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UNFOLDING OF TRUE DISTRIBUTIONS FROM EXPERIMENTAL DATA DISTORTED BY DETECTORS WITH FINITE RESOLUTIONS

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1. Introduction

Experimental distributions of events usually have certain distortions due to finite resolution of the detector. The detector resolution is defined by its technical characteristics and the precision of their parametrisations.

To obtain the true event distribution the distortions should be corrected. Certainly, information about the resolution of the detector, should be used. This information is obtained from calibration measurements and /or simulation of the measuring process.

Heuristic iterative methods of correction making use of the acceptance are applied in many experiments [1,2], where the acceptance is defined as the ratio of simulated reconstructed and simulated generated distributions. Initially, for the simulation some assumed distribution is used and further iterations proceed with respect to the corrected distribution or its parameters. The parameters are obtained by fitting the corrected distribution with the theoretical model.

There exist more rigorous approaches to this unfolding procedure, and the best known of them are based on solution of the Fredholm integral equation of the first kind [3,4]. This is an ill-posed problem and a stable solution of equation is obtained using additional information about its properties (smoothness, correlation relations and so on).

The main disadvantages of the known methods are:

a) difficulties in the analysis of biases caused by the method itself;

b) difficulties in the interpretation of statistical errors for the corrected distribution. For example, there may exist a discrepancy between the error values and the corrected distribution being smooth.

In this paper we propose a new unfolding method. It is, essentially, a procedure for choosing a stable result that may be obtained by the least squares method or by the maximum likelihood method.

Stability of results is achieved at the expense of its information content and / or using a priori information on the shape of the true distribution

The paper is organized as following. In section 2 the unfolding method consisting of two parts is described. The first part, detector identification, is described in section 3. The second part, unfolding the true distribution, is described in section 4. In section 5 the new method is illustrated by a numerical example.

2. The Unfolding Method

The distribution to be measured with a detector, is further termed "true". The measured distribution has statistical errors and it is distorted due to the resolution of the detector.

Without limiting the general nature of the discussion, we represent the true distribution with values at discrete points and the measured distribution as a histogram.

Let us denote the true distribution with a n-dimensional vector $\vec{\phi} = \{\phi_1, \phi_2, \ldots \phi_n\}$ and the measured distribution with a m-dimensional vector $\vec{f} = \{f_1, f_2, \ldots f_m\}$. The components of vector $\vec{\phi}$ are distribution values at some points, the components of vector $\vec{f}$, are channel contents of a histogram.

The true distribution is related to the measured one by the P-transformation (see Fig. 1)

$$\vec{f} = P\,\vec{\phi} + \vec{\varepsilon}, \qquad (1)$$

where $\vec{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \ldots \varepsilon_m\}$ is m -dimensional random vector (noise) with an average value $E\,\vec{\varepsilon} = 0$ and diagonal variance matrix V

$$V = \mathrm{Var}\,\vec{\varepsilon} = \mathrm{diag}\,(\sigma_1^2, \ldots \sigma_m^2),$$

where σ_i is the statistical error of the measured distribution for the i-th channel.

To restore (unfold) the true distribution from the measured one it is necessary, first, to study the relationship between the distributions and obtain the P-transformation and, second,

to solve equation (1) and obtain the true distribution $\vec{\phi}$.

3.Detector Identification

In this chapter we consider the first part of the method. It is classified as a system identification problem [5,6].

The P-transformation can be represented by its components P_i, i=1,m, where the P_i component transforms $\vec{\phi}$ into f_i (see Fig. 2).

The P_i transformation can be linear or nonlinear. For example, transformations

$$f_i = p_{i1}\,\phi_1 + p_{i3}\,\phi_3$$

and

$$f_i = p_{i1}\,\phi_1 + p_{i2}\,\phi_2^2$$

are linear and nonlinear,respectively.

In this paper we only discuss the linear case because:

a)the linear case is more studied;

b)the linear case is an approximation for the nonlinear case;

c)the linear case is a good way to describe distribution distortions related to particle transportation through the material of the detectors [7].

The identification problem for the linear case consists in finding p_{ij}, j=1,n for each P_i transformation.

To calculate P_i we use a set of equations obtained with the simulated data:

$$\begin{aligned}
f_{i,1} &= p_{i1}\phi_{1,1} + p_{i2}\phi_{2,1} \dots + p_{ik}\phi_{k,1} \dots + p_{in}\phi_{n,1} + \varepsilon_{i,1}\\
f_{i,2} &= p_{i1}\phi_{1,2} + p_{i2}\phi_{2,2} \dots + p_{ik}\phi_{k,2} \dots + p_{in}\phi_{n,2} + \varepsilon_{i,2} \qquad (2)\\
&\vdots\\
f_{i,t} &= p_{i1}\phi_{1,t} + p_{i2}\phi_{2,t} \dots + p_{ik}\phi_{k,t} \dots + p_{in}\phi_{n,t} + \varepsilon_{i,t}\\
&\vdots\\
f_{i,q} &= p_{i1}\phi_{1,q} + p_{i2}\phi_{2,q} \dots + p_{ik}\phi_{k,q} \dots + p_{in}\phi_{n,q} + \varepsilon_{i,q},
\end{aligned}$$

where $\phi_{k,t}$ is the k-th component of the t-th version of the generated distribution (true distribution analogue),

$f_{i,t}$ is the i-th component of the t-th version of the reconstructed distribution (measured distribution analogue),

$\varepsilon_{i,t}$ is the random value (noise) with an average value $E\,\varepsilon_{i,t} = 0$ and variance $Var\,\varepsilon_{i,t} = Var\,f_{i,t}$

Solving the set of equations (2) with the least square method we obtain P_i and its complete matrix of statistical errors D_i. The P transformation in this case is a n * m matrix with matrix elements p_{ij}.

We choose the versions of the generated distributions from the distribution set that may be known from theory,from kinematics ,from the physical sense, etc.

The points for the generated distribution and the channels for the reconstructed one must be chosen to provide for the relation between the distriibutions being linear. The χ^2 criterion can be used to check this .

The choice of the distribution set used for detector identification determines the correlations between the distribution values at different points.The correlations allow to describe the relation between $\vec{\phi}$ and f_i utilizing only a part of the elements p_{ij} of the P_i transformation.The elements not used,are equated to zero.

The choice of elements p_{ij} of the P_i transformation, in this case, is not unique.In the general case one can speak about a set of transformations $\{P_i\}$ that differ by the choice of non-zero elements p_{ij}.

4. Unfolding the True Distribution

With the P transformation,obtained in the previous chapter, equation (1) can be solved by the least squares method minimizing the χ^2 :

$$\chi^2 = (\vec{f} - P^*\vec{\phi})^* V^{-1}(\vec{f} - P^*\vec{\phi}).$$

The $\hat{\vec{\phi}}$ that yields the minimal χ^2, is taken as the estimator of $\vec{\phi}$ [8]:

$$\hat{\vec{\phi}} = (P^* V^{-1}P)^{-1}P^* V^{-1}\vec{f} , \qquad (3)$$

where * indicates matrix transposition

The estimator $\hat{\vec{\phi}}$ is unbiased and the complete matrix of statistical errors for $\hat{\vec{\phi}}$ is [8]

$$\mathrm{Var}\,\hat{\vec{\phi}} = (P^* V^{-1}P)^{-1} . \qquad (4)$$

Another solution of equation (1) taking into account statistical errors of P matrix elements can be obtained within the framework of the maximum likelihood method.

Considering the noise of matrix elements and the experimental distribution to be Gaussian the logarithm of the likelihood functional is the following:

$$\ln L = -\frac{1}{2}\sum_i (f_i - P_i\vec{\phi})^2/\sigma_i + (P_i - G_i) D_i^{-1} (P_i - G_i)^* + \text{const} ,$$

where G_i is the true value of P_i .

Maximizing the functional only for G_i, we obtain the functional

$$-\ln L = \frac{1}{2}\sum_i (f_i - P_i\vec{\phi})^2/ (\sigma_i + \vec{\phi}^* D_i \vec{\phi}) \qquad (5)$$

only for $\vec{\phi}$.

Minimization of (5) yields $\hat{\vec{\phi}}$ as an estimator of $\vec{\phi}$.

The complete matrix of statistical errors for $\hat{\vec{\phi}}$ may be calculated as the inverse matrix of the matrix of second derivatives of the functional (5) at the minimum [8].

Solving equation (1) we may obtain $\hat{\vec{\phi}}$ with extraordinary large errors ,or we may not be able to solve the equation at all.

For the χ^2 (-ln L) functional this signifies the existence of a direction, along which χ^2 (-ln L) varies too little and its minimum is, hence, very poorly determined.

For the given set of points of the true distribution and the given distribution set used for the identification these difficulties may be overcome by:

a) reducing the channel widths for the measured distribution histogram

and / or

b) choosing another P_i from the set $\{P_i\}$.

If no result is achieved one can:

a) reduce the density of the true distribution points

and / or

b) restrict the set of distributions used for detector identification

and / or

c) reduce the measured distribution to the distribution of a detector with a better resolution.

The latter reduction procedure means such correction of the measured distribution,that the result can be considered as the distribution obtained with some hypothetical detector having better resolution than the initial one. In this case we partly compensate the influence of the distortion on the true distribution.To perform such reduction, we should identify the detector that can transform the reduced distribution into the reconstructed one.The reduced distribution can be calculated on a set of points numerically.

One of these ways, or their combination, will lead to an equation that has a solution.

5. A Numerical Example

The method can be illustrated with a numerical example from ref.[3] where:

$$\phi(x) = A_1 \frac{C_1^2}{(x-B_1)^2+C_1^2} + A_2 \frac{C_2^2}{(x-B_2)^2+C_2^2}$$

is the true distribution with parameters $A_1=2$, $A_2=1$, $B_1=10$, $B_2=14$, $C_1=C_2=1$ and $\sigma=1.5$; x is defined on the interval [4, 16].

$$f(x) = \int_4^{16} \phi(x')E(x')K(x,x')dx'$$

is the measured distorted distribution,

$$E(x) = 1 - \frac{(x-10)^2}{6}$$

is the efficiency function, and

$$K(x,x') = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left[-\frac{1}{2}\left[\frac{x-x'}{\sigma}\right]^2\right]$$

is the detector resolution function.

For the detector identification the set of distributions $\phi_k(x)$,k=1,60 was used. A distribution is defined by random parameters generated uniformly on the intervals:

[1,3] for A_1; [0.5,1.5] for A_2;

[8,12] for B_1; [10,18] for B_2;

[0.5,1.5] for C_1; [0.5,1.5] for C_2.

The sample of events obtained with the generated distribution $\phi(x)$ was used for obtaining the reconstructed distribution $\vec{f}_k = \{f_{1,k}, f_{2,k}, \ldots f_{m,k}\}$. Reconstructed events are histogrammed for each $\vec{f}_k$ with weights $\phi_k(x)/\phi(x)$ [7], where x is the generated value of an event.

A histogram of the measured distribution $\vec{f}$ was obtained simulating 10^4 events with m = 90 channels (see Fig.3). The number of points is n = 40 for the true distribution.

For the detector identification 10^4 events were used in the first case, and $6*10^5$ in the second. Unfolded distribution $\vec{\hat{\phi}}$ and errors of $\hat{\phi}$ were calculated applying (3) and (4). Fig.4 shows the results in the first case and Fig.5 in the second ; the solid line is the true distribution. Comparison of the unfolded distribution and the true distribution yields $\chi^2 = 36.8$ in the first case and $\chi^2 = 32.1$ in the second one, for 40 points.

As compared to ref.[3], where from the statistical point of view, the agreement between the unfolded and true distributions is poor , in our case the agreement between two distributions is good. Due to the properties of the least squares method used by us no bias is introduced in the data, while the data from [3] are biased as a result of processing. Unlike [3] no difficulties arise in our procedure for the interpretation of errors, and due to the least squares method they are minimal.

6. Conclusion

The main results of this paper may be summarized as follows.

A new method for unfolding the true distribution from experimental data distorted by a detector with finite resolution is proposed. The method reduces to a procedure for choosing a stable result, that is obtained by the least squares method.

A generalization of the method is proposed, when identification of the detector is based on a limited number of simulated events.

The method can be applied for detectors introducing linear or nonlinear distortions.

Application of the method is illustrated by a numerical example.

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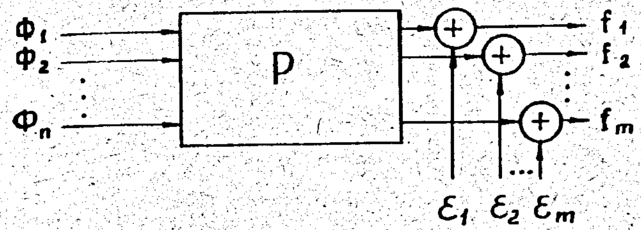

Fig. 1. Diagram of relations between true and measured distributions

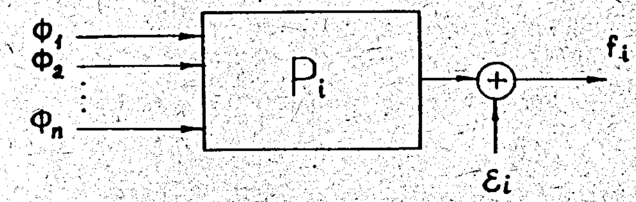

Fig. 2. Diagram of relations between the true distribution and the i-th channel of the measured distribution

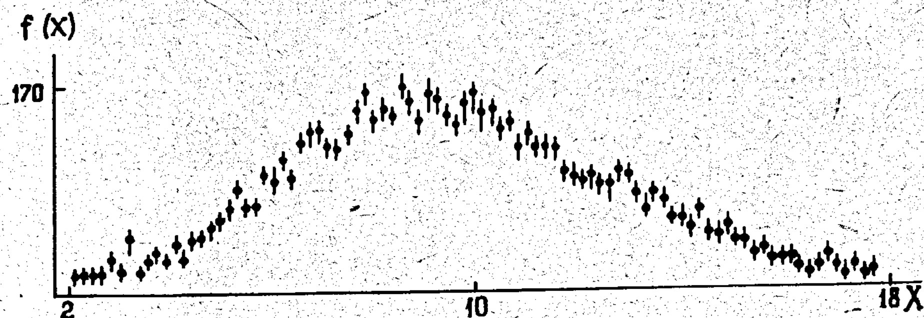

Fig. 3. Example of measured distribution $\vec{f}$

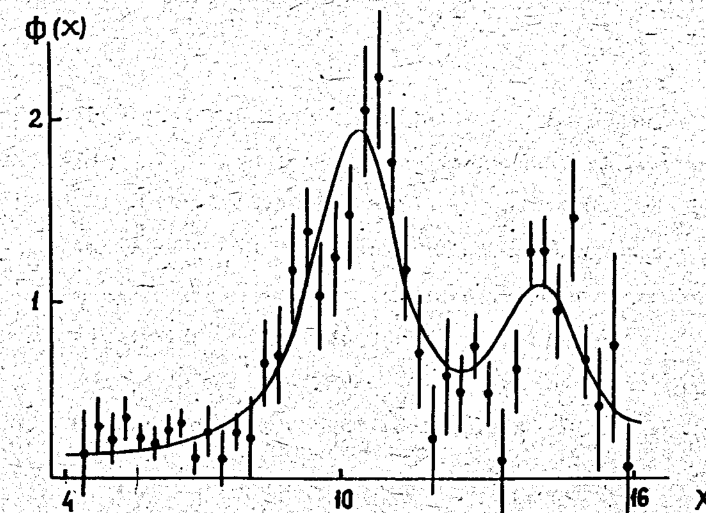

Fig. 4. Unfolded distribution $\vec{\Phi}$;the solid line is the true distribution $\Phi(x)$, the number of events used for detector identification is 10^4

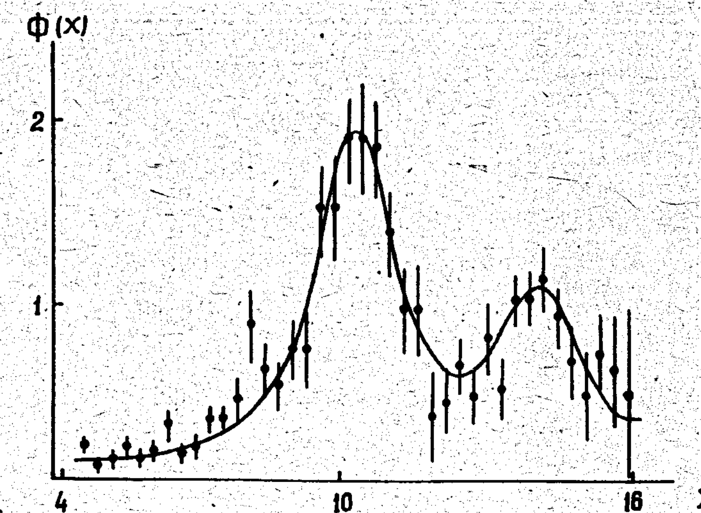

Fig. 5. Unfolded distribution $\vec{\Phi}$;the solid line is the true distribution $\Phi(x)$, the number of events used for detector identification is $6 * 10^5$

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