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RECONSTRUCTION OF PARTICLE MULTIPLICITY DISTRIBUTIONS USING THE METHOD OF STATISTICAL REGULARIZATION

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Summary

The problem of reconstruction of particle multiplicity distributions from experimental data is discussed. Because of statistical errors involved in the data the problem of reconstruction is "incorrectly posed" which results in the oscillatory behaviour of the direct solution when the detection efficiency ϵ is substantially lower than 100%.

It is shown that the method of statistical regularization used reconstructs the real distribution and allows one to estimate the rms errors of the results for $\epsilon \geq 25\%$. The possibilities of the method are examined on the basis of the measurements of the multiplicity distribution of neutrons from spontaneous fission of ^{244}Cm.

The application of the method to determination of the multiplicity distributions for three Fm isotopes is presented.

1. Introduction

Experiments on determining the average number of particles per interaction and their multiplicity distributions are quite common in low- and high-energy nuclear physics.

For a detector efficiency lower than 100%, or when an indirect method of registration is used, the observed multiplicity distribution is different from the real one and this difference should be suitably taken into account in an analysis of the data.

The problem of accounting for the efficiency of a measuring device consists, as a rule, in the solution of a system of linear algebraic equations of the type

$$\sum_{i=1}^{n} k_{ji} \phi_i = f_j , \quad j = 1,2,...,m , \qquad (1)$$

where: ϕ_i are the unknown components of the particle multiplicity distribution, f_j are the experimentally measured components of the registered multiplicity distribution, k_{ji} is the matrix of the coefficients, converting unknown components ϕ_i into measured ones (f_j).

Very often the errors involved in f_j cause difficulties in solving the system of equations. The direct solution of this system of equations gives reasonable results for detection efficiencies higher than approx. 70% while for lower efficiencies the solution has usually an incorrect and oscillating nature.

The aim of the present paper is to extract as much information as possible on the real multiplicity distribution of particles from the experimental data obtained with a low detection efficiency.

2. Fission Neutron Multiplicity. The Direct Reconstruction Method and its Incorrectness

As an example, the measurement of the multiplicity distribution of prompt fission neutrons emitted by the excited fission fragments, is discussed.

In these experiments neutrons are counted in coincidence with fragments. The neutrons moderated to thermal velocities are registered by proportional counters or scintillation detectors containing materials with high thermal neutron capture cross sections (Cd, Gd). The detection efficiency (ϵ) of one neutron varies from 20% to 80% depending on the type of the detector used.

It is reasonable to assume that neutrons from a fission act are registered independently. In this approximation the detection probability F_n of n neutrons is obtained by summing up the partial probabilities of detection for the emission of $\nu = n, n+1, \ldots, \nu_{max}$ neutrons:

$$\sum_{\nu=n}^{\nu_{max}} k_{n\nu} P_\nu = F_n \quad , \quad n = 0,1,2,\ldots, n_{max}$$

$$k_{n\nu} = \frac{\nu!}{n!(\nu-n)!} \epsilon^n (1-\epsilon)^{\nu-n}, \tag{2}$$

where: P_ν are the components of the real neutron distribution (the emission probability for ν neutrons), ν_{max} is the maximum possible number of neutrons emitted per fission.

The distributions F_n and P_ν are normalized as follows:

$$\sum_{n=0}^{n_{max}} F_n = 1, \qquad \sum_{\nu=0}^{\nu_{max}} P_\nu = 1.$$

The exact solution of the system (2) (which is the only possible one for an exactly known right-hand side) can be found according to [1,2] as follows:

$$P_\nu^d = \sum_{n=\nu}^{n_{max}} \frac{n!}{\nu!(n-\nu)!} \epsilon^{-\nu} (1-\epsilon^{-1})^{n-\nu} F_n \quad , \quad \nu = 0,1,2,\ldots,\nu_{max} \;. \tag{3}$$

It is clear from physical considerations that the real distribution of fission neutron multiplicity , reflecting the excitation energy distribution of the fragments, is the "smooth", non-negative function $P_\nu = f(\nu)$. At the same time, both the multiple production process and detection process are essentially statistical and, consequently, the measured values of F_n are burdened with errors. The system of equations (2) can be solved by the direct method using formulae (3). However, owing to the fact that the right-hand side of equations (2) is known only approximately, we can arrive at solutions containing large, oscillating, and sometimes even negative components of P_ν^d. The strong dependence of the direct solution on the errors involved in F_n is observed in this case. As a consequence, the problem of reconstruction of P_ν using the experimental values of F_n appears to be incorrectly posed, at least for $\epsilon \leq 60\%$ and not very large statistics. Under these conditions the "exact" solution is void of sense and has to be replaced by an approximate, "regularized" one.

3. Method of Statistical Regularization

We give a brief description of the main principles of the method (for convenience referred to as the "STREG" method). More detailed information can be found in [3,4] and in review [5].

The method consists in introducing an a priori information about the unknown function. In our case it is an information about the smoothness and non-negativity of the solution. The function to be reconstructed is dependent on the discrete integer argument. The mathematical methods developed in [3,4] concern, strictly speaking, only the systems of algebraic equations obtained as an approximation of the integral or differential equations. The method is however valid for our problem, as no assumptions on the necessity of transition to the continuous function were formally made.

The assumption on the smoothness of the unknown function is done in the STREG method by imposing the probabilistic restrictions on the value of a certain functional computed using the values of the function at support points. The commonly used functional is the finite-difference approximation of Euclidean norm of the second derivative:

$$\Omega(\vec{\phi}) = \sum_{i=3}^{n} \left[\frac{1}{h^2}(\phi_i - 2\phi_{i-1} + \phi_{i-2})\right]^2, \tag{4}$$

where: $\vec{\phi}$ is the vector whose components ϕ_i are the values of the unknown function at consecutive support points, h is a distance between neighbouring support points (a step). In our case $i = \nu + 1$, $\phi_i = P_{\nu+1}$, $h=1$. The value of ν_{max} was taken to be equal to 8, hence $n = 9$.

The approximate value of the functional $\Omega(\vec{\phi})$ is estimated in the following way. We consider in the space of $\vec{\phi}$ vectors the probability distribution with a density:

$$p_a(\vec{\phi}) = c_a \exp\{ - \frac{a}{2} \Omega(\vec{\phi}) \}, \qquad (5)$$

where: $a > 0$ is a parameter characterizing the smoothness of the unknown function, c_a is the normalizing coefficient dependent on a.

It can be shown that the average value of the functional $\Omega(\vec{\phi})$ over this distribution is n/a. The functions $\vec{\phi}$ for which $\Omega(\vec{\phi})$ is noticeably greater than n/a are suppressed by the exponent in $p_a(\vec{\phi})$. If the approximate value of the functional $\Omega(\vec{\phi})$ is known for the sought function $\vec{\phi}$, we can estimate a and take $p_a(\vec{\phi})$ as an a priori density of the probability for $\vec{\phi}$. Using the apparatus of mathematical statistics known as the Bayesian strategy, we can obtain a "regularized" solution and its rms errors. This is one of the versions of the STREG method. It requires an a priori information on parameter a, i.e. an a priori estimate of the value of the $\Omega(\vec{\phi})$ functional. If this information is not available, a more complicated variant of the method is used. In this case the a priori information about $\vec{\phi}$ is given in the form of a "laminar ensemble" (for more detailed explanation see ref. /5/):

$$p(\vec{\phi}) = const \int p_a(\vec{\phi})\, da. \qquad (6)$$

The "layers" are the ensembles of smooth functions with different fixed values of a, and the solution is obtained as their superposition. In other words, all the a priori values of a have equal probabilities. The solution in this ensemble reduces, in fact, to an a posteriori estimate of a from the experimental data, i.e. from equations (2). In the present paper the two above-mentioned variants of the method were combined. When the experiment was sufficiently informative, the parameter a was estimated a posteriori. The value of a found in this way was subsequently used as an a priori one for less informative experiments.

Errors in the values of F_n were considered to be independent and normally distributed. In reality, however, the main error

component which is a statistical one, has the Poisson distribution. For the components F_n computed on the basis of only a few and zero counts it would be more desirable to use this distribution, but this is unlikely to affect our results seriously.

For the reconstructions using the STREG method, the Algol and Fortran versions of programs have been used. The detailed description of the formulae of the method and the Algol version of the program are published elsewhere /6/. The calculations were made using a BESM-6 computer.

4. Some Examples of Regularized Solutions

To illustrate the different aspects of the STREG method, the data on the multiplicity distribution of spontaneous fission neutrons of ^{244}Cm were analysed. These data were obtained using devices /7,8/ with different efficiencies. The distributions reconstructed by the STREG method were discussed and compared with the results of the direct solution of equations (2) P_ν^d .

The experimental values of F_n in fig. 1a are taken from /7/. The neutron detection efficiency here is rather high (75.6%) and the total number of detected fission events is $M = 16200$. Under these conditions the error in the direct solution P_ν^d is reasonably small and, therefore, it is acceptable. The regularized solution P_ν^r coincides with high accuracy with the direct one. The errors involved in the regularized solutions are equal to those of the nonregularized ones.

Figure 1b shows the data obtained using the apparatus described in /8/. The registration efficiency was $\epsilon = 48.3\%$, the number of fissions analysed being $M = 7169$. In this case the direct solution P_ν^d is unacceptable. The regularized solution P_ν^r agrees with the curve P_ν in fig. 1a with an accuracy better than the error of reconstruction.

To estimate the extremal possibilities of reconstruction using the STREG method, the following experiment was made. From the real experimental data obtained in /8/ a small part ($M = 4039$ events taking account of pulses from only half of the neutron detectors) was used. This corresponds to a total efficiency of 23.7%. The resulting curves are shown in Fig. 2. The direct solution P_ν^d gives an absurd result. The regularized solution P_ν^r has noticeably larger errors than in the previous case (fig. 1b) but within the error limits it agrees again with the results of more precise experiments.

The regularized solution is not the exact solution of equations (2) if the real values of F_n are substituted by experimental

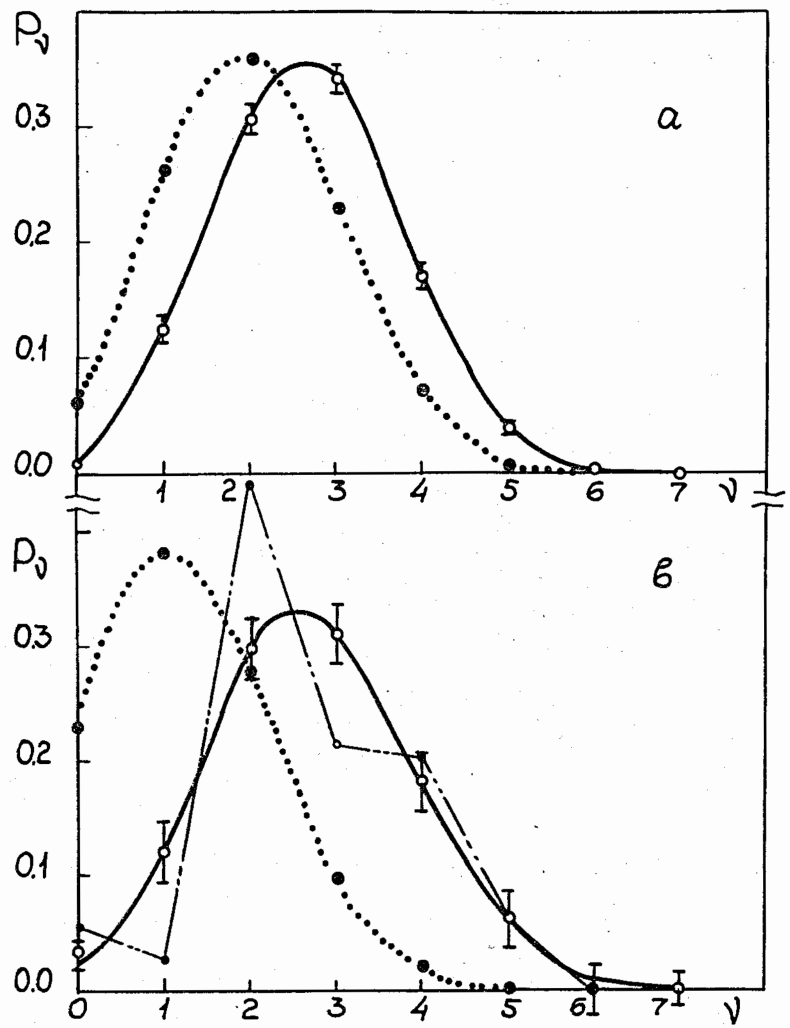

Fig. 1. Multiplicity distributions obtained in experiments at $\epsilon = 75.6\%$ (fig. 1a) and $\epsilon = 48.3\%$ (fig. 1b). The dotted line is experimental values of F_n , the dot-dash line is the result of reconstruction using direct formulae (P_ν^d), and the solid line-the results of regularized reconstruction (P_ν^r).

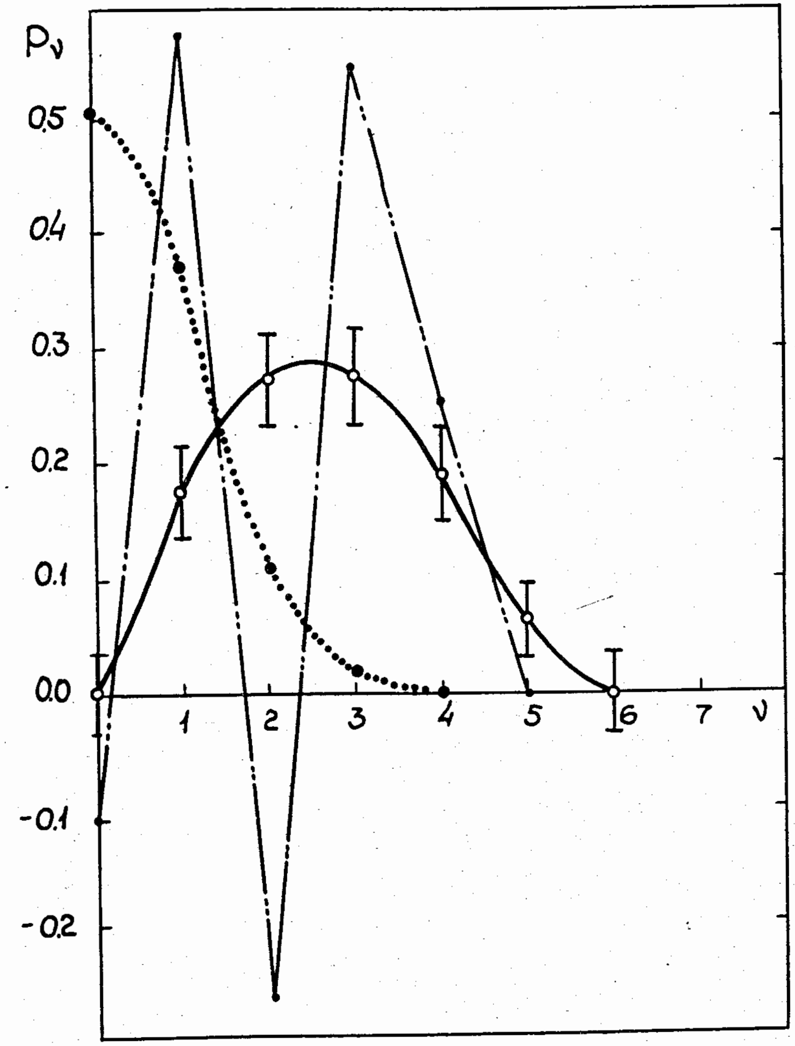

Fig. 2. Multiplicity distributions obtained in the experiment at $\epsilon = 23.7\%$ (the same notation as in fig. 1).

ones. It is interesting to verify with what accuracy this solution satisfies equations (2).

In table 1 the experimental values of F_n (and their rms errors s_n) for $\epsilon = 48.3\%$ are listed. In the first two columns the values of F_n^{comp} are listed, obtained using regularized P_ν^r in the left-hand side of equations (2). Note that the difference $|F_n - F_n^{comp}|$ is much smaller than s_n. The next two columns compare the regularized solutions for $\epsilon = 48.3\%$ and $\epsilon = 75.6\%$.

The prompt neutrons from the spontaneous fission of Fm isotopes were investigated by different authors /9–11/. However, only the experimental distributions of F_n and the integral characteristics $\bar{\nu}$ and σ_ν^2 of real distributions were quoted in these papers. The true distributions could not be obtained because of the incorrectness of the problem in the case of 48 - 61% efficiencies achieved in /9–11/. These distributions reconstructed using the STREG method are shown in fig. 3 and in table 2.

The value of $\bar{\nu} = 3.756$ for ^{252}Cf was used as a standard and the efficiency of the detectors from /9,11/ were accordingly renormalized. Data from /9,11/ were corrected only for a background, and for data from ref. /10/ corrections for the detector resolving time were also introduced.

5. The Effect of Errors Involved in F_n and ϵ

The reduction of rms errors s_n in the experimental values of F_n leads to a decrease in the error of the reconstructed function P_ν^r. However, this error does not decrease proportionally to s_n, as in the case of the direct solution P_ν^d, but considerably more slowly. For example, in one of the experiments with $\epsilon = 48.3\%$, a 9-fold increase in statistics (from $M = 7169$ to $M = 65015$), with the consequent lowering of the errors s_n by three times, the error of P_ν^r decreased only by about 30%. This effect is due to the fact that the significant contribution to the estimated error of reconstruction is made by the higher expansion components in the system of orthogonal functions, which are indefinite both for $M = 7169$ and $M = 65015$. Therefore, for a given value of ϵ (which determines the spectral properties of the kernel of equations (2)), even a large increase in experimental accuracy does not increase the accuracy of P_ν^r above a certain limit. At the same time, as it may be seen from the above figures, quite modest statistics is sufficient to obtain a reasonable, though not highly accurate, solution P_ν^r. These considerations could be useful in planning experiments.

Table 1

Comparison of experimental and computed distributions for $\epsilon = 48.3\%$; comparison of regularized solutions for $\epsilon = 48.3\%$ and $\epsilon = 75.6\%$

n, ν	F_n	F_n^{comp}	$P_\nu^r(\varepsilon = 48.3\%)$	$P_\nu^r(\varepsilon = 75.6\%)$
0	0.2296 ± 0.0063	0.2298	0.030 ± 0.011	0.007 ± 0.003
1	0.3789 ± 0.0087	0.3836	0.126 ± 0.023	0.126 ± 0.007
2	0.2777 ± 0.0071	0.2682	0.288 ± 0.025	0.306 ± 0.011
3	0.0917 ± 0.0040	0.0974	0.304 ± 0.025	0.342 ± 0.011
4	0.0204 ± 0.0018	0.0193	0.187 ± 0.023	0.173 ± 0.008
5	0.0017 ± 0.0005	0.0017	0.065 ± 0.018	0.040 ± 0.004
6	0.0000 ± 0.0001	0.0000	0.000 ± 0.017	0.006 ± 0.002

Table 2
The multiplicity distributions of fission neutrons for *Fm* isotopes, reconstructed using the STREG method

Isotope	^{254}Fm	^{256}Fm	^{257}Fm
Reference	9	10	11
M	870	204	1499
ε	61.1%	48.3%	51.0%
$\bar{\nu}$	3.98 ± 0.19 [1/]	3.73 ± 0.18	4.01 ± 0.13 [1/]
σ_ν^2	1.49 ± 0.20	2.30 ± 0.65	2.92 [2/] $^{+1.27}_{-1.68}$
P_0	0.003 ± 0.012	0.000 ± 0.036	0.059 ± 0.015
P_1	0.020 ± 0.027	0.080 ± 0.043	0.042 ± 0.029
P_2	0.095 ± 0.030	0.157 ± 0.048	0.077 ± 0.030
P_3	0.246 ± 0.034	0.217 ± 0.048	0.163 ± 0.035
P_4	0.317 ± 0.035	0.239 ± 0.048	0.232 ± 0.036
P_5	0.223 ± 0.033	0.201 ± 0.045	0.221 ± 0.036
P_6	0.076 ± 0.029	0.102 ± 0.040	0.146 ± 0.033
P_7	0.012 ± 0.026	0.004 ± 0.031	0.060 ± 0.033
P_8	0.008 ± 0.013	0.000 ± 0.013	0.000 ± 0.021

1/ renormalized using the value of $\bar{\nu}(^{252}Cf) = 3.765$
2/ value from ref. /11/.
All the other values in table 2 were calculated using experimental data from /9,10,11/.

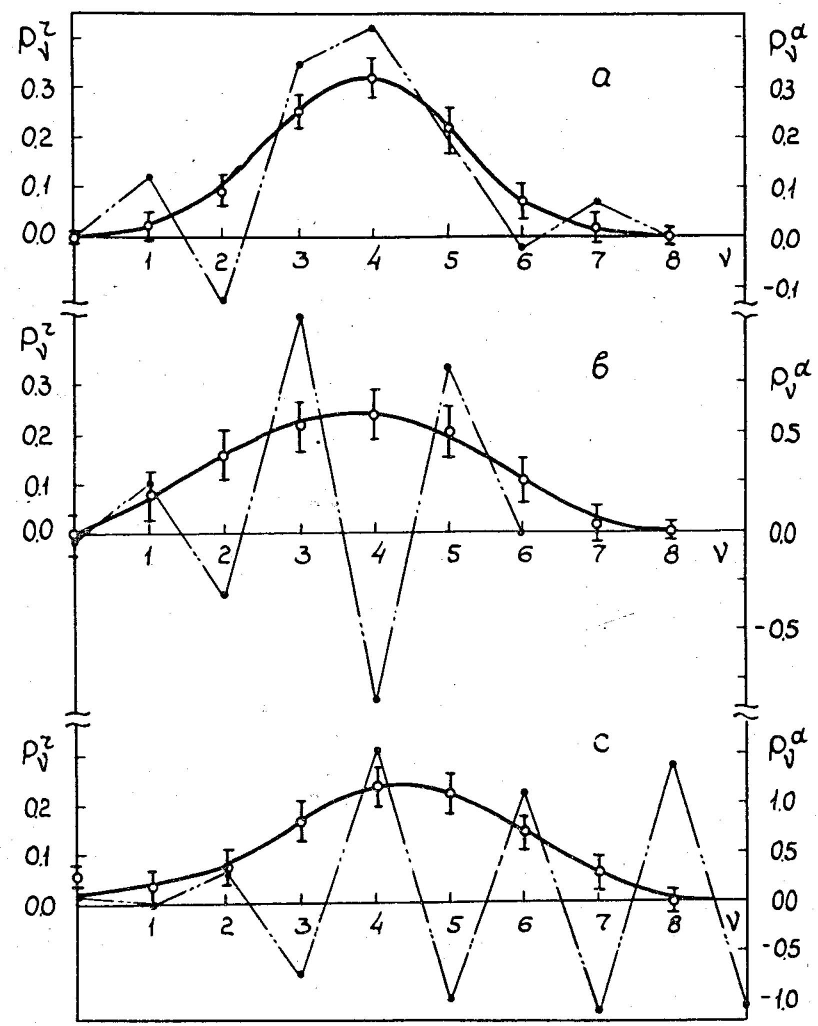

Fig. 3. Distribution of fission neutron multiplicities for *Fm* isotopes: fig. 3a - ^{254}Fm, $M = 870$, $\epsilon = 61.1\%$; fig. 3b - ^{256}Fm, $M = 204$, $\epsilon = 48.3\%$; fig. 3c - ^{257}Fm, $M = 1499$, $\epsilon = 51.0\%$ (the same notation as in fig. 1).

The regularized solution is less sensitive to the error of the kernel of equations (i.e., to the error in ϵ) than the direct one. This error is taken into account by the reconstruction of distributions for two values of ϵ (mean ϵ ± rms error of ϵ). In the experiment, the result of which is shown in fig. 1b, an error in ϵ was about 1%. The fluctuations of solutions for such a variation in ϵ are comparable with the line width. The variation in the nonregularized solution is many times larger.

6. The Integral Characteristics of Multiplicity Distributions

Two important integral characteristics of the distribution, namely the average number of emitted neutrons $\bar{\nu} = \sum_{\nu=0}^{\nu_{max}} \nu P_\nu$ and its dispersion $\sigma_\nu^2 = \sum_{\nu=0}^{\nu_{max}} (\nu-\bar{\nu})^2 P_\nu$ can be determined directly from the experimental data:

$$\bar{\nu} = \frac{1}{\epsilon} \sum_{n=0}^{n_{max}} n F_n = \frac{\bar{n}}{\epsilon}, \quad \sigma_\nu^2 = \frac{\langle n^2 \rangle - \bar{n}^2 - \bar{n}(1-\epsilon)}{\epsilon^2} \qquad (7)$$

Evidently, $\bar{\nu}$ and σ_ν^2, computed using direct solution P_ν^d , agree with these values. These parameters obtained by the STREG method (let us call them $\bar{\nu}_r$ and $\sigma_{\nu r}^2$) are, generally speaking, different from $\bar{\nu}$ and σ_ν^2. How large can these differences be?

In table 3 the values of $\bar{\nu}$, $\bar{\nu}_r$, σ_ν^2, $\sigma_{\nu r}^2$ are listed for six measured sets of F_n , all for ^{244}Cm. The first four are the results of real experiments, the last two are obtained by dividing the results of a real experiment into two parts, as mentioned above. The values of $\bar{\nu}$ and σ_ν^2 are given with their errors. The value of $\bar{\nu} = 2.690$ for ^{244}Cm is used as a standard, so the errors of $\bar{\nu}$ reflect only the accuracy of determining ϵ.

From table 3 it may be seen that the differences $|\bar{\nu}-\bar{\nu}_r|$ are much smaller than the errors in $\bar{\nu}$. The differences $(\sigma_{\nu r}^2 - \sigma_\nu^2)$ are positive. This may be explained by the cut-off of the higher harmonics of the sought function which generally leads to a small broadening of the distribution. However, as for $\bar{\nu}$, all differences $(\sigma_{\nu r}^2 - \sigma_\nu^2)$ account for only a small part of the errors in σ_ν^2. It should be noted that these errors are quite large, which is confirmed by the large scatter of values of σ_ν^2 for the different sets.

Table 3
Regularized and directly obtained parameters of distributions for different experiments (^{244}Cm).

No	ε,%	M	$\bar{\nu}$	$\bar{\nu}_r$	σ_ν^2	$\sigma_{\nu r}^2$
1	48.3	7169	2.690 ± 0.036	2.687	1.388 ± 0.076	1.410
2	48.2	65015	2.690 ± 0.015	2.691	1.290 ± 0.025	1.296
3	44.4	6928	2.690 ± 0.038	2.688	1.212 ± 0.084	1.226
4	39.9	20359	2.690 ± 0.025	2.688	1.173 ± 0.057	1.187
5	23.7	4039	2.690 ± 0.071	2.684	1.230 ± 0.272	1.287
6	22.0	4039	2.690 ± 0.075	2.684	1.587 ± 0.311	1.661

7. Summation of Data from Different Experiments

Let us consider some independent experiments carried out to determine one particular multiplicity distribution. During reconstruction of the unknown function from some sets of data using the STREG method, the usual weighted averaging procedure assuming the statistical independence of errors cannot be followed. This is due to the fact that during reconstruction of different versions the same a priori information is used. The theoretical error involved in the regularized solution is mainly an estimate of the possible influence of these higher harmonics of an unknown function, which in the experiment remain quite indefinite. The other component of the error originates from the harmonics which are, more or less successfully, determined from experiment. Only the latter component decreases with increasing number of similar experiments (i.e., with a similar value of ϵ) while the former one does not vary. Therefore, as the number of experiments increases, or the experimental error decreases, the error in the regularized solution decreases at a slower rate than in the case of correctly posed equations (and their solutions).

The question arises as to how to combine the results of different experiments, obtained using the STREG method, and how to combine the regularized results with the nonregularized ones? This can obviously be done by taking into account as independent, only the really independent data, i.e., the measured values of F_n . Then, the number of equation in (2) should be increased proportionally to the number of experiments, preserving the number of unknown quantities P_ν which describe the same unknown function. A similar procedure is used in combining the regularized results with those of correctly posed equations. The only difference lies in the fact that the function P_ν^d is used as input data for the correctly posed problem with a kernel in the form of an identity matrix. Figure 4 shows the summarized result of three experiments with comparable informativity (versions 1, 3 and 4 in table 3). The error in the result is about 20% less than those in the components. Note that the combined "curve" is somewhat narrower than the partial ones as with increasing informativity the broadening of the distribution, mentioned in the preceding section, decreases.

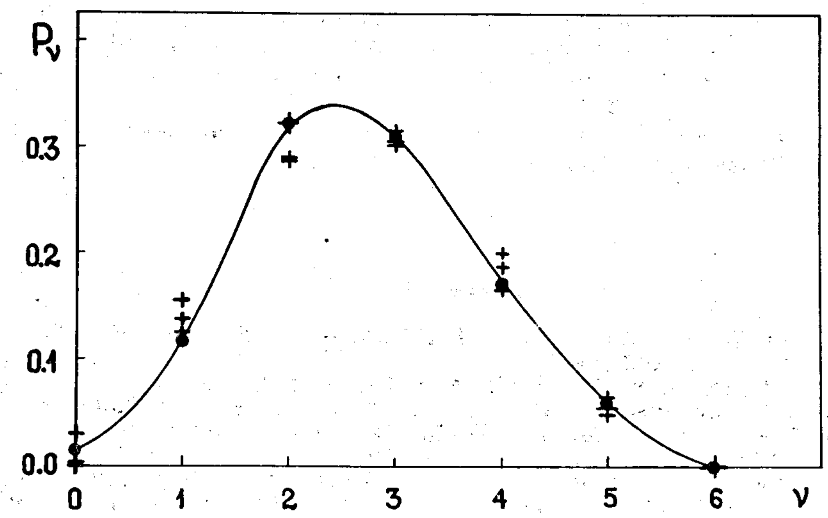

Fig. 4. Combined result of multiplicity reconstruction on the basis of three experiments: crosses correspond to partial results, points are a combined result.

8. Conclusion

For the measurements of multiplicity distributions with detection efficiencies substantially lower than unity, the direct solutions of equations (2) connecting the real distribution P_ν with the measured one F_n, appear to be unreasonable because of the incorrectness of equations. In these cases, where owing to a high efficiency and small experimental error, the direct method of solution is acceptable, the STREG method gives identical results and errors. Thus we can conclude that the STREG method is more general and allows one to find P_ν with reasonable errors for an efficiency of $\epsilon \geq 25\%$.

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