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THE GENERAL STRUCTURE OF SOLUTIONS OF UNSTABLE INVERSE PROBLEMS

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1.Introduction

The discrete-discrete mathematical model of unstable inverse problem (UIP) can be reduced to a system of linear algebraic equations (SLAE) like

$$At = f \tag{1}$$

where A – the coefficient (design) nxn-matrix, t – a true solution vector and f – a noisy r.h.s. input data vector.

Let us assume the input vector f to be composed of a structured component, f_S, consisting of relatively broad (Gaussian, Breit-Wignerian etc) peaks and a smooth background support, f_B, due to a previous incorrect background subtraction and the smoothing effect of the l.h.s. convolution, At.

Then the r.h.s. vector can written down as

$$f = f_S + f_B + f_Z \tag{2}$$

where

$$f_Z = z = (0, 0....0)^T \tag{3}$$

2.Possible information losses in convolution

The conceptual analysis of the l.h.s. convolution in (1) demonstrates the two following sources of information losses [1].

One of the most important sources seems to be due to the co-called orthogonal erosion, when

$$t = t_I + t_V \tag{4}$$

where t_I – an invisible A-orthogonal component and t_V – a visible A-nonorhtogonal component [2].

A clear indication of this erosion to be a real danger is suggested by the fundamental Riemann-Lebesgue lemma [3]:

$$\int_{\alpha\to\infty} a(x, s)\sin(\alpha s)ds \to 0 \tag{5}$$

which states that an orthogonal nonzero factor, $\sin \alpha s$, when multiplied by a nonzero one, $a(x, s)$, can result in a pure zero r.h.s., i.e. in a noninformative product.

With this controversal situation at hand we will try to analyze the structure of the column vector t corresponding to the complete true input information.

In general, we name an informative element any nonzero r.h.s. term from (1).

3.The structure of t-vector

3.1. The first source of information losses – the spectrum component, t_S

The A-matrix is a discrete image of the linear operator, A, to yield

$$At = A(t_S + t_B + t_Z) = f_S + f_B + f_Z \tag{6}$$

that closely resembles (2).

The smoothing action of the A-matrix is the first source of peak broadening and information loss, observed in f_S.

Under ideal conditions, the corresponding t_S-component can be viewed as a set off δ-function-like peaks, p_I, interspersed with zeros;

$$t_S = (p_1, 0, 0, p_2, ...p_N) \tag{7}$$

3.2. The second source of information losses – the background component, t_B

In a matrix formulation the orthogonality relation looks like

$$AA^T = AA^- = I \tag{8}$$

where A^T – the transposed matrix, A^- – the inverse matrix and I – the diagonal identity matrix;

$$I = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \tag{9}$$

A scalar matrix, S, is defined as

$$S = \alpha I \tag{10}$$

where α – a scalar factor.

In our standard matrix-vector formulation an equivalent "quasiorthogonal" version of the I-matrix is the following unit column vector;

$$u = (1, 1, ..., 1)^T \tag{11}$$

so that the corresponding "scalar" vector is

$$t_B = \alpha u \tag{12}$$

It must be noted that the unit column vector, u, acts in (1) as an A-matrix row summator, thus producing a smoothed (background) part, f_B.

The quasiorthogonal convolution, At_B, becomes the second source of information losses. This is especially clear in the all-matrix version of (1), where $T_B = \alpha I$.

3.3.The third source of information losses – t_Z

The third source of information losses in a transfer from an l.h.s. to an r.h.s. in (1) can be due to

$$t_Z \neq 0 \tag{13}$$

In a close analogy to (5) this situation mirrors the problem of finding a general solution of (1) which is composed of both homogeneous and inhomogeneous terms. The homogeneous term corresponds to

$$At_Z = 0 \tag{14}$$

It is known [4] that a nontrivial solution of (14):

$$t_Z \neq z \tag{15}$$

is feasible only for an A-matrix with an incomplete rank,

$$r < n \tag{16}$$

where n is the dimension of the A-matrix.

Thus, to check for third source of information losses in the SLAE (1) needs regular control computations of the rank, r. The latter problem is known to be the most difficult to solve in linear algebra at large.

3.4. Some model alternatives

The first alternative for the above solution model can be derived from the linear nature of the operator A:

$$A(t + d) = f + n \tag{17}$$

where d – the t-error and n – the f-error. Other noise models can be incorporated in the same simple way.

Another model can be adapted from the modular arithmetic approach widely used to describe error-correcting codes [5]:

$$s = A(t + d) = z + e \tag{18}$$

where s-the so-called syndrome term, z-the term corresponding to an error-free code and e – the nonzero error-detecting term. The latter model is directly linked to the error-free computation strategy developed within the framework of modular arithmetic and P-addic numbers approaches [6].

Conclusions

The true solution of the SLAE (1) is shown to be composed of three main components. One of these components is due to a solution of the homogeneous version of (1), while the two remaining ones stem from a solution of the inhomogeneous version of (1).

Any real losses of the information contained in a complete l.h.s. t-vector can proceed through three interdependent channels.

To sum up, the author would like to acknowledge the late Profs. C.Lanczos and A.S.Householder for their cognitive and inventive insight into the fundamental problems under study.

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Общая структура решений неустойчивых обратных задач

Показано, что общее векторное решение системы линейных алгебраических уравнений (СЛАУ), $At = f$, можно представить в виде вектора-столбца $t = t_S + t_B + t_Z$, где t_S – спектральная компонента, t_B – фоновая компонента и t_Z – решение однородной СЛАУ, $At_Z = z = (0, 0, ..., 0)^T$. Компонента t_S состоит из δ-образных пиков, $t_B = \alpha u$, где α – скаляр и $u = (1, 1, ..., 1)^T$, при этом $t_Z = z$ только для A-матриц полного ранга, с $r = n$. Таким образом, для контроля возможных потерь информации при решении СЛАУ необходимо проверять величину ранга A-матрицы в течение всего процесса вычислений.

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The General Structure of Solutions of Unstable Inverse Problems

The general vector solution of systems of linear algebraic equations (SLAE), $At = f$, is shown to be expressed as a column vector, $t = t_S + t_B + t_Z$, where t_S – a spectral term, t_B – a background term and t_Z – a solution of the homogeneous SLAE, $At_Z = z = (0, 0, ..., 0)^T$. The term t_S consists of δ-function-like peaks, the next term, $t_B = \alpha u$ with a scalar α and $u = (1, 1, ..., 1)^T$, while $t_Z = z$ only for a full-rank A-matrix ($r = n$). Thus, to control information losses needs a regular check of the A-matrix rank during the whole computation process.

The investigation has been performed at the Laboratory of High Energies, JINR.

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