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THE BUSY PERIOD OF ORDER n IN THE GI/D/ ∞ QUEUE

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1. INTRODUCTION

Let us suppose that the customers arrive at the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, where $\tau_1 < \tau_2 < \ldots < \infty$. We assume that τ_1 is an arbitrary positive random variable and the interarival times $T_n = \tau_n - \tau_{n-1}$, $n = 2, 3, \ldots$, are identically distributed independent positive random variables which are independent also of $T_1 = \tau_1$. Let us have

$$F(t) = P(T_n \le t), \quad n = 2,3,\ldots, \tag{1.1}$$

$$\hat{F}(t) = P(T_1 \le t). \tag{1.2}$$

The queueing system is assumed to have the infinitely many servers with a fixed time $p > 0$. Denote by B_n the duration of the busy period of order n, that is, the time when at least n customers has been served (not necessarily simultaneously). The duration of the period between two neighbouring busy periods of order n, I_n, is called the idle period of order n.

The busy period of other kind has been defined by Glaz [1]. In his sense it is a period when at least n servers are simultaneously busy. He derived bounds for the distribution and expected value of the number of customers served during that period, and the bounds for expected length of that busy period. For the special M/D/∞ queue he derived the exact distribution of B_n.

These two notions are the same for n=1, and our busy period is always longer than the Glaz period, in general. Hence our moments are upper bounds for the moments of Glaz's period.

As follows from the paper of Dvurečenskij et al. [2] the queueing system with infinitely many servers may be used as a model for description of blob length measurements in the track chambers in high energy physics. The blob length corresponds to the busy period of the first order. The other papers dealing with the blob length determination are, for example, papers by Dvurečenskij [2], Glückstern [3], Kuljukina et. al. [4].

In Section 2 of the present paper the busy period of the first order is studied. The number of customers served in that period, integral equation, distribution, characteristic function and all moments are derived.

In Section 3 the same characteristics as in Section 2 are given for higher orders and the relationships between them are shown.

Finally, in Section 4 the waiting time for the first period of order n and the idle period of order n are investigated. We derive the probability law and the generating function of the indexes of customers arrived at that time, and the Laplace transform of the idle period of order n.

2. THE BUSY PERIOD OF THE FIRST ORDER

We assume that for the quantity

$$I = P(T_2 > p) \tag{2.1}$$

we have $0 < I < 1$. The case $I = 0$ corresponds to the infinitely long busy period and $I = 1$ corresponds to a sequence of the busy period when one customer is served. Define

$$\bar{F}(t) = P(T_2 \le t / T_2 \le p),$$
$$\bar{\bar{F}}(t) = P(T_2 \le t / T_2 > p). \tag{2.2}$$

Let N_n, $n = 1,2,\ldots$, be the index of customer from which the busy period of order n begins, and ν_n, $n = 1,2,\ldots$, be the number of customers served during that period. Let

$$G_n(t) = P(B_n \le t). \tag{2.3}$$

be the distribution function of B_n, $n = 1, 2, \ldots$.

Lemma 2.1. There hold

$$N_1 = 1, \tag{2.4}$$

$$P(\nu_1 = k) = I(1-I)^{k-1}, \quad k = 1,2,\ldots,$$
$$E(\nu_1) = 1/I. \tag{2.5}$$

Proof. It obviously holds that $P(\nu_1 = k) = P(T_2 \le p, \ldots, T_k \le p, T_{k+1} > p)$. Q.E.D.

Theorem 2.2. For the distribution function of B_1 we have

$$G_1(t) = \sum_{k=1}^{\infty} I(1-I)^{k-1} \bar{F}^{*(k-1)}(t-p), \tag{2.6}$$

where $\bar{F}^{*k}$ denotes the k-th iterated convolution of $\bar{F}$ with itself, $\bar{F}^{*0}(t) = \epsilon(t)$, $\epsilon(t) = 1$ if $t > 0$, else $\epsilon(t) = 0$.

Moreover $G_1(t)$ satisfies the linear integral equation

$$G_1(t) = I\epsilon(t-p) + (1+I)\int_0^p G_1(t-x)\,d\bar{F}(x) . \qquad (2.7)$$

Proof. Since we have $P(B_1 \le t/\nu_1 = 1) = \epsilon(t-p)$ and $P(B_1 \le t/\nu_1 = k) = \bar{F}^{*(k-1)}(t-p) = \int_0^p P(B_1 \le t-x/\nu_1 = k-1)\,d\bar{F}(x)$, $k = 2,3,\dots$,

the formulae (2.6) and (2.7) are proved. Q.E.D.

We use the following notations

$$\phi_0(s) = \int_0^\infty e^{its}\,d\bar{F}(t), \quad s \in R_1 , \qquad (2.8)$$

$$m^o_q = \int_0^\infty t^q\,d\bar{F}(t), \quad q = 0,1,2,\dots . \qquad (2.9)$$

Theorem 2.3. For the characteristic function ϕ_1 of B_1 we have

$$\phi_1(s) = Ie^{isp}/(1-(1-I)\phi_0(s)), \quad s \in R_1 . \qquad (2.10)$$

The moments $E(B_1^q)$, $q = 0,1,2,\dots$, are finite and they can be evaluated from the following relations

$$E(B_1) = p + (1+I)\,m_1^o/I , \qquad (2.11)$$

$$E(B_1^q) = p^q + (1-I)I^{-1}\sum_{j=0}^{q-1} E(B_1^j)\,m^o_{q-j} , \quad q = 2,3,\dots . \qquad (2.12)$$

Proof. Multiplying the formula (2.7) by e^{ist} and then integrating both sides, the formula (2.10) can be obtained. Analogically we proceed with the moments replacing e^{ist} by t^q, $q = 0,1,2,\dots$. Q.E.D.

The similar results valid for the case $M/D/\infty$ can be find in refs. [2,4]. In ref. [3] the moments of B_1 in $M/D/\infty$ queue are evaluated by a known way of a derivation of the Laplace transform of B_1 which is, of course, more complicated than the formula (2.12).

3. THE GENERAL CASE

In this section we will study the same characteristics as in Section 2 for the busy period of order n. For the number of customers served in that period we have the next theorem.

Theorem 3.1. We have

$$P(\nu_n = k) = I(1-I)^{k-n} , \quad k = n, n+1, \dots, \; n = 1,2,\dots , \qquad (3.1)$$

$$E(\nu_n) = n - 1 + 1/I , \quad n = 1,2,\dots . \qquad (3.2)$$

Proof. Due to the theorem of total probability we have

$$P(\nu_n = k) = \sum_{j=1}^\infty P(\nu_n = k/N_n = j)P(N_n = j) .$$

But

$$P(\nu_n = k/N_n = j) = P(T_{j+1} \le p, \dots, T_{j+k-1} \le p, T_{j+k} \ge p/f_j(T_2, \dots, T_j) ,$$

$$T_{j+1} \le p, \dots, T_{j+n-1} \le p) = I(1-I)^{k-n} ,$$

where f_j is a some suitable function of $T_2, \dots, T_j$ and p. Hence this conditional probability does not depend on j.

Q.E.D.

Theorem 3.2. The distribution function G_n of B_n is of the form

$$G_n(t) = \sum_{k=n}^\infty I(1-I)^{k-n}\,\bar{F}^{*(k-1)}(t-p), \quad n = 1,2,\dots , \qquad (3.3)$$

and for the characteristic function we have

$$\phi_n(s) = Ie^{isp}\,\phi_0^{n-1}(s)/(1-(1-I)\phi_0(s)), \quad s \in R_1 , \; n = 1,2,\dots . \qquad (3.4)$$

Proof. The theorem of total probability implies that

$$P(B_n \le t) = \sum_{j=1}^\infty \sum_{k=n}^\infty P(B_n \le t/N_n = j, \nu_n = k)P(\nu_n = k/N_n = j)P(N_n = j) = \sum_{k=n}^\infty \bar{F}^{*(k-1)}(t-p)\,I(1-I)^{k-n} .$$

The formula (3.4) results from a multiplication of (3.3) by e^{ist} and an integration. Q.E.D.

Theorem 3.3. We have

$$G_n = G_{n-1} * \bar{F}, \quad n = 2,3,\dots , \qquad (3.5)$$

$$G_n = G_1 * \bar{F}^{*(n-1)} , \quad n = 2,3,\dots , \qquad (3.6)$$

$$G_n(t) = I\bar{F}^{*(n-1)}(t-p) + (1-I)\int_0^p G_n(t-x)\,d\bar{F}(x) , \qquad (3.7)$$

$n = 1,2,\dots$,

$$G_{n+1}(t) = (1-I)^{-1}G_n(t) - I(1-I)^{-1}\bar{F}^{*(n-1)}(t-p) , \qquad (3.8)$$

$n = 1,2,\dots$,

$$G_n(t) = (1-I)^{1-n}\; G_1(t) - \sum_{j=0}^{n-1} I(1-I)^{-j}\; \bar{F}^{*(n-j)}(t-p), \qquad (3.9)$$
$$n = 1,2,\ldots .$$

Proof. The formulae (3.5) and (3.6) may be obtained, for example, from the forms of the characteristic functions (3.4) and (2.10). The formula (3.7) follows from (3.3). The last proved formula and (3.5) give us (3.8). The successive repetition of (3.8) implies the formula (3.9). Q.E.D.

Theorem 3.3. The expected length of B_n is finite and there hold

$$E(B_n) = p + m_1^{\circ}((n-2)I+1)/I, \quad n = 1,2,\ldots . \qquad (3.10)$$

Proof. The above formulae can be proved by using the formulae (3.4) or (3.6) and (2.11). Q.E.D.

Now, multiplying the formula (3.9) or (3.7), by t^q, $q = 2,3,\ldots$, we can derive the following expressions for $E(B_n^q)$:

Theorem 3.4. All the moments of B_n are finite and we have

$$E(B_n^q) = E(B_1^q)(1-I)^{n-1} - I\sum_{j=0}^{n-1}\sum_{r=0}^{n-1}\sum_{C_{n,j}} (1-I)^{-j}\binom{q}{r} p^{q-r} \times$$
$$\times\; p(r_1,\ldots,r_{n-j})\, m_{r_1}^{\circ}\ldots m_{r_{n-j}}^{\circ}, \qquad (3.11)$$
$$q = 1,2,\ldots ,$$

$$E(B_n^q) = \sum_{r=0}^{q}\sum_{C_{n,n-1}} \binom{q}{r} p^{q-r} p(r_1,\ldots,r_{n-1})\, m_{r_1}^{\circ}\ldots m_{r_{n-1}}^{\circ} +$$
$$+\;(1-I)I^{-1}\sum_{r=0}^{q-1}\binom{q}{r} E(B_n^r)\, m_{q-r}^{\circ}, \qquad (3.12)$$
$$q = 2,3,\ldots ,$$

where $C_{n,j}$ denotes the set of nonnegative integers $r_1,\ldots,r_{n-j}$ such that $r_1+\ldots+r_{n-j} = r$, and

$$p(r_1,\ldots,r_{n-j}) = (r_1+\ldots+r_{n-j})!/(r_1!\ldots r_{n-j}!), \qquad (3.13)$$
$$j = 0,1,\ldots,n-1.$$

Example. Let the customers arrive to the service facility according to a Poisson process with a rate g. Then

$$E(B_n) = (n-1)(1-gpe^{-gp}-e^{-gp})\,g^{-1}(1-e^{-gp})^{-1} +$$
$$+\;(e^{gp}-1)g^{-1} . \qquad (3.14)$$

4. IDLE PERIOD OF ORDER

The idle period of order n, I_n, is defined as a period between two neighbouring busy periods of order n. It is strongly connected with the waiting time I_n° for the first period of order n.

Theorem 4.1. For $n = 2,3,\ldots$ there holds

$$P(N_n = k) = \begin{cases} (1-I)^{n-1} & \text{if } k = 1, \\ I(1-I)^{n-1} & \text{if } 2 \le k \le n, \\ (1-I)^{n+k-2}\sum_{D_{n,k}} p(k_1,\ldots,k_{n-1}) \times & \\ \times (I/(1-I))^{k_1+\ldots k_{n-1}}, & \text{if } k > n, \end{cases} \qquad (4.1)$$

where $D_{n,k}$ denotes the set of nonnegative integers $k_1,\ldots,k_{n-1}$ such that $k_1 + 2k_2 + \ldots + (n-1)k_{n-1} = k-1$.

Proof. Let $k>n$. From the definition of N_n we have that from the first $k-1$ served customers k_1 periods must be formed from a service of one customer, k_2 periods must be formed from services of two customers, etc., k_{n-1} periods must be formed from services of $n-1$ customers, where $k_1 + 2k_2 + \ldots + (n-1)k_{n-1} = k-1$. Let $A(k_1,\ldots,k_{n-1})$ be just described event. Hence we have

$$P(A(k_1,\ldots,k_{n-1})) = p(k_1,\ldots,k_{n-1})P^{k_1}(\nu_1 = 1)\ldots P^{k_{n-1}}(\nu_1 = n-1),$$

and $P(\nu_1 = i) = I(1-I)^{i-1}$ by (2.4).

Q.E.D.

If $n=2$, then from what we said above we have

$$P(N_2 = k) = (1-I)I^{k-1}, \quad k = 1,2,\ldots . \qquad (4.2)$$

Now for $n \ge 3$ we define

$$A_k = \{T_{k+1} \le p,\ldots,T_{k+n-1} \le p\}, \quad k = 1,2,\ldots . \qquad (4.3)$$

Theorem 4.2. Let $h_n(z) = \sum_{k=1}^{\infty} z^k P(N_n = k)$, $|z|<1$, be the generating function of N_n, $n \ge 3$. Then

$$h_n(z) = g_n(z)/(1 - z + z^{n-2} g_n(z)), \qquad (4.4)$$

where

$$g_n(z) = \sum_{i=1}^{\infty} a_i z^i , \quad a_i = (1-I)^{n-2+i} \sum_{k_1 \dots k_{j-1}, i} (-1)^i , \tag{4.5}$$

where the summation is over all integers k_s such that

$$0 < k_1 - 1 \le n-2, \quad 0 < k_2 - k_1 \le n-2, \dots, 0 < k_{j-1} - k_{j-2} \le$$
$$\le n-2, \quad 0 < i - k_{j-1} \le n-2 .$$

Moreover

$$E(N_n) = g_n^{-1}(1) - n + 2 . \tag{4.6}$$

Proof. It is evident that $P(N_n = k) = P(\bar{A}_1 \dots \bar{A}_{k-1} A_k)$, where $\bar{A}$ denotes the complement of A. The sequence of dependent events $\{A_k\}_{k=1}^{\infty}$ fulfils the Solov´ev conditions from ref. /5/ with $m = n-2$. Using his combinatorial identity we can prove the identity (4.4). Q.E.D.

As particular cases we have

$$h_2(z) = (1-I)z/(1-Iz) ,$$
$$E(N_2) = (1-I)^{-1} , \tag{4.7}$$

$$h_3(z) = (1-I)^2 z/(1 - Iz - Iz^2 + I^2 z^2) ,$$
$$E(N_3) = (2-I)(1-I)^{-2} - 1 . \tag{4.8}$$

Theorem 4.3. For the distribution function of I_n° we have

$$P(I_1^\circ \le t) = \hat{F}(t) ,$$

$$P(I_n^\circ \le t) = (1-I)^{n-1} \hat{F}(t) + I(1-I)^{n-1} \sum_{k=2}^{n} (\hat{F} * F^{*(n-2)} * \bar{\bar{F}})(t) +$$
$$+ \sum_{k=n+1}^{\infty} \sum_{D_{n,k}} p^2(k_1, \dots, k_{n-1}) \times$$
$$\times (I/(1-I))^{k_1 + \dots + k_{n-1}} (\hat{F} * \bar{\bar{F}}^{*(k_1 + \dots + k_{n-1})} * \tag{4.9}$$
$$* \bar{F}^{*(k_2 + 2k_3 + \dots + (n-2)k_{n-1})})(t) ,$$
$$n \ge 2 .$$

Proof. By aid of the theorem of total probability it suffices to prove the following. Let $k > n \ge 2$, then

$$P(I_n^\circ \le t / A(k_1, \dots, k_{n-1})) = P(T_1 + \dots + T_k \le t / A(k_1, \dots, k_{n-1})) =$$
$$= p(k_1, \dots, k_{n-1}) (\hat{F} * \bar{\bar{F}}^{*k_1} * \bar{\bar{F}}^{*k_2} * \bar{F}^{*k_2} * \dots * \bar{\bar{F}}^{*k_{n-1}} *$$
$$* \bar{F}^{*(n-2)k_{n-1}})(t) .$$

Q.E.D.

Theorem 4.4. For the distribution function of the idle period I_n we have

$$P(I_1 \le t) = \bar{\bar{F}}(t+p) ,$$

$$P(I \le t) = (1-I)^{n-1} \bar{\bar{F}}(t+p) + I(1-I)^{n-1} \sum_{k=2}^{n} (\bar{\bar{F}}^{*2} *$$
$$* F^{*(k-2)})(t+p) + \sum_{k=n+1}^{\infty} \sum_{D_{n,k}} p(k_1, \dots, k_{n-1})^2 \times \tag{4.10}$$
$$\times (I/(1-I))^{k_1 + \dots + k_{n-1}} (\bar{\bar{F}}^{*(1+k_1+\dots+k_{n-1})} *$$
$$* \bar{F}^{*(k_2 + 2k_3 + \dots + (n-2)k_{n-1})})(t+p) ,$$
$$n \ge 2 .$$

Proof. This theorem follows from Theorem 4.3 if we replace I_n° by I_n, $\hat{F}$ by $\bar{\bar{F}}$, and t by t+p in the right-hand side of (4.9) Q.E.D.

Theorem 4.5. Let $a_1(s) = E(e^{-sT_1})$, $a(s) = E(e^{-sT_2})$, $s \ge 0$, $g_n^*(s)$ be the function from (4.5) corresponding to

$$I = I(s) = 1 - a(s)^{-1} \int_0^p e^{-st} dF(t) . \tag{4.11}$$

Then the Laplace transform of I_n°, H_n°, for $n \ge 3$, is of the form

$$H_n^\circ(s) = \frac{a_1(s) a(s)^{n-2} g_n^*(a(s))}{1 - a(s) + a(s)^{n-2} g_n^*(a(s))} , \quad s \ge 0 . \tag{4.12}$$

Moreover, if $a = \int_0^\infty t dF(t) < \infty$, $a_1 = \int_0^\infty t d\hat{F}(t) < \infty$, then

$$E(I_n^\circ) = a_1 + a/g_n(1) . \tag{4.13}$$

Proof. There holds

$$H_n^\circ(s) = E(\exp(-I_n^\circ s)) = \sum_{k=1}^{\infty} \int \dots \int_{\{N_n = k\}} \exp(-(t_1 + \dots + t_k)s) d\hat{F}(t_1) \times$$
$$\times dF(t_2) \dots dF(t_{k+n-1}) .$$

Now we define the distribution function

$$F^*(t,s) = a(s)^{-1} \int_0^t e^{-sx}\, dF(x)$$

and let N_n^* be a random variable corresponding to $F^*(t,s)$ for $I = I(s)$ from (4.11). Using the Solov´ev method and analogical reasonings as in the proof of Theorem 4.2 we conclude

$$H_n^\circ(s) = a_1(s) \sum_{k=1}^{\infty} a(s)^{n+k-2}\, P(N_n^* = k). \qquad \text{Q.E.D.}$$

Let us put

$$a_2(s) = \int_0^\infty e^{-st}\, d\bar{\bar{F}}(t) \quad \text{and} \quad a_2 = \int_0^\infty t\, d\bar{\bar{F}}(t).$$

Then by a similar way as in the two last theorems we may prove the next theorem.

Theorem 4.6. The Laplace transform of the idle period of order n, H_n, is of the form

$$H_2(s) = (1-I)\, e^{sp} / (1 - I a_2(s)), \tag{4.14}$$

$$H_n(s) = \frac{e^{sp} a_2(s) a(s)^{n-2} g_n^*(a(s))}{1 - a(s) + a(s)^{n-2} g_n^*(a(s))}, \quad n \geq 3. \tag{4.15}$$

Moreover, if $a < \infty$, then

$$E(I_2) = a_2/(1-I) - p, \tag{4.16}$$

$$E(I_n) = a_2 - p + a/g_n(1), \quad n \geq 3. \tag{4.17}$$

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Период занятости порядка n в GI/D/∞ очереди

В работе исследуются задача определения функции распределения, характеристической функции, интегральные уравнения и все моменты периода занятости порядка n, т.е. периода, когда заняты по крайней мере n из бесконечного числа обслуживающих устройств системы массового обслуживания GI/D/∞. Также изучается период простоя порядка n, т.е. период между двумя соседними периодами порядка n. Эти проблемы возникают при определении длины сгустков в трековых камерах в физике высоких энергий.

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The Busy Period of Order n in the GI/D/∞ Queue

In the paper we investigate the problem of determination of the distribution function, integral equation and all moments of the busy period of order n, that is, the period when at least n servers are busy from infinitely many servers of the GI/D/∞ queueing system. We are studying also the idle period of order n, i.e., the period between two neighbouring busy periods of order n. Those problems arise in the blob length determination in track chambers in high energy physics.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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