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A NONSTANDARD MATRIX DERIVATIVE OBTAINED FROM A STANDARD SCALAR DERIVATIVE

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1. Introduction

One of the first historical definitions of the derivative, dy/dx, is based on the notion of the limiting transition developed by I.Newton (1642-1727):

$$dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}, \qquad (1)$$

where $\Delta y = (y_2 - y_1)$ and $\Delta x = (x_2 - x_1)$ are finite differences for $y = y(x)$. The second principal step in developing this limiting transition approach is due to the theorem on function's mean value advanced by A.Cauchy (1789-1857).

The second approach to the problem of derivative is associated with the name of G.W.Leibniz (1646-1716) who considered, as a primary element, the differential dx being a composite part of the abscissa axis, i.e. a value smaller than any finite value but not equal to zero (an actual infinitesimal value).

The third approach to the same problem was advanced by L.Lagrange (1736-1813) who rejected both Newtonian and Leibnizian definitions in favour of functions, f(x), representable by a power series like

$$y = f(x) = a_0 + a_1 x + \dots + a_n x^n \qquad (2)$$

Then the Lagrangian derivative function, $f'(x)$, is defined as

$$y' = f'(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}. \qquad (3)$$

2. Modern developments

The new ideas concerning the notion of derivative were provoked by the so-called non-standard analysis based on hyperreal numbers and developed by A.Robinson (1918-1974) in the early 1960's [1].

The hyperreal number field, R*, and the usual number field, R, are known to be ordered, i.e. these fields satisfy the inequality relation, $a > b$. The principal difference between R* and R is determined by the Archimedean axiom stating that if

$$|x| < \frac{1}{n} \qquad (4)$$

for any natural number, n, then x = 0.

The field R satisfies this axiom, while the field R* does not.

Thus the novel non-standard definition of Robinson's derivative can be treated as an extension of the primary "atomistic" Leibnizian formulation.

3. Reversible mathematics [2]

Our studies of global optimization[3], unstable inverse problems[4] as well as differentiation and integration[5] indicate a widespread - and widely noncritical - use of the equality sign in "equations" like a=b as a mere substitution for a left-to-right or right-to-left reduction signs, $a \to b$ or $a \leftarrow b$, respectively. This nonevident logical discrepancy between semiotic and semantic sides of one of the most universal mathematical operations, i.e. equalization, leads to irreversible losses of information in going from one side of any "equation" to another. Moreover, in a long chain of associated mathematical transformations these losses may result even in logical contradictions in the form of the well-known mathematical paradoxes[6].

4. Matrix derivative, [D]

Let us begin with an "atomistic" definition of the function's derivative taken as

$$d = f'(x) = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (5)$$

This reduced equation (requation) can be rewritten as

$$\dot{D} = \frac{\begin{vmatrix} y_2 & 1 \\ y_1 & 1 \end{vmatrix}}{\begin{vmatrix} x_2 & 1 \\ x_1 & 1 \end{vmatrix}} \tag{6}$$

in a close analogy to the eigenproblem formulation

$$Av = v\lambda\ , \tag{7}$$

where A - a matrix, v - an eigenvector and λ - a scalar eigenvalue. Let us consider the scalar, d, as the determinant of a matrix

$$[D] = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}\ . \tag{8}$$

Then the elements of this matrix will be found to equal

$$d_{11} = + \frac{y_2 - x_1}{x_2 - x_1} \tag{9}$$

$$d_{12} = - \frac{y_2 - x_2}{x_2 - x_1} \tag{10}$$

$$d_{21} = + \frac{y_1 - x_1}{x_2 - x_1} \tag{11}$$

$$d_{22} = - \frac{y_1 - x_2}{x_2 - x_1}\ . \tag{12}$$

All usual checks show these matrix derivative components to be correct algebraic values.

By taking a sample function like $y = x^2$, it is possible to obtain an evident geometrical interpretation of each component of $[D]$.

5. A non-standard illustrative example

Non-standard devices need non-standard applications.

The so-called nowhere differentiable function studied in 1871 by K.Weierstrass (1815-1897) can be presented parametrically as[7]:

$$x = \sin(\theta) \tag{13}$$

$$y = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n\theta) \tag{14}$$

so that the derivative $y'(\theta) = dy/d\theta$ does not exist anywhere within $0 \leq \theta \leq 2\pi$, because the corresponding derivative series is divergent.

It can be easily shown that all the matrix derivatives (9-12) do exist for $y(\theta)$ shown in (14). For example,

$$d_{11} = + \frac{y_2 - \theta_1}{\theta_2 - \theta_1}\ . \tag{15}$$

Notwithstanding the non-standard overall behaviour of the Weierstrass function, which initially has rejected any feasible application at all, now it is widely used as a seed function in fractal mathematics, in particular, to desribe the Wiener diffusion process inherent in charged particle tracking.

6. Conclusions

The methods of reversible mathematics have been applied to obtain a novel algebraic definition of function's derivative in a matrix form. The unusual algebraic form of this derivative appeals to unusual mathematical applications.

7. Acknowledgements

To sum up, the author would like to acknowledge the late Prof. F.Klein (1849-1925) for his deep and inventive insight into the fundamental mathematical problems under study.

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Нестандартная матричная производная, полученная из стандартной скалярной производной

Показано, что из математической формулы для вычисления стандартной скалярной производной $d = (y_2 - y_1)/(x_2 - x_1)$, где $y = y(x)$, можно получить нестандартную матричную производную [D]. Элементы матрицы [D] имеют нестандартную форму — например, $d_{11} = (y_2 - x_1)/(x_2 - x_1)$. Матричная производная [D] может найти применение в анализе таких экзотических объектов, как нигде недифференцируемые функции. На примере известной функции Вейерштрасса показана конкретная применимость матричной производной.

Наиболее вероятной областью применения [D] может служить фрактальная математика и математика разрывных функций. Новый математический объект допускает очевидную геометрическую интерпретацию.

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A Nonstandard Matrix Derivative Obtained from a Standard Scalar Derivative

It has been shown that from the mathematical formula for a standard scalar derivative, $d = (y_2 - y_1)/(x_2 - x_1)$, where $y = y(x)$, one can obtain a nonstandard matrix derivative [D]. The elements of the matrix [D] possess a nonstandard from — e.g., $d_{11} = (y_2 - x_1)/(x_2 - x_1)$. The matrix derivative [D] can be used for analyzing such exotic objects as nowhere differentiable functions. As a specific application example the author used the well-known Weierstrass function. More extended domains for application of the matrix derivative [D] can be provided by fractal mathematics and singular functions. The novel mathematical object allows an evident geometrical interpretation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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