac{P_0(x \to x')P_1(x \to x')}{\bar{p}(x \to x')} M[\bar{\xi}_1^2(x')]dx'. \tag{34}$$

The moments $M[\bar{\xi}_1^2(x)]$ and $M[\bar{\xi}_1(x)\,\bar{\xi}_1(y)]$ appearing in equations (29) through (34) are determined by the transport game of the perturbatons. We have not yet studied correlated games for both types of perturbatons. For the uncorrelated biasing games we have

$$M[\bar{\xi}_1^2(x)] = D(x)\big(2W_1(x) - D(x)\big) + \frac{G_1(x)}{\bar{G}_1(x)}\int K_1(x \to x')M[\bar{\xi}_1^2(x')]dx' \tag{35}$$

and

$$M[\bar{\xi}_1(x)\,\bar{\xi}_1(y)] = W_1(x)W_1(y) \quad , \quad x \neq y. \tag{36}$$

With the help of equation (23) we now analyse the general structure of the variance (14) of the PSM event point estimator (11). For this end we look at three different states of the PSM which differ by the degree of knowledge on the solution (evaluation of λ) utilized in the Monte-Carlo calculation.

a) <u>$M[\xi(x)]$ is assumed to be known</u>

This means that with $W_1(x)$ the solution of the field equation (4) is known and, furthermore, that the quadratures (20) and the subtraction (19) are carried out deterministically. Using equations (4) through (8) λ may be represented

The knowledge on the solution is reduced from $M[\xi(x)]$ to $W_1(x)$, i.e., the quadratures (20) and the subtraction (19) are now performed statistically in the Monte-Carlo procedure. This loss in knowledge causes the increase in variance.

In comparison with the true PSM procedure in this variant b) λ is estimated by using the expected value $M[\bar{\xi}_1(x)] = W_1(x)$ instead of the random variable $\bar{\xi}_1(x)$, i.e.,

$$\bar{\xi}_1(x) \longrightarrow M[\bar{\xi}_1(x)] . \qquad (42)$$

With (42) equation (36) is valid and, furthermore,

$$M[\bar{\xi}_1^2(x)] = W_1^2(x) . \qquad (43)$$

With that the expressions for the variance functions $Var_b(\xi(x))$ for all generation procedures may be derived from equations (27) through (34) /1/. It should yet be pointed out that the additional variance part also depends on the transport game of the basic particles via the weighting function $F_0(x)$ in the quadrature of $Var(\xi(x))$.

c) Only the detector function D(x) is known

In this case we have to perform the complete PSM procedure. The general structure of the variance is the same as in b)

$$Var(\eta^*) = Var(\eta_a^*) + \int Var(\xi(x)) \bar{F}_0(x)\, dx \quad , \qquad (44)$$

however, the additional variance term is increased because of

$$Var(\xi(x)) \geq Var_b(\xi(x)) \qquad (45)$$

what is in consequence of

$$M[\bar{\xi}_1^2(x)] \geq W_1^2(x) \qquad (46)$$

if the random variable $\bar{\xi}_1(x)$ is used instead of its expected value as it was in b). As the $\bar{\xi}_1(x)$ we use the event point estimator with D(x) as the contribution function in a history starting at x and taking place in the system K_1. In comparison with a) not only the integrals (20) and their difference (19) but also $W_1(x)$ itself is unknown. The extraction of all these informations is now included in the Monte-Carlo procedure. Comparing with b), the further loss in knowledge results in a further increase of variance.

Next we show which elements of the PSM in what way help in positive correlating both estimators η^*_j. For this purpose we compare the variance of the PSM estimator with that of a difference estimator (11) with statistically independent estimators η^*_j. The latter means that we have two statistically independent, modified PSM calculations each of them containing only a single generation process, namely, for the estimation of λ^*_0 (λ^*_1) the generation of "zero" ("one") perturbatons. The variances Var (η^*_j) are easily calculated with the help of the formulas derived in /1/ for the case of a single generation process. Then, in accordance with equation (12), we find for the covariance

$$\mathrm{Cov}(\eta^*_0, \eta^*_1) = \frac{1}{2}\int\Big(M_1(2W_0^0 - M_0) + M_0(2W_0^1 - M_1)\Big)\bar{F}_0(x)\,dx + \int \mathrm{Cov}\big(\mathfrak{Z}_0(x), \mathfrak{Z}_1(x)\big)\bar{F}_0(x)\,dx - \lambda^*_0\lambda^*_1 . \tag{47}$$

where we used the abbreviations

$$M_j = M[\bar{\mathfrak{Z}}_j(x)] \quad , \quad W_0^j = W_0^j(x) \quad , \quad j = 0,1 \tag{48}$$

and defined the $W_0^j(x) \geq 0$ as solutions of

$$W_0^j(x) = M[\bar{\mathfrak{Z}}_j(x)] + \int K_0(x \to x')\,W_0^j(x')\,dx' \quad , \quad j = 0,1. \tag{49}$$

The integral terms on the right side of equation (47) announce two features of the PSM which may help in positive correlating the estimators η^*_j. The term containing the covariance function Cov($\mathfrak{Z}_0(x)$, $\mathfrak{Z}_1(x)$) is determined by the simulation procedure of the generation process and by the following random walk of the perturbatons. The behaviour of this function is immediately discussed in more detail for different cases. The other, nonnegative integral term stems from that fact that in the PSM the generation processes of both types of perturbatons are based on the same basic histories. Both constituents of the covariance enlighten that the common generation process of the perturbatons and their following random walk in the same system state are the promising features of the PSM for introducing positive correlation between subtracting estimators and, therefore, are recisive with respect to the usefulness of this method at all.

Now let us discuss the effects of the different generation procedures considered above on the variance function Var($\mathfrak{F}(x)$). Because in the generation procedure (A) both perturbatons are independently generated according to $\bar{p}_j(x)$ and $\bar{p}_j(x \to y)$ these functions influence the variances Var($\mathfrak{F}_j(x)$) only and not the covariance Cov($\mathfrak{F}_0(x)$, $\mathfrak{F}_1(x)$). The dependences of the second moments $M[\mathfrak{F}_j^2(x)]$ on the biasing functions $\bar{p}_j(x)$ and $\bar{p}_j(x \to y)$ are those as usually in a biasing of a transition kernel /2/. Obviously,

$$\bar{p}_j(x) = 1 \qquad , \qquad j = 0, 1 \tag{50}$$

results in minimum variance.

From equations (27) through (30) we see that variance reduction by a positive covariance function Cov($\mathfrak{F}_0(x)$, $\mathfrak{F}_1(x)$) requires positive correlated transport games of the "zero" and "one" perturbations so that

$$M[\bar{\xi}_1(x)\,\bar{\xi}_1(y)] \geq W_1(x)\,W_1(y) \quad , \quad x \neq y . \tag{51}$$

Such possibilities should be theoretically and practically investigated. For uncorrelated transport games the equality is valid (equation (36)), hence,

$$Cov(\bar{\xi}_0(x), \bar{\xi}_1(x)) = 0 . \tag{52}$$

The generation procedure (B) comprises a positive correlation in the generation of both perturbatons and thus gives variance reduction. It is obvious that for

$$\bar{p}_j(x) = \bar{p}(x) = 1 \qquad , \qquad j = 0, 1 \tag{53}$$

(A) and (B) become identical. However, for

$$\bar{p}_j(x) = \bar{p}(x) < 1 \quad , \qquad j = 0,1 \tag{54}$$

we gain in (B) by a positive covariance function Cov($\bar{\mathfrak{I}}_0(x)$, $\bar{\mathfrak{I}}_1(x)$) and this even in the case if the transport games of both perturbatons are uncorrelated, i.e.,

$$Cov\left(\bar{\mathfrak{I}}_0(x), \bar{\mathfrak{I}}_1(x)\right) = \left(\frac{1}{\bar{p}(x)} - 1\right) M\left[\bar{\mathfrak{I}}_0(x)\right] M\left[\bar{\mathfrak{I}}_1(x)\right] . \tag{55}$$

The <u>generation procedure (C)</u> could be quite favourable with respect to variance reduction provided there are sufficiently extended phase space regions where the generation kernels are overlapping, i.e.,

$$P_0(x \to y)\, P_1(x \to y) > \vartheta . \tag{56}$$

Then, there would be a great portion of common generation processes for the "zero" and "one" perturbatons which have equal, expected contributions to their estimators η^*_0 and η^*_1 , respectively. Just those contributions are identically simulated by one resulting perturbaton of the procedure (C). Of course, furthermore we would have a considerably reduced numerical expense by economizing the transport game of one perturbaton. On the other hand, nothing will be gained by (C) if there are no such overlapping regions. On the contrary, then the common biasing by the transfer function $\bar{p}(x \to y)$ would split in a generation of either a pure "zero" or a pure "one" perturbaton. With $\bar{p}(x) = 1$ we would get the variances Var ($\bar{\mathfrak{I}}_j(x)$) from the generation procedure (A) but additionally a negative covariance function

$$Cov\left(\bar{\mathfrak{I}}_0(x), \bar{\mathfrak{I}}_1(x)\right) = - M\left[\bar{\mathfrak{I}}_0(x)\right] M\left[\bar{\mathfrak{I}}_1(x)\right] \tag{57}$$

what is caused by the "either - or" generation.

Next we point out the dependence of Var(η^*) on the distribution of the random variable $\mathfrak{I}(x)$ which takes a decisive part with respect to the efficiency of the PSM. The first variance part Var (η^*_a) is completely determined by its mean value $M[\mathfrak{I}(x)]$ (see equation (38)), but the second by its variance Var($\mathfrak{I}(x)$). The first part decreases when $M[\mathfrak{I}(x)] \to 0$ whereas the second goes to a limit determined by $M[\mathcal{F}^2(x)] \geq 0$. Obviously, this fact has mischievous consequences for the PSM in the tendency of decreasing effect by decreasing $M[\mathcal{F}(x)]$. Thus, in dependence of the distribution of this variable $\mathcal{F}(x)$ the PSM may show quite different efficiencies in cases with the same order of the effect λ .

Two further general properties of the PSM are worth to note. The first concernes the dependence of the variance on the transport games chosen for the basic particles and for the perturbatons. As in ordinary transport problems with nonmultiplying transition kernels the EV-biasing game among all survival biasing games leads to the smallest variance /3/. Its application in the transport of the perturbatons minimizes the $M[\mathcal{F}_1^2(x)]$ from equation (35) and by that the variance function Var($\mathcal{F}(x)$) in the second term of equation (23). The EV-biasing in the basic game minimizes $\bar{F}_0(x)$ from equation (24) and in that way both parts of the second moment $M[\xi^{*2}]$. Though, the basic game influences both terms, in practice we are kept to handle very carefully the application of the EV-biasing game because the prolongation of the basic histories increases the number of generated perturbatons and by that considerably the entire expense. Doubtless we have to seek for appropriate procedures for selecting real generation events from all possible ones of a basic history. The other general property to be pointed out is the asymmetry between both system states with respect to the variance of the PSM estimator, i.e., an interchange of the states for the basic and the perturbaton transport games (see /1/), in general, will result in a different variance. It seems to be difficult but paying to deduce recommendations on the disposition of the system states to the basic and perturbaton game. Model investigations should help to enlighten this problem. Furthermore, it should be noted that a calculation of $-\lambda$ instead of λ with the same disposition of the system states gives the same variance.

We want yet to hint at the application of a general variance reduction method, the importance function method (IF-method), in the PSM+. In ordinary particle

+: A detailed discussion will be given in another paper.

transport problems the IF - method is known to be one of the most powerful means in variance reduction. It even allows to construct two direct zero-variance solutions in the special case if the value function is used as the importance function /3,4,5/. Therefore, it is of great interest to ask for the efficiency of this variance reduction method in the case of the PSM.

At first let us consider the effect of its application on the generation process and transport game of the perturbatons. We assume to know the value function $W_1(x)$ of the state K_1 and the normalization functions $M[\mathfrak{Z}_j(x)]$ of the modified generation kernels $P_j(x \rightarrow y)\ W_1(y)$, respectively. Keeping on the biasing technique used in the analysis of the generation process and denoting all biasing entities involved for this special case by tildes, we would have to use the biasing transfer functions

$$\tilde{p}_j(x \rightarrow y) = P_j(x \rightarrow y) W_1(y) / M[\tilde{\mathfrak{Z}}_j(x)] \quad , \qquad j = 0,1 \tag{58}$$

in the generation procedures (A) or (B) for the case (53)[+] .

Hence,

$$M[\tilde{\mathfrak{Z}}_j^2(x)] = M[\mathfrak{Z}_j(x)] \int \frac{P_j(x \rightarrow x')}{W_1(x')} M[\mathfrak{Z}_1^2(x')]\, dx' \, , \qquad j = 0,1. \tag{59}$$

Both, with the value function $W_1(x)$ modified perturbaton transport games corresponding to the zero-variance solutions mentioned previously are characterized by a deterministic variable $\mathfrak{Z}_1(x) = M[\mathfrak{Z}_1(x)] = W_1(x)$. Thus,

$$M[\tilde{\mathfrak{Z}}_j^2(x)] = M^2[\tilde{\mathfrak{Z}}_j(x)] \quad , \qquad j = 0,1 \tag{60}$$

and therefore

$$Var(\mathfrak{Z}(x)) = 0 . \tag{61}$$

Of course, this result is in accordance with the degenerated case a) of the PSM discussed above. The proper utilization of the knowledge of $W_1(x)$ and $M[\mathfrak{Z}_j(x)]$ $(j=0,1)$ allows to zero the variance part caused by generation pro-

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+: Of course, mean values remain the same, e.g., $M[\mathfrak{Z}_j(x)] = M[\tilde{\mathfrak{Z}}_j(x)]$.

cesa and perturbaton transport game. Here is the deciding fact that both types of perturbatona have the aame value function $W_1(x)$ becauae they live in the same ayatem state and are counted by the aame detector. In that way the general usefulneas of the application of the IF - method in this part of the PSM is demonstrated.

Now we turn to the application of the importance function method in the transport game of the basic particles having in view the reduction of the first variance portion Var(η_a^*) in equation (44). At first we consider the degenerated case a) and assume a nonnegative function $M[\mathcal{F}(x)]$ +. Then, knowing not only $M[\mathcal{F}(x)]$ as in a), but alao $W(x)$ and λ we are able to construct both modified zero-variance aolutiona of the ordinary particle transport problem. If $M[\mathcal{F}(x)]$ is an alternating function then, in general, it is $W(x)$ too and a zero-variance solution for λ could only be constructed as a difference of the two zero-variance solutions for λ_j^* , both modified with the nonnegative constituents $W_o^j(x)$ of $W(x)$, respectively. Similarly, in the apecial case of two separate PSM calculations each of them containing only a aingle gsneration process for the estimation of the λ_j^* we could also conatruct a zero-variance solution of λ . For that we would have to modify both basic transport games with the importance functions $W_o^j(x)/\lambda_j^*$, but the transport games of the perturbatons with $W_1(x)$ and using the normalization functions $M[\mathcal{F}_j(x)]$ in the generation processes for j=0,1, respectively. The principle of such a modified zero-variance solution for the λ_j^* is that the weight of each basic particle is fully transfered to the perturbatons generated during its lifetime and then fully converted by them into the eatimate. In general, thia zero-variance principle cannot be realized in the true PSM procedure with one basic transport for both generation processes. Thia is only possible in the csse of a nonnegative function $M[\mathcal{F}(x)]$ and using the single generation procedure (C) for the modified generation process, i.e., if we unite both generation proceasea in an analytical way. For that we would have to use $W(x)/\lambda$ as the importance function

+: The case of a nonpositive function $M[\mathcal{F}(x)]$ may be reduced to that case.

in the basic game and $W_1(x)$ in the perturbaton game. Here ia of importance that the single, nonnegetive generation process involves the appearence of only one value function $W(x)$ for the basic particles wheress two generation processee result in two different value functions $W_0^j(x)$ which demand different modified basic games.

The fact that the true PSM has lost the ideal zero-variance solutions of the IF - method should not be taken too seriously. In practical applicationa the use of this variance reduction method both in the basic and in the perturbaton game will doubtless yield a substantial improvement of the PSM. However, the efficiency of the IF - method in the PSM ahould be investigated in more detail by model calculations.

Last we outline an approximative version (bin-version) of the PSM which should turn out to be quite favourable in practical applications. We have not yet taken trouble to show the improvements by this version, but some "phyaical" arguments ssem doubtless to apeak in its favour. We start from the adjoint representations of the counting rates λ_j^* from equations (6)

$$\lambda_j^* = \int W_1(x) P_j(x) dx \qquad , \qquad j = 0,1. \tag{62}$$

Let ua suppose the phase spacs of the system to be devided in G subregions (bins). In this grained phase space the integrals (62) may be repreaented as sums

$$\lambda_j^* = \sum_{g=1}^{G} W_{1,j}^g * P_j^g \qquad , \qquad j = 0,1, \tag{63}$$

where we have defined

$$P_j^g = \int^g P_j(x) dx \qquad , \qquad j = 0,1 \tag{64}$$

and bin mean values

$$W_{1,j}^g \equiv \int^g W_1(x) P_j(x) dx / P_j^g \qquad , \qquad j = 0,1. \tag{65}$$

If the bins are sufficiently small and properly chosen so that in a bin g the mean values $W_{1,j}^g$ for j=0,1 are approximately of the same magnitude W_1^g, i.e.,

$$W_{1,0}^{g} \cong W_1^{g} \quad , \quad W_{1,1}^{g} \cong W_1^{g} \quad , \quad g=1,2,\ldots,G, \tag{66}$$

then λ may be calculated from

$$\lambda = \sum_{g=1}^{G} W_1^{g}\left(P_1^{g} - P_0^{g}\right). \tag{67}$$

A complete Monte-Carlo calculation of λ on the basis of equation (67) would look as follows:

At the first event points of the generated "j" perturbatons the P_j^g are estimated by summing up the particle weights for all bins g and perturbaton types j. The following transport games of all perturbatons starting in a bin g would give the estimates of the W_1^g using the event point estimator D(x) as the contribution function.

Likewise as in the PSM variant b) with the known value function $W_1(x)$ the variance reducing feature of this approximative bin-method consists in the circumstance that here all "zero" and "one" particles having their first events in the same bin g would have the same contribution W_1^g to the estimate. The applicability of this approximative PSM version should be tested in practice.

As a next step, with the help of the theoretical foundation presented in this paper, we intend to investigate the variance behaviour of different PSM versions in simple but practical models.

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