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A NUMERICAL ALGORITHM FOR MODELLING BOSON-FERMION STARS IN DILATONIC GRAVITY

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1 Main model

Boson stars are gravitationally bound macroscopic quantum states made up of scalar bosons [19,24,6,14]. They differ from the usual fermionic stars in that they are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle. For self-interacting boson field the mass of the boson star, even for small values of the coupling constant, turns out to be of the order of Chandrasekhar's mass when the boson mass is similar to a proton mass. Thus, the boson stars arise as possible candidates for non-baryonic dark matter in the universe and consequently as a possible solution of the one of the outstanding problems in modern astrophysics - the problem of nonluminous matter in the universe. Most of the stars are of primordial origin being formed from an original gas of fermions and bosons in the early universe. That is why it should be expected that most stars are a mixture of both, fermions and bosons in different proportions.

Boson-fermion stars are also a good model for learning more about the nature of strong gravitational fields not only in general relativity but also in the other theories of gravity.

The most natural and promising generalizations of general relativity are the scalar-tensor theories of gravity [4,9,30,8]. In these theories gravity is mediated not only by a tensor field (the metric of space-time) but also by a scalar field (the dilaton). These dilatonic theories of gravity contain arbitrary functions of the scalar field that determine the gravitational "constant" as a dynamical variable and the strength of the coupling between the scalar field and matter. It should be stressed that specific scalar-tensor theories of gravity arise naturally as low energy limit of the string theory [15,5,12,25,21,20], which is the most promising modern model of unification of all fundamental physical interactions.

Boson stars in scalar-tensor theories of gravity with massless dilaton have been widely investigated recently both numerically and analytically [26,16,27,28,7], [2,31]. Mixed boson-fermion stars in scalar tensor theories of gravity, however, have not been investigated so far in contrast to the general relativistic case, in which boson-fermion stars have been investigated [17].

In the present paper we consider boson-fermion stars in the most general scalar-tensor theory of gravity with massive dilaton.

In the Einstein frame the field equations in the presence of fermion and boson matter are:

$$
\begin{aligned}
G_m^n = \kappa_* \left(\overset{B}{T_m^n} + \overset{F}{T_m^n} \right) + 2\partial_m\varphi\,\partial^n\varphi - \partial^l\varphi\partial_l\varphi\,\delta_m^n + \frac{1}{2}U(\varphi)\,\delta_m^n, \\
\nabla_m\nabla^m\varphi + \frac{1}{4}U'(\varphi) = -\frac{\kappa_*}{2}\alpha(\varphi)\left(\overset{B}{T} + \overset{F}{T}\right), \\
\nabla_m\nabla^m\Psi + 2\alpha(\varphi)\partial^l\varphi\partial_l\Psi = -2A^2(\varphi)\frac{\partial\tilde{W}(\Psi^+\Psi)}{\partial\Psi^+}, \\
\nabla_m\nabla^m\Psi^+ + 2\alpha(\varphi)\partial^l\varphi\,\partial_l\,\Psi^+ = -2A^2(\varphi)\frac{\partial\,\tilde{W}(\Psi^+\Psi)}{\partial\,\Psi^+},
\end{aligned}
\tag{1}
$$

where ∇_m is the Levi-Civita connection with respect to the metric g_{mn}, $(l = 0, ..., 3,$ $m = 0, ..., 3$, $n = 0, ..., 3)$. The constant κ_* is given by $\kappa_* = 8\pi G_*$, where G_* is the bare Newtonian gravitational constant. The physical gravitational "constant" is $G_* A^2(\varphi)$, where $A(\varphi)$ is a function of the dilaton field φ depending on the concrete scalar-tensor theory of gravity. $\tilde{W}(\Psi^+\Psi)$ is the potential of the boson field. The dilaton potential $U(\varphi)$ can be written in the form $U(\varphi) = m_D^2 V(\varphi)$, where m_D is the dilaton mass and $V(\varphi)$ is a given dimensionless function of φ.

The function $\alpha(\varphi) = \partial\left[\ln A(\varphi)\right]/\partial\varphi$ determines the strength of the coupling between the dilaton field φ and the matter. The functions $\overset{B}{T}$ and $\overset{F}{T}$ are correspondingly the trace of the energy-momentum tensor of the fermionic matter[1] $\overset{F}{T_m^n}$ and bosonic matter $\overset{B}{T_m^n}$. Their explicit forms are:

$$
\begin{aligned}
\overset{B}{T_m^n} = \frac{1}{2}A^2(\varphi)\left(\partial_m\Psi^+\partial^n\Psi + \partial_m\Psi\partial^n\Psi^+\right) \\
- \frac{1}{2}A^2(\varphi)\left[\partial_l\Psi^+\partial^l\Psi - 2A^2(\varphi)\tilde{W}(\Psi^+\Psi)\right]\delta_m^n,
\end{aligned}
\tag{2}
$$

$$
\overset{F}{T_m^n} = (\varepsilon + p)\; u_m\, u^n - p\,\delta_m^n\,. \tag{3}
$$

Here Ψ is a complex scalar field, describing the bosonic matter, while Ψ^+ is its complex conjugated function. The energy density and the pressure of the fermionic fluid in the Einstein frame are $\varepsilon = A^4(\varphi)\tilde{\varepsilon}$ and $p = A^4(\varphi)\,\tilde{p}$, where $\tilde{\varepsilon}$ and $\tilde{p}$ are the physical energy density and pressure. Instead of giving the equation of state of the fermionic matter in the form $\tilde{p} = \tilde{p}(\tilde{\varepsilon})$, it is more convenient to write it in a parametric form

$$
\tilde{\varepsilon} = \tilde{\varepsilon}_0 g(\mu)\,, \qquad \tilde{p} = \tilde{\varepsilon}_0\, f(\mu), \tag{4}
$$

where $\tilde{\varepsilon}_0$ is a properly chosen dimensional constant and μ is the dimensionless Fermi momentum.

The physical four-velocity of the fluid is denoted by u_μ. The potential for the boson

[1] In the present article we consider fermionic matter only in macroscopic approximation, *i.e.*, after averaging quantum fluctuations of the corresponding fermion fields. Thus, we actually consider standard classical relativistic matter.

field has the form:

$$\tilde{W}(\Psi^+\Psi) = -\frac{m_D^2}{2}\Psi^+\,\Psi - \frac{1}{4}\tilde{\Lambda}\,(\Psi^+\Psi)^2.$$

The field equations together with the Bianchi identities lead to the local conservation law of the energy-momentum of matter

$$\nabla_n \overset{F}{T}{}^n_m = \alpha(\varphi)\,\overset{F}{T}\,\partial_m\varphi \ . \tag{5}$$

We will consider a static and spherically symmetric mixed boson-fermion star in asymptotic flat space-time. This means that the metric g_{mn} has the form

$$d\,s^2 = \exp\left[\nu(\mathrm{R})\right]d\,t^2 - \exp\left[\lambda(\mathrm{R})\right]d\,\mathrm{R}^2 - \mathrm{R}^2\left(d\,\theta^2 + \sin^2\theta\,d\,\psi^2\right), \tag{6}$$

where R, θ, ψ are usual spherical coordinates. The field configuration is static when the boson field Ψ satisfies the relationship:

$$\Psi = \tilde{\sigma}(\mathrm{R})\,\exp(\mathrm{i}\omega t).$$

Here ω is a real number and $\tilde{\sigma}(\mathrm{R})$ is a real function. Taking into account the assumption that has been made the system of the field equation is reduced to a system of ordinary differential equations (ODEs). Before we explicitly write the system, we are going to introduce a dimensionless radial coordinate by $r = m_B\mathrm{R}$, where m_B is the mass of the bosons. From now on, a prime will denote a differentiation with respect to the dimensionless radial coordinate r. After introducing the dimensionless quantities

$$\Omega = \frac{\omega}{m_B}, \quad \sigma = \sqrt{\kappa_*}\,\tilde{\sigma}, \quad \Lambda = \frac{\tilde{\Lambda}}{\kappa_* {m^2}_B}, \quad \gamma = \frac{m_D}{m_B},$$

and defining the dimensionless energy-momentum tensors as $T^n_m = \frac{\kappa_*}{m_B^2}T^n_m$, the components of the dimensionless energy-momentum tensor of the fermionic and bosonic matter become correspondingly:

$$\overset{F}{T}{}^0_0 = b\,A^4(\varphi)\,g(\mu), \qquad \overset{F}{T}{}^1_1 = -bA^4(\varphi)\,f(\mu),$$

$$\overset{B}{T}{}^0_0 = \frac{1}{2}A^2(\varphi)\left[\Omega^2\sigma^2(r)\exp[-\nu(r)] + \left(\frac{d\,\sigma}{d\,r}\right)^2\exp[-\lambda(r)]\right] - A^4(\varphi)W(\sigma),$$

$$\overset{B}{T}{}^1_1 = -\frac{1}{2}A^2(\varphi)\left[\Omega^2\sigma^2(r)\exp[-\nu(r)] + \left(\frac{d\,\sigma}{d\,r}\right)^2\exp[-\lambda(r)]\right] - A^4(\varphi)W(\sigma).$$

Here, the parameter $b = \kappa_*\tilde{\varepsilon}_0/m_B^2$ describes the relation between the Compton length of dilaton and the usual radius of neutron star in general relativity.

The functions $\overset{B}{T}$ and $\overset{F}{T}$, describing the trace of energy-momentum tensor, have the form:

$$\overset{B}{T} = -A^2(\varphi)\left[\Omega^2\sigma^2(r)\exp[-\nu(r)] - \left(\frac{d\,\sigma}{d\,r}\right)^2\exp[-\lambda(r)]\right] - 4A^4(\varphi)\,W(\sigma),$$

$$\overset{F}{T} = b\, A^4(\varphi)\ [g(\mu) - 3\, f(\mu)]\ .$$

For the independent dimensionless radial coordinate r we have

$$r \in [0, R_s] \cup [R_s, \infty)\,,$$

where $0 < R_s < \infty$ is the unknown radius of the fermionic part of the mixed boson-fermion star.

With all definitions we have given, the main system of differential equations governing the structure of static and spherically symmetric boson-fermion stars can be presented in the following form:

1. In the interior of the fermionic part of the star (the functions in this domain are subscribed by i)

$$\begin{aligned}
\frac{d\lambda}{dr} &= F_{1,i} \equiv \frac{1-\exp(\lambda)}{r} + r\left\{\exp(\lambda)\left[\overset{F}{T^0_0} + \overset{B}{T^0_0} + \frac{1}{2}\gamma^2 V(\varphi)\right] + \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d\nu}{dr} &= F_{2,i} \equiv -\frac{1-\exp(\lambda)}{r} - r\left\{\exp(\lambda)\left[\overset{F}{T^1_1} + \overset{B}{T^1_1} + \frac{1}{2}\gamma^2 V(\varphi)\right] - \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d^2\varphi}{dr^2} &= F_{3,i} \equiv -\frac{2}{r}\frac{d\varphi}{dr} + \frac{1}{2}\left(F_{1,i} - F_{2,i}\right)\frac{d\varphi}{dr} \\
&\quad + \frac{1}{2}\exp(\lambda)\left[\alpha(\varphi)\,(\overset{F}{T} + \overset{B}{T}) + \frac{1}{2}\gamma^2 V'(\varphi)\right], \qquad (7) \\
\frac{d^2\sigma}{dr^2} &= F_{4,i} \equiv -\frac{2}{r}\frac{d\sigma}{dr} + \left[\frac{1}{2}\left(F_{1,i} - F_{2,i}\right) - 2\alpha(\varphi)\frac{d\varphi}{dr}\right]\frac{d\sigma}{dr} \\
&\quad - \sigma\exp(\lambda)\left[\Omega^2\exp(-\nu) + 4\sigma A^2(\varphi) W'(\sigma)\right], \\
\frac{d\mu}{dr} &= F_{5,i} \equiv -\frac{g(\mu)+f(\mu)}{f'(\mu)}\left[\frac{1}{2}F_{2,i} + \alpha(\varphi)\frac{d\varphi}{dr}\right].
\end{aligned}$$

Here $\lambda(r), \nu(r), \varphi(r), \sigma(r)$, and $\mu(r)$ are unknown functions of r, and Ω is an unknown parameter. Having in mind the physical assumptions, we have to solve the equations (7) under the following boundary conditions:

$$\lambda(0) = \frac{d\varphi}{dr}(0) = \frac{d\sigma}{dr}(0) = 0, \quad \sigma(0) = \sigma_c, \quad \mu(0) = \mu_c\,, \qquad (8)$$

$$\mu(R_s) = 0 \qquad (9)$$

where σ_c and μ_c are the values of density of, respectively, the bosonic and fermionic matter at the star's center. The first BC in (8) ensures the existence of local Lorentzian system in some vicinity of the star's center. The second and third conditions in (8) guarantee the nonsingularity of the dilaton field and the boson matter at the star's center. As for the quantities σ_c and μ_c in the last two BCs (8) they must be given. At last the quantity R_s in (9) is the radius of the fermionic part of the star, where the pressure of the fermionic matter vanishes (for more physical details see [3]).

2. In the external domain (subscribed by e) there is no fermionic matter, *i.e.*, it can be formally supposed that the function $\mu(r) \equiv 0$ if $x \geq R_s$. The fermionic part of the energy-momentum tensor also vanishes identically and, thus, the differential equations with respect to the rest four unknown functions $\lambda(r)$, $\nu(r)$, $\varphi(r)$, and $\sigma(r)$ are:

$$
\begin{aligned}
\frac{d\lambda}{dr} &= F_{1,e} \equiv \frac{1-\exp(\lambda)}{r} + r\left\{\exp(\lambda)\left[\overset{B}{T}{}_0^0 + \frac{1}{2}\gamma^2 V(\varphi)\right] + \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d\nu}{dr} &= F_{2,e} \equiv -\frac{1-\exp(\lambda)}{r} - r\left\{\exp(\lambda)\left[\overset{B}{T}{}_1^1 + \frac{1}{2}\gamma^2 V(\varphi)\right] - \left(\frac{d\varphi}{dr}\right)^2\right\}, \\
\frac{d^2\varphi}{dr^2} &= F_{3,e} \equiv -\frac{2}{r}\frac{d\varphi}{dr} + \frac{1}{2}\left(F_{1,e} - F_{2,e}\right)\frac{d\varphi}{dr} \\
&\quad + \frac{1}{2}\exp(\lambda)\left[\alpha(\varphi)\,\overset{B}{T} + \frac{1}{2}\gamma^2 V'(\varphi)\right], \\
\frac{d^2\sigma}{dr^2} &= F_{4,e} \equiv -\frac{2}{r}\frac{d\sigma}{dr} + \left[\frac{1}{2}\left(F_{1,e} - F_{2,e}\right) - 2\alpha(\varphi)\frac{d\varphi}{dr}\right]\frac{d\sigma}{dr} \\
&\quad -\sigma\exp(\lambda)\left[\Omega^2\exp(-\nu) + 4\sigma A^2(\varphi)\,W'(\sigma)\right].
\end{aligned}
\tag{10}
$$

As it is required by the asymptotic flatness of space-time (see [3]), the boundary conditions (BCs) at the infinity are:

$$
\nu(\infty) = 0, \quad \varphi(\infty) = 0, \quad \sigma(\infty) = 0\,, \tag{11}
$$

where we denote $(\cdot)(\infty) = \lim_{r\to\infty}(\cdot)(r)$.

We seek for a solution $\{\lambda(r), \nu(r), \varphi(r), \sigma(r), \mu(r), R_s, \Omega\}$ subjected to the nonlinear ODEs (7) and (10), satisfying the BCs (8), (9), and (11). At that we assume the function $\mu(r)$ is continuous in the interval $[0, R_s]$, while the functions $\lambda(r)$, $\nu(r)$ are continuous and the functions $\varphi(r)$, $\sigma(r)$ are smooth in the whole interval $[0, \infty)$, including the unknown internal boundary $r = R_s$.

The so-posed boundary value problem (BVP) represents a nonlinear two-parametric eigenvalue problem with respect to the quantities R_s and Ω.

Let us emphasize that a number of methods for solving the free-boundary problems are considered in detail in [29,22].

Here, we aim at applying the new solving method to the above-formulated problem.

2 Method of solution

At first we scale the variable r using the Landau transformation [29] and, in this way, we obtain a fixed computational domain. Namely,

$$x = \frac{r}{R_s}, \quad x \in [0,1] \cup [1, \infty).$$

For our further considerations, it is convenient to present the systems (7) and (10) in the following equivalent forms as systems of first order ODEs:

$$-\mathbf{y}_i{}' + R_s\,\mathbf{F}_i(R_s x, \mathbf{y}_i, \Omega) = 0, \tag{12a}$$

$$-\mathbf{y}_e{}' + R_s\,\mathbf{F}_e(R_s x, \mathbf{y}_e, \Omega) = 0 \tag{12b}$$

with respect to the unknown vector functions

$$\mathbf{y}_i(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x), \mu(x))^T,$$

$$\mathbf{y}_e(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x))^T,$$

and right hand sides $\mathbf{F}_i \equiv (F_1, F_2, \xi, F_3, \eta, F_4, F_5)^T$, $\mathbf{F}_e \equiv (F_1, F_2, \xi, F_3, \eta, F_4)^T$, where $(.)'$ stands for differentiation towards the new variable x.

For given values of the parameters R_s and Ω, the independent solving of the internal system (12a) requires seven BCs. At the same time we have at disposal only six conditions of the kind (8) and (9). In order to complete the problem, we set additionally one more parametric condition (the value of someone from among the functions $\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x)$, or $\eta(x)$) at the point $x = 1$). Let us set for example:

$$\varphi_i(1) = \varphi_s, \tag{13}$$

where φ_s is a parameter. Then, the boundary conditions (8), (9), and (13) of the internal BVP can be presented in the form:

$$B_{0,i}\,\mathbf{y}_i(0) - D_{0,i} = 0\,, \quad B_{1,i}\,\mathbf{y}_i(1) - D_{1,i}(\varphi_s) = 0\,. \tag{14}$$

Here, the matrices $B_{0,i} = \mathrm{diag}(1,0,0,1,1,1,1)$, $D_{0,i} = \mathrm{diag}(0,0,0,0,\sigma_c,0,\mu_c)$, $B_{1,i} = \mathrm{diag}(0,0,1,0,0,0,1)$, $D_{1,i} = \mathrm{diag}(0,0,\varphi_s,0,0,0,0)$.

Obviously, the solution in the internal domain $x \in [0,1]$ depends not only on the variable x, but it also is a function of the three parameters R_s, Ω, φ_s, *i.e.*, $\mathbf{y}_i = \mathbf{y}_i(x, \Omega, R_s, \varphi_s)$.

In the external domain $x \geq 1$ the vector of solutions

$$\mathbf{y}_e(x) \equiv (\lambda(x), \nu(x), \varphi(x), \xi(x), \sigma(x), \eta(x))^T$$

is 6D. Thereupon, six BCs are indispensable for solving the equation (12b). At the same time only the three BCs (11) are known. Let us consider that the solution $y_i(x)$

in the internal domain $x \in [0,1]$ is known. Then, we postulate the rest three deficient conditions to be the continuity conditions at the point $x = 1$. The first of them is similar to the condition (13) and the other two we assign to two arbitrary functions from among $\lambda(x)$, $\nu(x), \xi(x)$, $\sigma(x)$, and $\eta(x)$; for example:

$$\lambda_e(1) = \lambda_i(1), \quad \varphi_e(1) = \varphi_s, \quad \sigma_e(1) = \sigma_i(1).$$

It is convenient to present the BCs in the external domain in matrix form again:

$$B_{1,e}\,\mathbf{y}_e(1) - D_{1,e}(\varphi_s) = 0, \quad B_{\infty,e}\,\mathbf{y}_e(\infty) = 0\,, \tag{15}$$

where the matrices

$$B_{1,e} = \mathrm{diag}(1,0,1,0,1,0)\,,$$
$$D_{1,e} = \mathrm{diag}(\lambda_i(1), 0, \varphi_s, 0, \sigma_i(1), 0\,,)$$
$$B_{\infty,e} = \mathrm{diag}(0,1,1,0,1,0)\,.$$

Let the solutions $\mathbf{y}_i = \mathbf{y}_i(x,\Omega,R_s,\varphi_s)$ and $\mathbf{y}_e = \mathbf{y}_e(x,\Omega,R_s,\varphi_s)$ be supposed known. Generally speaking, for given arbitrary values of the parameters R_s, Ω, and φ_s the continuity conditions with respect to the functions $\nu(x), \xi(x)$, and $\eta(x)$ at the point $x = 1$ are not satisfied. We choose the parameters R_s, Ω, and φ_s in such manner that the continuity conditions for the functions $\nu(x)$, $\xi(x)$, and $\eta(x)$ are held, *i.e.*,

$$\begin{aligned} \nu_e(1,R_s,\Omega,\varphi_s) - \nu_i(1,R_s,\Omega,\varphi_s) &= 0, \\ \xi_e(1,R_s,\Omega,\varphi_s) - \xi_i(1,R_s,\Omega,\varphi_s) &= 0, \\ \eta_e(1,R_s,\Omega,\varphi_s) - \eta_i(1,R_s,\Omega,\varphi_s) &= 0. \end{aligned} \tag{16}$$

These conditions should be interpreted as three nonlinear algebraic equations in regard to the unknown quantities R_s, Ω, and φ_s. The usual way for solving the above-mentioned kind of equations (16) is by means of various iteration methods, for example Newton's methods. The traditional technology similar to methods like shutting [33], requires separate treatment of the BVPs and the algebraic continuity equations and brings itself to additional linear ODEs for elements of the corresponding to (16) Jacobi matrix. These elements are functions of the variable x and they have to be known actually only at the point $x = 1$. The solving of both the nonlinear BVPs (12a), (14) and (12b), (15), and the attached linear equations is another hard enough task.

In the present work, using the Continuous Analogue of Newton Method (CANM) [13] (see the comprehensive reviews [18], [23]) we propose a common treatment of both differential and algebraic problems.

We suppose that the nonlinear spectral problem (12a), (14), (12b), (15), and (16) has a "well separated" [18] exact solution. Let the functions $\mathbf{y}_{i,0}(x), \mathbf{y}_{e,0}(x)$ and the parameters $R_{s,0}, \Omega_0, \varphi_{s,0}$ be initial approximations of this solution. CANM leads to the following iteration process:

$$\mathbf{y}_{i,k+1}(x) = \mathbf{y}_{i,k}(x) + \tau_k \mathbf{z}_{i,k}(x), \tag{17a}$$
$$\mathbf{y}_{e,k+1}(x) = \mathbf{y}_{e,k}(x) + \tau_k \mathbf{z}_{e,k}(x), \tag{17b}$$
$$R_{s,k+1} = R_{s,k} + \tau_k \rho_k, \tag{17c}$$
$$\Omega_{k+1} = \Omega_k + \tau_k \omega_k, \tag{17d}$$
$$\varphi_{s,k+1} = \varphi_{s,k} + \tau_k \phi_k. \tag{17e}$$

Here $\tau_k \in (0,1]$ is a parameter, which can rule the convergence of iteration process. The increments $\mathbf{z}_{i,k}(x)$, $\mathbf{z}_{e,k}(x), \rho_k, \omega_k$, and ϕ_k, $k = 0, 1, 2, \ldots$ satisfy the linear ODEs (for sake of simplicity we will henceforth omit the number of iterations k):

$$-\mathbf{z}_i{}' + R_s \frac{\partial \mathbf{F}_i}{\partial \mathbf{y}_i} \mathbf{z}_i + \left(R_s \frac{\partial \mathbf{F}_i}{\partial R_s} + \mathbf{F}_i \right) \rho + R_s \frac{\partial \mathbf{F}_i}{\partial \Omega} \omega = \mathbf{y}_i' - R_s \mathbf{F}_i, \tag{18a}$$

$$-\mathbf{z}_e{}' + R_s \frac{\partial \mathbf{F}_e}{\partial \mathbf{y}_e} \mathbf{z}_e + \left(R_s \frac{\partial \mathbf{F}_e}{\partial R_s} + \mathbf{F}_e \right) \rho + R_s \frac{\partial \mathbf{F}_e}{\partial \Omega} \omega = \mathbf{y}_e{}' - R_s \mathbf{F}_e. \tag{18b}$$

In the above two equations all coefficients and right-hand sides as well are known functions of the arguments x, R_s, Ω by means of the solution from the previous iteration. We seek for the unknowns $\mathbf{z}_i(x)$ of equation (18a) and $\mathbf{z}_e(x)$ of equation (18b) as linear combinations with coefficients ρ, ω and ϕ:

$$\mathbf{z}_i(x) = \mathbf{s}_i(x) + \rho\, \mathbf{u}_i(x) + \omega \mathbf{v}_i(x) + \phi \mathbf{w}_i(x)\,, \tag{19a}$$
$$\mathbf{z}_e(x) = \mathbf{s}_e(x) + \rho\, \mathbf{u}_e(x) + \omega \mathbf{v}_e(x) + \phi \mathbf{w}_e(x)\,. \tag{19b}$$

Here $\mathbf{s}_i(x)$, $\mathbf{u}_i(x)$, $\mathbf{v}_i(x)$, $\mathbf{w}_i(x)$ and $\mathbf{s}_e(x), \mathbf{u}_e(x)$, $\mathbf{v}_e(x)$, $\mathbf{w}_e(x)$ are new unknown functions, which are defined in either, internal or external domains. Substituting for the decomposition (19a) into equation (18a) after reduction we obtain:

$$-\mathbf{s}_i{}' + Q_i(x)\, \mathbf{s}_i = \mathbf{y}_i{}' - R_s \mathbf{F}_i,$$
$$-\mathbf{u}_i{}' + Q_i(x)\, \mathbf{u}_i = -\left(\mathbf{F}_i + R_s \frac{\partial \mathbf{F}_i}{\partial R_s} \right),$$
$$-\mathbf{v}_i{}' + Q_i(x)\, \mathbf{v}_i = -R_s \frac{\partial \mathbf{F}_i}{\partial \Omega},$$
$$-\mathbf{w}_i{}' + Q_i(x)\, \mathbf{w}_i = 0$$

where $Q_i(x) \equiv R_s \partial \mathbf{F}_i (R_s x, \mathbf{y}_i, \Omega) / \partial \mathbf{y}_i$ stands for a square matrix (7×7), which consists of the Frechét derivatives of operator $\mathbf{F}_i$ at the point $\{\mathbf{y}_i(x), R_s, \Omega\}$.

Similarly, applying CANM to the BCs (14) and taking into account the dependence of matrix $D_{1,i}$ on the parameter φ_s yields:

$$B_{0,i}\, \mathbf{z}_i(0) = D_{0,i} - B_{0,i}\, \mathbf{y}_i(0), \quad B_{1,i}\, \mathbf{z}_i(1) = D_{1,i} - B_{1,i}\, \mathbf{y}_i(1) - D_{1,i}{}' \phi.$$

By means of the representation (19a) we obtain the following eight BCs (four left + four right) for the equations (2):

$$
\begin{aligned}
&B_{0,i}\,\mathbf{s}_i(0) = D_{0,i} - B_{0,i}\,\mathbf{y}_i(0)\,, && B_{1,i}\,\mathbf{s}_i(1) = D_{1,i} - B_{1,i}\,\mathbf{y}_i(1),\\
&B_{0,i}\,\mathbf{u}_i(0) = 0, && B_{1,i}\,\mathbf{u}_i(1) = 0,\\
&B_{0,i}\,\mathbf{v}_i(0) = 0, && B_{1,i}\,\mathbf{v}_i(1) = 0,\\
&B_{0,i}\,\mathbf{w}_i(0) = 0, && B_{1,i}\,\mathbf{w}_i(1) = -{D_{1,i}}'(\varphi_s).
\end{aligned}
\tag{20}
$$

Let us now substitute for decomposition (19b) into the linear equations for external domain (18b). As a result, we obtain the following four vector equations with regard to the unknown functions $\mathbf{s}_e(x), \mathbf{u}_e(x)$, $\mathbf{v}_e(x)$, and $\mathbf{w}_e(x)$ with eight BCs (four left + four right):

$$
\begin{aligned}
-{\mathbf{s}_e}' + Q_e(x)\,\mathbf{s}_e &= {\mathbf{y}_e}' - R_s\,\mathbf{F}_e\,,\\
-{\mathbf{u}_e}' + Q_e(x)\,\mathbf{u}_e &= -\left(\mathbf{F}_e + R_s\frac{\partial\,\mathbf{F}_e}{\partial R_s}\right),\\
-{\mathbf{v}_e}' + Q_e(x)\,\mathbf{v}_e &= -R_s\frac{\partial \mathbf{F}_e}{\partial \Omega},\\
-{\mathbf{w}_e}' + Q_e(x)\,\mathbf{w}_e &= 0.
\end{aligned}
\tag{21}
$$

Here, $Q_e(x) \equiv R_s\,\partial\,\mathbf{F}_e\,[R_s x, \mathbf{y}_e(x), \Omega]\,/\partial\,\mathbf{y}_e$ is a square matrix (6×6) whose elements are Frechét's derivatives of the operator $\mathbf{F}_e$ at the point $\{\mathbf{y}_e(x), R_s, \Omega\}$.

The corresponding linear BCs are obtained in the same way as (20) and they become:

$$
\begin{aligned}
&B_{1,e}\,\mathbf{s}_e(1) = D_{1,e} - B_{1,e}\,\mathbf{y}_e(1), && B_{\infty,e}\,\mathbf{s}_e(\infty) = -B_{\infty,e}\,\mathbf{y}_e(\infty),\\
&B_{1,e}\,\mathbf{u}_e(1) = 0, && B_{\infty,e}\,\mathbf{u}_e(\infty) = 0,\\
&B_{1,e}\,\mathbf{v}_e(1) = 0, && B_{\infty,e}\,\mathbf{v}_e(\infty) = 0,\\
&B_{1,e}\,\mathbf{w}_e(1) = -{D_{1,e}}'(\varphi_s), && B_{\infty,e}\,\mathbf{w}_e(\infty) = 0.
\end{aligned}
\tag{22}
$$

In the end, to compute the increments $\{\rho, \omega, \phi\}$ of parameters R_s, Ω, and φ_s we use the three continuity conditions (16).

Let the solutions of linear BVPs (2), (20), and (21), (22) at the kth iteration stage be assumed as known. For simplicity, we introduce the vector $\tilde{\mathbf{y}}(x) \equiv (\nu(x), \xi(x), \eta(x))^T$. For two arbitrary functions $h_i(x)$ and $h_e(x)$, defined in left and right vicinity of the point $x = 1$, we set $\Delta h \equiv h_e(1) - h_i(1)$. Then, applying CANM to the equations (16) and having in mind the decompositions (19a), (19b), we attain the vector equation:

$$
\Delta\,\tilde{\mathbf{u}}\,\rho + \Delta\,\tilde{\mathbf{v}}\,\omega + \Delta\,\tilde{\mathbf{w}}\,\phi = -\left(\Delta\,\tilde{\mathbf{y}} + \Delta\,\tilde{s}\right)\,, \tag{23}
$$

which represents an algebraic system consisting of three linear scalar equations with respect to the three unknowns ρ, ω, and ϕ.

The general sequence of the algorithm can be recapitulated in the following way. Let us assume that the functions $\mathbf{y}_{i,k}(x)$, $\mathbf{y}_{e,k}(x)$, and parameters $R_{s,k}$, Ω_k, $\varphi_{s,k}$ are given for $k \geq 0$. We solve the linear BVPs (2), (20) and, thus, we compute the functions $\mathbf{s}_{i,k}(x), \mathbf{u}_{i,k}(x)$, $\mathbf{v}_{i,k}(x)$, $\mathbf{w}_{i,k}(x)$ in the internal domain $x \in [0, 1]$. Then, we solve the linear BVPs (21),(22) in the external domain $x \in [1, \infty]$ and compute the functions $\mathbf{s}_{e,k}(x)$, $\mathbf{u}_{e,k}(x)$, $\mathbf{v}_{e,k}(x)$, and $\mathbf{w}_{e,k}(x)$.

Next, to obtain the increments ρ_k, ω_k, and ϕ_k we solve the linear algebraic system (23). Using the decompositions 19a, 19b and then the formulae (17a) (17e), we calculate the functions $\mathbf{y}_{i,k+1}(x)$, $\mathbf{y}_{e,k+1}(x)$, the radius of the star $R_{s,k+1}$, the frequency Ω_{k+1}, and the parameter $\varphi_{s,k+1}$ as well at the new iteration stage $k+1$.

At the every iteration k an optimal time step τ_{opt} is determinated in accordance to the Ermakov&Kalitkin formula [10] $\tau_{opt} \approx \delta(0)/\left[\delta(0)+\delta(1)\right]$, where the residual $\delta(\tau)$ is calculated as follows

$$\delta(\tau_k) = \max\left[\delta_f, (R_{s,k}+\tau_k\rho_k)^2, (\Omega_k+\tau_k\omega_k)^2, (\varphi_{s,k}+\tau_k\phi_k)^2\right]$$

and δ_f is the Euclidean residual of the right-hand sides of the first equations in the systems (2), (20), and (21), (22).

The criterion for termination of the iterations is $\delta(\tau_{opt}) < \varepsilon$, where $\varepsilon \sim 10^{-8} \div 10^{-12}$ for some k.

Taking into account the smoothness of sought solutions, we solve the linear BVPs (2), (20), and (21, 22, employing spline collocation scheme of fourth order of approximation [32]. At that, we utilize essentially the important feature that each of the above-mentioned two groups vector BVPs (internal and external) has one and the same left-hand side.

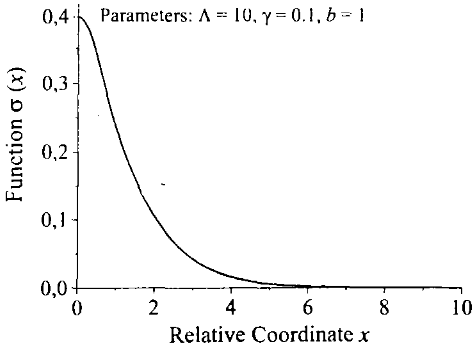

Fig. 1. The function $\sigma(x)$ for $\sigma_c = 0.4$; $\mu_c = 1.2$.

It is worth noting that the algebraic systems of linear equations and the system (23) as well become ill-posed in the vicinity of the "exact" solution, *i.e.*, for sufficiently small residuals δ. That is why for small δ, for example if $\delta < 10^{-3}$ (then $\tau_{opt} \sim 1$ usually), it is expedient to use the Newton-Kantorovich method when the respective matrices are fixed for some $\delta \geq 10^{-3}$.

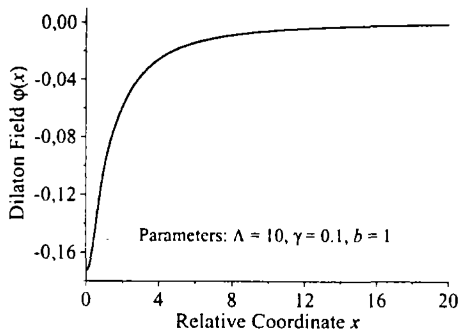

Fig. 2. The function $\varphi(x)$ for $\sigma_c = 0.4$; $\mu_c = 1.2$.

3 Some numerical results

For the purpose of illustrating we will shortly discuss some results obtained from numerical experiments.

In the present article, we consider concrete scalar-tensor model characterized by the functions (see Section 1):

$$A(\varphi) = \exp\left(\frac{\varphi}{\sqrt{3}}\right), \quad V(\varphi) = \frac{3}{2}[1 - A^2(\varphi)]^2,$$

$$f(\mu) = \frac{1}{8}\left[(2\mu - 3)\sqrt{\mu + \mu^2} + 3\ln\left(\sqrt{\mu} + \sqrt{1+\mu}\right)\right],$$

$$g(\mu) = \frac{1}{8}\left[(6\mu + 3)\sqrt{\mu + \mu^2} - 3\ln\left(\sqrt{\mu} + \sqrt{1+\mu}\right)\right],$$

$$W(\sigma) = -\frac{1}{2}\left(\sigma^2 + \frac{1}{2}\Lambda\sigma^4\right).$$

The quantities b, Λ are given parameters. For completeness, we note that in the concrete case the functions $f(\mu)$ and $g(\mu)$ represent the equation of state of noninteracting neutron gas in parametric form, while the function $W(\sigma)$ describes the boson field with quadratic self-interaction.

The calculated eigenfunctions $\sigma(x)$, $\varphi(x)$, $\nu(x)$, and $\mu(x)$ are plotted correspondingly in Fig. 1, 2, 3, and 4 for the values of the parameters $\gamma = 0.1$, $\Lambda = 10$ and $b = 1$. The behaviour of the mentioned functions is typical for a wider range of the parameters not only for those values presented in the figures. The function $\sigma(x)$ decreases rapidly from its central value $\sigma_c = 0.4$ (in the case under consideration) to zero, at that when dimensionless coordinate $x > 6$, the function does not exceed 10^{-4}. Similarly the function $\nu(x)$ has the largest derivative for $x \in (0, 9)$. After that it approaches slowly zero at infinity like $1/x$. For example, when $x \approx 9$ the derivative $\nu'(x) \approx 10^{-2}$, while for $x > 27$ we have $\nu'(x) < 10^{-4}$, *i.e.*, the asymptotical behavior of calculated grid function and its

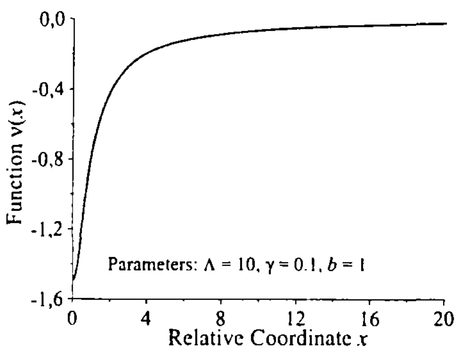

Fig. 3. The function $\nu(x)$ for $\sigma_c = 0.4$; $\mu_c = 1.2$.

derivative agrees very well with the theoretical prediction (see [3]). The function $\varphi(x)$ increases rapidly for $x < 4$; besides that it trends asymptotically to zero. Obviously, the quantitative behavior of $\varphi(x)$ for central value $\sigma_c = 0.4$ is determinated by the dominance of the term $\overset{B}{T}$ over the term $\overset{F}{T}$ (see [3]). At last the function $\mu(x)$ is nontrivial in the internal domain $x \in [0, 1]$, *i.e.* inside the star. Here, it varies monotonously and continuously from its central value (in the case under consideration) $\mu_c = 1.2$ until zero at $x = 1$, corresponding to the radius of the star.

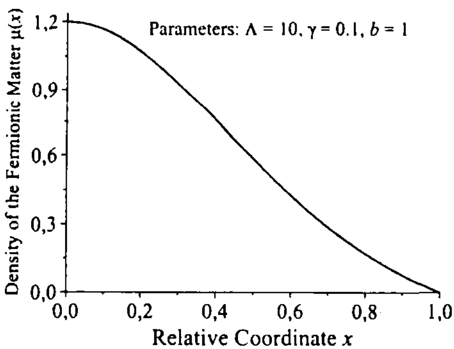

Fig. 4. The function $\mu(x)$ for $\sigma_c = 0.4$; $\mu_c = 1.2$.

The computed grid functions $\sigma(x)$, $\varphi(x)$, $\nu(x)$ are compared with their theoretical predictions when $x \to \infty$

$$\sigma_{as}(x) \sim \frac{1}{xR_s} \exp\left(-x\,R_s\sqrt{1-\Omega^2}\right), \quad \varphi_{as}(x) \sim \frac{1}{x\,R_s} \exp\left(-\frac{\gamma\varphi''(0)}{2} xR_s\right),$$

$$\nu_{as}(x) \sim \frac{M}{xR_s},$$

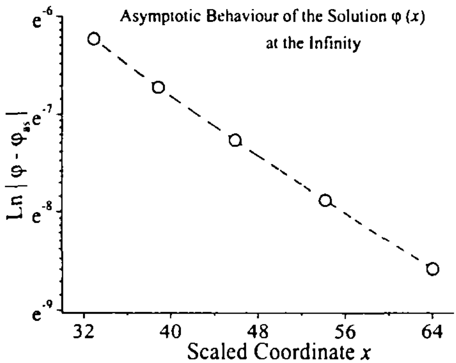

Fig. 5. The calculated function $\varphi(x)$ compared with its asymptotical behaviour.

where M is the star mass (see below). The following results are obtained:

$|\sigma(x) - \sigma_{as}(x)| = 0$ within 10^{-7};

$\ln|\varphi(x) - \varphi_{as}(x)| = Cx + D$, where the constants $C < 0$, $D > 0$ (Fig. 5);

$|\nu(x) - \nu_{as}(x)| = \nu_0$. In the case of the solution under consideration the constant $\nu_0 \simeq 4.03 \times 10^{-4}$.

From physical point of view, it is important to know the mass of the boson-fermion star and the total number of particles (bosons and fermions) making up the star.

The dimensionless star mass can be calculated via the formula:

$$M = \int_0^\infty \left[\overset{B}{T_0^0} + \overset{F}{T_0^0} + \exp(-\lambda)\left(\frac{d\varphi}{dr}\right)^2 + \frac{\gamma^2}{2} V(\varphi) \right] r^2 dr.$$

The dimensionless rest mass of the bosons (total number of bosons times the boson mass) is given by

$$M_{RB} = \Omega \int_0^\infty A^2(\varphi) \exp\left(\frac{\lambda - \nu}{2}\right) \sigma^2 r^2 dr.$$

The dimensionless rest mass of the fermions is correspondingly:

$$M_{RF} = b \int_0^\infty A^3(\varphi) \exp\left(\frac{\lambda}{2}\right) n(\mu) r^2 dr$$

where $n(\mu)$ is the density of the fermions. In the case we consider we have $n(\mu) = \mu^{3/2}(x)$.

The dependencies of the star mass M and the rest mass of fermions M_{RF} on the central value μ_c of the function $\mu(x)$ are shown in a configuration diagram on Fig. 6 for $\Lambda = 0$,

$\gamma = 0.1$, $b = 1$, and $\sigma_c = 0.002$. It should be pointed that for such small central value σ_c we have in practice "pure" fermionic star. On the figure, it is seen that from small values of μ_c to values near beyond the peak the rest mass is greater than the total mass of the star, which means that the star is potentially stable.

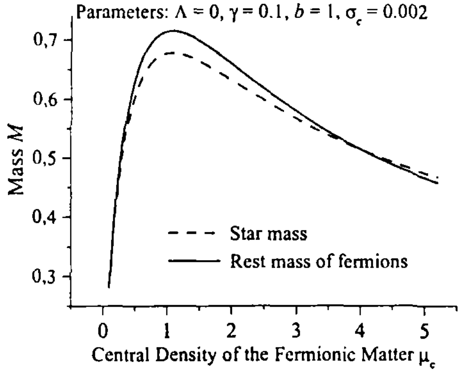

Fig. 6. The star mass M and the rest fermion mass M_{RF} as functions of μ_c

On Fig. 7 the binding energy of the star $E_b = M - M_{RB} - M_{RF}$ is drawn against the rest mass of fermions M_{RF} for $\Lambda = 0$, $\gamma = 0.1$, $b = 1$, and $\sigma_c = 0.002$. Fig. 7 is actually a bifurcation diagram. By increasing the central value of the function $\mu(x)$ one meets a cusp. The appearance of a cusp shows that the stability of the star changes - a perturbation mode develops instability. Beyond the cusp the star is unstable and may collapse eventually forming a black hole. The corresponding physical results for pure boson stars are considered in our recent paper [11].

In the case of a mixed boson-fermion star with approximately equal parts of bosons and fermions, the total mass of the star is plotted on Fig. 8 as a function of the central

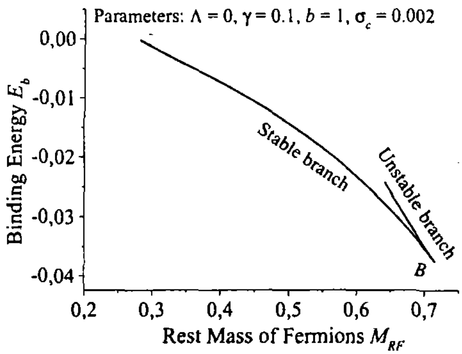

Fig. 7. The binding energy E versus the rest fermion mass M_{RF}

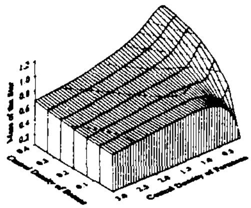

Fig. 8. The star mass as a function of central values μ_c and σ_c.

densities σ_c and μ_c at the corresponding ranges of these parameters. The projection of the mass surface on the appropriate plane gives, correspondingly, the pure fermionic and bosonic case we have discussed above.

4 Concluding Remarks

In the present paper a two-parametric nonlinear BVP about spherically symmetric mixed boson-fermion stars is solved. The computational domain is divided into two parts, in which two systems of ODEs with different number of equations are considered and treated numerically. Through CANM they are reduced to linear systems of seven ODEs in the inner part (inside the star) and six ODEs in the outer part (outside the star). In order to solve the internal system an additional parametric BC about one of the sought function is introduced (say $\varphi(1) = \varphi_s$). Three BCs on the left side are necessary to complete the outer BVP. To this end we choose the calculated values of two functions $\lambda(x)$ and $\sigma(x)$ at the point $x = 1$, and φ_s, as well. In this way the continuity of the above quantities at the radius of star is ensured. Generally, the continuity of the rest free functions $\nu(x)$, $\varphi'(x)$, and $\sigma'(x)$ in the point $x = 1$ is not guarantied. The continuity requirement for them leads to three algebraic equations, depending on the parameters R_s, Ω, and φ_s. Applying CANM to these nonlinear continuity conditions we solve completely both the differential and the algebraic problems.

This implementation of the original BVP is more convenient and common with regard to that presented in [3] because it does not depend on choice of the concrete model of a fermionic matter (the functions $f(r)$ and $g(r)$), and enables to avoid the separate integration of the equation about fermionic matter $\mu(r)$.

The uniform structure of matrices of the left-hand sides of linearized systems in both domains, inside and outside the star, is the second principal advantage of the presented numerical algorithm, which makes easier and accelerates its numerical treatment.

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