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ON ONE PROBLEM OF THE BUSY PERIOD DETERMINATION IN QUEUES WITH INFINITELY MANY SERVERS

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1. INTRODUCTION

As was shown in the paper [1] a queueing system

$$\langle E, 1, GI/\infty, 1 \rangle \tag{1}$$

may be used as a model of the streamer track in high energy physics, where the blob length and the gap length correspond to the busy period and the idle period, respectively, of the system (1).

Due to the automatical scanning in track chambers we deal with discretized values. Therefore the problem arised of exact determination of the busy period of some queueing system with infinitely many servers as a discretized blob length. This problem was solved with some simplifications which are true, for example, in systems $\langle E,1,M/\infty,1\rangle$, see refs. [1,2]. Determination of the nondiscretized busy period (or nondiscretized blob length) was treated in the papers [1-4].

In the present paper this problem is treated without any simplification. We derive the formulae for the probability of discretized system and some stability properties of the busy and idle periods are given.

2. DESCRIPTION OF DISCRETIZED QUEUEING SYSTEM

Let on the probability space $(\Omega,\mathcal{S},P)$ the queueing system $\langle E,1,GI/\infty,1\rangle$ with infinitely many servers be defined. It means that at the instants $t_n = \ell_1+...+\ell_n$, $n=1,2,...$, $t_0=0$, the customers arrive. The number of customers at each arrival instant is assumed to be one. Let the interarrival times $\{\ell_n\}_{n=1}^{\infty}$ be a sequence of positive independent identically distributed random variables with the distribution function

$$P(\ell_n \le t) = 1 - e^{-gt}, \quad t \ge 0, \quad n = 1,2,... . \tag{2}$$

Denote by D_n the service time of the n-th customer. We assume that $\{D_n\}_{n=1}^{\infty}$ is a sequence of independent identically distributed positive random variables which are independent also of the random variables $\{\ell_n\}_{n=1}^{\infty}$, and

$$F(t) = P(D_n \le t), \quad n = 1,2,... . \tag{3}$$

Those models of queues are described, in detail, for example, in ref. [5].

The queueing system generates a sequence of the idle periods, that is, the periods when none of the servers is busy (with the distribution function (2)), and a sequence of the busy periods, that is, the periods when at least one server is busy.

Our discretization of a system (1) may be presented as a putting of the lattice with the step h on the time axis, and setting a small time part sh, $0 \le s < 1/2$ (this restriction on s is not important), see Fig.1. Hence we obtain a sequence of cells of the same length h. In this process, the arrival instants of customers and the service times will be changed in such a way: a customer is regarded as an arriving in the k-th cell, when it arrived in the time interval $((k-1-s)h, (k-s)h\rangle$ and it has the discretized service time ih, when the service is finished in the time interval $((k+i-1+s)h, (k+i+s)h\rangle$. The physical motivation of such discretization is given in refs. [1,2].

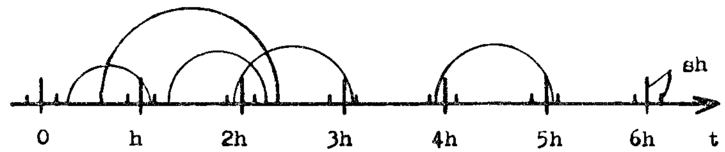

Fig.1

For the busy period probability determination we replace $(k-s)h$ by kh for any k, that is, the cells are shifted to the left-hand side. Hence, the service of a customer begins (finishes) in the k-th cell if it begins (finishes) in the time interval $((k-1)h, kh\rangle$ $(((k-1+2s)h, (k+2s)h\rangle)$ by the new designation, see Fig.2.

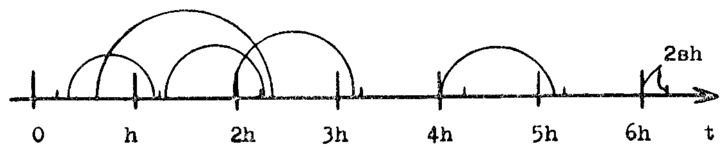

Fig.2

In this way the original queueing system (1) is transformed in the discretization queue

$$\langle GI, GI, GI/\infty, GI\rangle \tag{4}$$

with infinitely many servers where the interarrival times and the service times are the multipliers of h.

We see that there appears here a new real practically useful infinite-server queue different of those in ref. [4].

Proposition 1. The discretization system (4) is described by the following probabilities:

1. The probability that the interarrival times have the lengths kh for

a. the first customer

$$P_1^1(k) = \begin{cases} 1/gh\,(gh - 1 + e^{-gh}), & k = 0, \\ 1/gh\,(e^{gh} - 2 + e^{-gh}), & k \geq 1, \end{cases} \tag{5}$$

b. the n-th customer, $n \geq 2$,

$$P_n^1(k) = (e^{gh} - 2 + e^{-gh})\, e^{-kgh}/(\ell - e^{-gh}), \quad k \geq 1. \tag{6}$$

2. The probability of the number of customers in the series arriving into an arbitrary cell is

$$P^2(k) = e^{-gh}(gh)^k/((1 - e^{-gh})k!), \quad k \geq 1. \tag{7}$$

3. The probability of the service time of one customer is

$$P^3(k) = 1/h \int_{-sh}^{(1-s)h} (F(h(k+s) - t) - F(h(k-1+s) - t))\, dt, \tag{8}$$

$k \geq 1$.

The proof of the Proposition 1 follows from the results of ref. [2].

3. BUSY PERIOD OF DISCRETIZED QUEUEING SYSTEM

In the discretized queueing system (4) the sequences of the busy and idle periods, respectively, are formed. Our main aim is to determine the probability of the busy period b of this queue. We will assume that at the origine $t = 0$ the system (1) is idle.

The busy period b essentially depends on a random variable $q(t, x)$ – the number of the busy servers at the instant $t + x$ which was busy at the instant t. Denote by $R = 1 - F$.

Theorem 2.

$$P(q(t, x) = k) = \exp\left(-g \int_x^{t+x} R(u)\, du\right)\left(g \int_x^{t+x} R(u)\, du\right)^k / k! \tag{9}$$

$k = 0, 1, 2, \dots$.

Proof. Let X_t be the number of customers arriving in the time interval $(0, t\rangle$. Let $X = (x_1, \dots, x_n)$ be a vector from $D_n(t) = \{(x_1, \dots, x_n) : 0 < x_1 < \dots < x_n \leq t\}$, and (X, t, n) be an event assigning that in the time interval $(0, t\rangle$ n customers arrive at the instants $t_i\ (0 < t_1 < \dots < t_n \leq t)$ for which we have $x_i < t_i \leq x_i + dx_i$, $i = 1, \dots, n$. Therefore

$$P(q(t, x) = k) = \sum_{n=k}^{\infty} P(q(t, x) = k/(X, t, n), X_t = n) \times$$

$$\times P((X, t, n)/X_t = n)\, P(X_t = n) =$$

$$= \sum_{n=k}^{\infty} \sum_{C} \int \dots \int_{D_n(t)} P\Big(\bigcap_{i \in C} \{x_i + D_i > t + x\} \cap \bigcap_{i \notin C} \{x_i + D_i \leq$$

$$\leq t + x\}\Big) \cdot n!/t^n\, dx_1 \dots dx_n \cdot e^{-gt} g^n t^n / n!,$$

where C is an arbitrary permutation $(i_1, \dots, i_k)$ of k integers of $(1, \dots, n)$.

Hence

$$P(q(t, x) = k) = \sum_{n=k}^{\infty} e^{-gt} g^n \binom{n}{k} \left(\int_0^t R(t + x - u)\, du\right)^k \times$$

$$\times \left(\int_0^t (1 - R(t + x - u))\right)^{n-k} / n!\,.$$

Q.E.D.

Lemma 3.

$$P(q(t, x_1 + x_2) = 0,\ q(t, x_1) \geq 1) =$$

$$= e^{-gt}\left(\exp\left(g \int_{x_1 + x_2}^{t + x_1 + x_2} F(u)\, du\right) - \exp\left(g \int_{x_1}^{t + x_1} F(u)\, du\right)\right). \tag{10}$$

Proof. The property (10) follows from Theorem 2 and from the following simple relations:

$$P(q(t, x_1+x_2) = 0) = P(q(t, x_1+x_2) = 0,\ q(t, x_1) \geq 1) +$$

$$+ P(q(t, x_1+x_2) = 0,\ q(t, x_1) = 0) = P(q(t, x_1+x_2)=0,\ q(t, x_1) \geq 1) +$$

$$+ P(q(t, x_1) = 0\ .$$

Q.E.D.

Theorem 4. The probability $P_0(k), k=1,2,\dots,$ that the service of a series of customers arriving into the first cell will be finished in the k-th cell is

$$P_0(k) = e^{-gh}(e^{gF_k} - e^{gF_{k-1}})\ , \tag{11}$$

where

$$F_k = \int_{h(k-1+2s)}^{h(k+2s)} F(u)\, du\ . \tag{12}$$

The probability $P_0(k)$ for $k=0$ is defined as

$$P_0(0) = e^{-gh}\ . \tag{13}$$

Proof. This result may be obtained from the Lemma 3 when we put $t=h$, $x_1 = (k-1+2s)h$, $x_2 = h$. Hence

$$P_0(k) = P(q(t, x_1+x_2) = 0,\ q(t, x_1) \geq 1)\ .$$

The expression (13) follows from an observation that $P_0(0)$ is the probability of a nonarrival of customers into the first cell.
Q.E.D.

Now we will study the busy period b of the queueing system (4). Let us denote by A the event that into the first cell at least one customer arrives, and let B_n be the length of the busy period which begins in the first cell and finishes in the n-th cell, that is, $b = nh$. We denote the conditional probability in question, $P(B_n/A)$, by $P(n)$, and the joint distribution of those events, $P(A, B_n)$, by $PP(n)$. Clearly, $P(n) = PP(n)/(1-e^{-gh})$, $n = 1,2,\dots$.

The probability $PP(n)$ is obtained by the aid of the theorem of total probability, and for this aim we use the tables with two inputs. Here all possible cases of the arrivals of customer series into the concrete cells and the service lengths of this discrete queue are found. The indexes of cells are shown in the right-hand corners of each cell in the table heading. The integers $1-i\ (i=1,\dots,n)$ or $0-i\ (i=0,1,\dots,n)$ denote all possible service lengths of the series of customers arriving into the given cell or nonarrival (the sign 0). The sign $*k\ (k=1,\dots,n-1)$ denotes that in this cell none of the customer, which arrived into the previous cells is served, and from the series of customers arriving into this cell the busy period of the length kh is created.

Let $W(n,k)$, $n=1,2,\dots$, $k=1,\dots,n$, be the probability that from the series of customers arriving into the first cell and finishing in the k-th cell, the busy period of the length nh be created. Then

$$PP(n) = \sum_{k=1}^{n} W(n, k),\ n = 1,2,\dots\ . \tag{14}$$

In the tables below the probabilities $PP(n)$ for $n = 1,2,n$ with recurrent relationships between them are shown in the general form. Due to the independence of the table columns we have the following properties

Table 1

	1 1	0 2
W(1,1)	1	0

Table 2

	1-2 1	0-1 2	0 3
W(2,1)	1	*1	
W(2,2)	2	0-1	0

$$\left.\begin{aligned} W(1,1) &= P_0(1)\, P_0(0)\ , \\ PP(1) &= W(1,1)\ , \end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned} W(2,1) &= P_0(1)\, PP(1)\ , \\ W(2,2) &= P_0(2)\,(P_0(1) + P_0(2))\, P_0(0)\ , \\ PP(2) &= W(2,1) + W(2,2)\ . \end{aligned}\right\} \tag{16}$$

Let us put

$$S(k) = \sum_{i=0}^{k} P_0(i),\ k = 0,1,2,\dots\ , \tag{17}$$

$$SS(k) = \prod_{i=0}^{k} S(i),\ k = 1,2,\dots\ . \tag{18}$$

Table 3

	1-n 1	0-(n-1) 2	0-(n-2) 3	...	0-1 n	0 n+1
W(n,1)	1	*(n-1)				
W(n,2)	2	0-1	*(n-2)			
		W(n-1,2) ⋮ W(n-1,n-1)				
W(n,3)	3	0-2	0-1	*(n-3)		
			W(n-2,2) ⋮ W(n-2,n-2)			
		W(n-1,3) ⋮ W(n-1,n-1)		...		
⋮						
W(n,n)	n	0-(n-1)	0-(n-2)	...	0-1	0

For the general case $n \geq 3$ we define

$$W(n,1) = P_0(1)\,PP(n-1)\,. \tag{19}$$

As follows from tables 1-3, the table for $n \geq 2$ is composed of the subblocks of the tables for $k = n-1, n-2, \dots, 2, 1$.

Therefore by the recurrent formulae for

$2 \leq k \leq n-1$

we introduce the following helpfull notations

$$B(n,k-1,1) = S(1)\,PP(n-k) + \sum_{i=2}^{n-k+1} W(n-k+1,i)\,, \tag{20}$$

and for

$2 \leq j \leq k-1$

$$B(n,k-1,j) = S(j)\,B(n,k-1,j-1) + \sum_{i=j+1}^{n-k+j} W(n-k+j,i)\,. \tag{21}$$

Finally, we have

$$\left.\begin{aligned} W(n,k) &= P_0(k)\,B(n,k-1,k-1)\,,\\ W(n,n) &= P_0(n)\,SS(n-1)\,, \end{aligned}\right\} \tag{22}$$

$$P(n) = PP(n)/(1-e^{-gh})\,, \tag{23}$$

and we have proved the following theorem:

Theorem 5. The probability of the busy period of the queueing system (4) with infinitely many servers is given by the formulae (14)-(23).

4. STABILITY OF BUSY AND IDLE PERIODS OF THE QUEUEING SYSTEM

The authors of the present paper regret that they do not know the busy period distribution of a queue $\langle E,1,GI/\infty,1\rangle$, yet, in this part we prove that for any $h_k \searrow 0$, $0 \leq s_k < 1/2$ the corresponding discretized busy period distributions converge weakly to the one mentioned before. That is, this characteristic is stable.

Theorem 6. Let $\{h_k\}_{k=0}^{\infty} \searrow 0$, $0 \leq s_k < 1/2$, $k = 0,1,2,\dots$, $F_k^{s_k}(G_k^{s_k})$ be the corresponding probability distributions of the busy (idle) period of the queueing system (1).

Then

$$F_k^{s_k} \Rightarrow F\,,$$

$$G_k^{s_k} \Rightarrow G \quad (G(t) = 1 - e^{-gt}\,,\ t \geq 0)\,.$$

Proof. Let $\{B_n\}_{n=1}^{\infty}$, $\{L_n\}_{n=1}^{\infty}$ be sequence of the busy and idle periods, respectively, of the queueing system (1). As random variables they are mutually independent and their probability laws do not depend on n. After the discretizations with the steps h_k and with the time parts $h_k s_k$, $0 \leq s_k < 1/2$, $k = 0,1,\dots$, from the above sequences the sequences of the discretized busy periods $\{b_k^{s_k}\}_{k=0}^{\infty}$ and the idle periods $\{\ell_k^{s_k}\}_{k=0}^{\infty}$ of the systems (4) are generated.

The proof of the Theorem is devided into three steps.

1. Firstly we assume that $h_k = h/2^k$, $h > 0$, $s_k = 0$, $k = 0,1,2,\dots$.

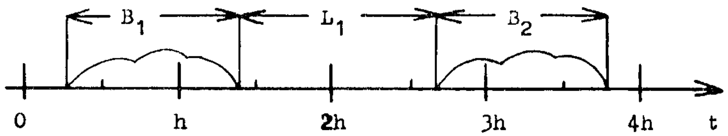

Fig.3

From Fig.3 we have $b_k^o(\omega) \searrow B_1(\omega)$ and $\ell_k^o(\omega) \nearrow L_1(\omega)$, $\omega\in\Omega$, (by $\tilde{b}_k^{s_k}$, $\tilde{\ell}_k^{s_k}$ we denote the discretized variables for $h_k = h/2^k$).
Therefore

$$F_k^o(t) = P(\tilde{b}_k^o \le t) \nearrow P(B_1 \le t) = F(t) .$$

2. Now let $0 < s < 1/2$, $s_k = s$, $h_k = h/2^k$, $k = 0,1,\dots$. Let $\tilde{b}_k^o(\omega) = m_k h/2^k$, where m_k be the suitable integer. Then $b_k^s(\omega) = m_k h/2^k$ if the $B_1(\omega)$ finishes in the time interval $((m_k - 1 + 2s)h/2^k, m_k h/2^k>$, and $b_k^s(\omega) = (m_k - 1)h/2^k$ if the $B_1(\omega)$ finishes in the time interval $((m_k - 1)h/2^k, (m_k - 1 + 2s)h/2^k>$.

Therefore $b_k^s(\omega) \le \tilde{b}_k^o(\omega)$, $\omega\in\Omega$, and $b_k^s \to B$ with the probability one. Let us put $\eta_k = b_k^s - \tilde{b}_k^o$, then due to the Egorov theorem ref. /6/ we have $\eta_k \xrightarrow{P} 0$ and

$$\lim_k F_k^s(t) = \lim_k P(b_k^s \le t) = \lim_k P(\tilde{b}_k^o + \eta_k \le t) =$$

$$= \lim_k P(\tilde{b}_k^o \le t) = F(t) .$$

3. The general case may be obtained by the similar way.
Analogically we may prove the weak convergence of the $G_k^{s_k}$, or we may use the following Note:

Note. In paper /2/ it is proved that the probability of the discretized idle period of the queueing system (4) for the step h and the time part sh is equal to

$$P_\ell(k) = (1 - e^{-gh})e^{-kgh}, \quad k = 1,2,\dots .$$

Q.E.D.

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Об одной проблеме определения периода занятости в системах обслуживания с бесконечным числом каналов обслуживания

В работе рассматривается задача определения вероятности дискретизованного сгустка по данным сканирования следов в трековых камерах. Эта проблема решается в рамках теории массового обслуживания с бесконечным числом каналов как определение вероятности дискретного периода занятости этой системы. Получены точные вероятности и доказывается, что они в пределе слабо сходятся к распределению недискретизованного сгустка при уменьшении шага дискретизации.

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On One Problem of the Busy Period Determination in Queues with Infinitely Many Servers

In the paper the problem of the discretized blob length probability determination based on the scanning in the track chambers is considered. This problem is solved in the frame of the queueing system with infinitely many servers as a discretized busy period probability determination of this system. The precise formulae of a probability are given and it is proved that those probabilities converge weakly to the probability distribution of the nondiscretized blob when the discretization steps are diminished.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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