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STATISTICAL MODELING IN PHENOMENOLOGICAL DESCRIPTION OF ELECTROMAGNETIC CASCADE PROCESSES PRODUCED BY HIGH-ENERGY GAMMA QUANTA

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1. INTRODUCTION

In high energy physics and its applications a problem arises often to build a simple model describing reliably enough general features of a given physical process when an analytical approach to this problem fails completely on the practical level. Such a model is required, for example, to predict a behavior of some characteristics of this process within some intervals of values of parameters depending on external conditions which can be changed. If a complex process under consideration may be expressed as a superposition or a sequence of some numbers of more elementary processes having exact analytical description or being determined by another way, then the former process can be reconstructed using computer techniques. But in this case one can get concrete numerical results only (numbers, distributions) at some fixed conditions and to obtain another results of this kind at another conditions it is necessary to perform such calculations all over again, etc. This procedure being trivial in principle often needs much time of big computers, as well as the confrontation of final results with experimental data is sometimes expedient, especially when the process is complicated (i.e., intranuclear cascade process). On the other hand, experimental investigations of the processes under consideration are not possible within sufficiently wide intervals of external parameter values.

As an example let us discuss an electromagnetic cascade process (ECP or shower) produced by high-energy gamma quanta in heavy media. This process is one of the oldest being studied in high energy and cosmic ray physics. It consists of three leading elementary phenomena: radiation and pair production, which make the shower possible, and ionization as a stopping factor. The simplest one-dimensional approach to this statistical problem first put forward 50 years ago by Bhabha and Heitler nowadays has no practical applications because of its limitations. On the other hand, the three-dimentional cascade theory as based on the equations (Ramakrishnan(1962)):

$$\frac{\partial P}{\partial t} = - A\cdot P + B\cdot \Gamma + \frac{E_s}{4E^2}\nabla_\theta^2 + \beta - \theta_x \frac{\partial P}{\partial x} - \theta_y \frac{\partial P}{\partial y}, \quad (1')$$

$$\frac{\partial \Gamma}{\partial t} = C\cdot P - D\cdot \Gamma + \theta_x \frac{\partial \Gamma}{\partial x} - \theta_y \frac{\partial \Gamma}{\partial y}, \quad (1'')$$

even in the small angle θ limitation when the process can be treated as markovien does not exhaust the problem providing information about average ECP characteristics only without fluctuations and correlations which are very important in practice. Moreover, the resolution of these equations is a matter of great difficulties. Here P and Γ are the numbers of electrons and photons with the (E, E+dE) energy at the distance (ρ, ρ +dρ) from a shower axis (SA), correspondingly (x and y are the coordinates of ρ); A, B, C, D are some quantities taking into account probabilities of partial elementary processes; the shower depth t is measured along the SA, θ_x and θ_y are the components of the shower particle angle θ in the (x, y) plane.

So, although all elementary processes making up the ECP are strictly known long ago the user can't be satisfied at present to have a sufficiently convenient tool in his practice (for example, to design high-energy particle detectors). Therefore attempts to obtain analytical approximations of the behavior of some general characteristics of the ECP using both experimantal and computer simulated data are undertaken, Longo(1975), Słowiński(1981). But up to last years these attempts give mainly a fragmentary picture of the ECP only for lack of complete information about ECP (experiment) and because the problem is complicated enough.

2. SPATIAL DISTRIBUTION OF AVERAGE IONIZATION LOSSES

In ECP modeling it is more interesting than P and Γ in (1′) and (1″) the dependence on primary gamma quanta energy E_γ of average electron and positron (later: electron) ionization losses (AIL), $\Delta E(E_\gamma, E_o, t, \rho)/\Delta V$, released inside a ring having ρ as the radius and $\Delta\rho$

as its thickness in the (x, y) plane at the shower depth t, and with the thickness Δt along the SA, so that the volume is equal to $\Delta V = 2\pi\rho\Delta\rho\Delta t$. Here media isotropy is supposed to take place, and the predetermined cut-off energy E_o of electrons being observed may be put to be equal to zero. To build a simple ECP model one must find out such function $F(t,\rho|E_\gamma)$ which satisfies the equation

$$\frac{\Delta E(E_\gamma, t, \rho)}{2\pi\rho\Delta\rho\Delta t} = F(t,\rho|E_\gamma), \quad (2)$$

where notation $F(t,\rho|E_\gamma)$ means that spatial AIL distribution is regarded as a function of primary gamma quanta energy E_γ (the parameter E_o is omitted taking into account that $E_o = 0$). This may be done in principle by three ways: 1/ as the solution of an equation similar to (1′) and (1″); 2/ by means of the computer ECP simulation and, as a next step, a statistical approximation of obtained numerical data; 3/ the same way as in 2/ but with experimental data as former information. The first approach involves the difficulties mentioned above. The second one is universal in principle and has no shortcomings pointed out, but as has been already remarked verification of computer simulation results with experimental data is needed. Therefore in our works, Słowiński(1981), empirical information as a basis of statistical modeling of (2) has been used. Unfortunately in such general form the function $F(t,\rho|E_\gamma)$ is difficult to be determined from a sample of experimental data which, for example, can consist of a set of random numbers $Z_l^{(i,j)}(E_\gamma) = (\Delta E(E_\gamma, t_i, \rho_j)/\Delta V_{ij})_l$ obtained each taken separately at fixed values of coordinates t_i and ρ_j, and at the fixed value of energy E_γ for each l-th ECP event registered by the detector. Here $l=1,\ldots,N$, N being the sample size, i and j are the numbers of the lattice cell within which the measurement of $Z_l^{(i,j)}(E_\gamma)$ has been made. Therefore let us write the function $F(t,\rho|E_\gamma)$ as a product of the function $F_t(t|E_\gamma)$ describing the longitudinal AIL distribution and the conditional distribution $F_\rho(\rho|t;E_\gamma)$ for lateral AIL spreading:

$$F(t,\rho|E_\gamma) = F_t(t|E_\gamma)\cdot F_\rho(\rho|t;E_\gamma). \quad (3)$$

Now the one-dimensional functions F_t and F_ρ can be easily fitted to the data using standard methods.

2.1. Longitudinal disribution

We have approximated the longitudinal AIL distribution $F_t(t|E_\gamma)$ by the Weibull function:

$$F_t(t|E_\gamma) = a_o \frac{a_1}{a_2}\left(\frac{t}{a_2}\right)^{a_2-1} \exp/-\left(\frac{t}{a_2}\right)^{a_1}/, \qquad (4)$$

where a_o is the normalization coefficient, i.e.

$$\int_0^\infty F_t(t|E_\gamma)\cdot dt = E_\gamma, \qquad (5)$$

a_1 and a_2 are the parameters depending on E_γ which in one's turn are to be estimated from sample.

There are certain reasons of the general matter to choose the Weibull function as the F_t distribution. Firstly, according to some model considerations based on the Central Limit Theorem one can expect that at high enough values of E_γ when a number of shower electrons is sufficiently large the function F_t must tend to the normal distribution at high t values. Secondly, from the practical point of view it is desirable to have such approximations for F_t which depends on minimal number of parameters only and their dependence on energy E_γ may be traced reliably enough within sufficiently large range of E_γ values to make possible to predict the AIL behavior at such energy E_γ when both measurements and calculations are too difficult to be realised or even problematic. Thirdly, for some practical purposes it is convenient to have not only a simple form of the function F_t but also of its moments and even of the integral

$$E(t) = \int_0^t F_t(t'|E_\gamma)\cdot dt'. \qquad (6)$$

In particular, the majority of shower detectors can register just the E(t) value which in the case of the Weibull function has very simple form:

$$E(t) = E_\gamma\left\{1 - \exp/-\left(\frac{t}{a_2}\right)^{a_1}/\right\}. \qquad (7)$$

In Fig. 1 the function (7) is compared with experimantal data. Then, using the values $E_m(t)$ measured inside the t_m thickness of a shower absorbend, it is easly to estimate by the iteration method the value E_γ of a primary gamma quantum producing shower. Simple approximations of E_γ dependence of the parameters a_1 and a_2 has been obtained too:

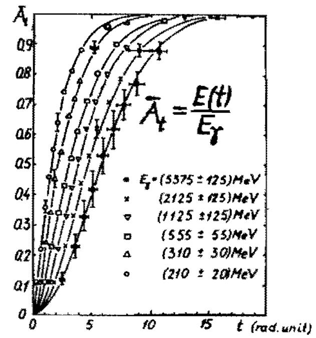

Figure 1

$$a_1 = \begin{cases} \alpha_1 + \beta_1\cdot \ln E_\gamma, & \text{if } 200 \le E_\gamma \le 3500 \text{ MeV}, \\ 2, & \text{if } E_\gamma \ge 3500 \text{ MeV}, \end{cases} \qquad (8')$$

$$a_2 = \alpha_2 + \beta_2\cdot \ln E_\gamma, \text{ if } E_\gamma \ge 200 \text{ MeV}, \qquad (8'')$$

where α_1, α_2, β_1 and β_2 are the numbers estimated for sample of experimental data (Table 1). Because at values greater than some hundreds of MeV the leading partial elementary processes of ECP reach their asymptotical regime and not any other significant process includes at all one can expect that longitudinal AIL distribution in ECP of the form (4) and having the parameters given by the fits (8′) and (8″) is trustworthy for practice with a good enough accuracy when $E_\gamma \ge 200$ MeV in heavy media.

2.2. Lateral distribution

More complicated is a problem of estimation of lateral ECP distribution, i.e. the function $F_\rho(\rho|t;E_\gamma)$ in (3). The thing is that in reality experimentally observed values are rather not the numbers $Z_1^{(i,j)}(E_\gamma)$ at fixed i but the plane distribution of ionization losses (IL) $f_r(r|t;E_\gamma)$, where r is the projection of ρ on the SA plane (being at the same time the projection plane). The functions F_ρ and f_r are connected by means of the following integral equation

$$f_r(r|t;E_\gamma) = 2\int_r^\infty F_\rho(\rho|t;E_\gamma)\frac{d\rho}{\sqrt{1-(r/\rho)^2}} \qquad (9)$$

when the xenon bubble chamber is used as a shower detector. The solution of this equation has the form:

$$F_{\varrho}(\varrho|t;E_{\gamma}) = \frac{1}{\pi\varrho^2}\int_{\infty}^{\varrho}\frac{d}{dr}(r\cdot f_r(r|t;E_{\gamma}))\frac{dr}{\sqrt{1-(\varrho/r)^2}}. \quad (10)$$

The lateral AIL distribution $f_r(r|t;E_{\gamma})$ has been approximated by the exponential function normalized to 1:

$$f_r(r|t;E_{\gamma}) = \frac{1}{\bar{r}(t)}\exp(-\frac{r}{\bar{r}(t)}), \quad (11)$$

where the parameter $\bar{r}(t)$ is found to be proportional to the shower depth t:

$$\bar{r}(t) = \alpha_3 + \beta_3\cdot t \quad (12)$$

within all range of E_{γ} values investigated, i.e., E_{γ} = 200 - 3500 MeV and independent on E_{γ}. Here α_3 and β_3 are the constants presented in the Table.

3. FLUCTUATIONS

It is expedient to determine this important characteristics of ECP as a fractional variation of some part A of IL released within a layer of absorbent inside which in average the part $\bar{A}$ of IL is registered. So, for longitudinal shower development we have $\sigma_{A_t(t)}/\bar{A}_t(t)$ and

$$\bar{A}_t(t) = E(t)/E_{\gamma}, \quad (13)$$

and for lateral ECP spreading - $\sigma_{A_r(r)}/\bar{A}_r(r)$, where $\bar{A}_r(r)$ may be determined from (11) taking into account that in the SA plane AIL have the form

$$f(t,r|E_{\gamma}) = F_t(t|E_{\gamma})\cdot f_r(r|t;E_{\gamma}), \quad (14)$$

whence

$$\bar{A}_r(r) = 2\left\{1 - \int_0^{\infty}F_t(t|E_{\gamma})\cdot\frac{\exp(-r/\bar{r}(t))}{\bar{r}(t)}dt\right\}. \quad (15)$$

It has been ascertained that A_t is a normally distributed random variable with mean $\bar{A}_t(t)$ and standard deviation $\sigma^2_{A_t}$, i.e.,

$$A_t(t) \sim N(\bar{A}_t(t);\ \sigma^2_{A_t}) \quad (16)$$

at $\bar{A}_t(t) \gtrsim 0.5$. The dependence of the ratio $\sigma_{A_t(t)}/\bar{A}_t(t)$ on $\bar{A}_t(t)$ is described by the relation

$$\sigma_{A_t(t)}/\bar{A}_t(t) = \left\{\frac{1}{b}\ln\frac{a}{\bar{A}_t(t)}\right\}^{1/2}, \quad (17)$$

which is valid within the interval $\bar{A}_t(t)$ = 0.1-0.95 at confidence level 0.3. Here

$$a = 1.02 - 1.3\cdot 10^{-5}E_{\gamma} \quad (18)$$

and b = 13.6 ± 1.3.

The behavior of the fractional variation coefficient for lateral ECP spreading $\sigma_{A_r(r)}/\bar{A}_r(r)$ is more complicated than in the case of longitudinal ECP development. This is displayed in Figure 2. But values of $A_r(r)$ are also normally distributed like $A_t(t)$.

It is of great practical interest to know how the spatial shape of ECP evolves in a general way when energy E_{γ} of primary gamma quanta increases. Relative information is contained in a dependence on E_{γ} of the following quantities: σ_r, σ_t, $\sigma_r/\bar{r}_o$ and $\sigma_t/\bar{t}$, where σ denotes a root mean square deviation: $\bar{r}_o$ and $\bar{t}$ being corresponding average values which can be estimated from (14), (4) and (11). It turned out to fit these dependences by the simple formulae of the same form:

$$\sigma_r = \alpha_4 + \beta_4\cdot\ln E_{\gamma}, \quad (19)$$

$$\sigma_t = \alpha_5 + \beta_5\cdot\ln E_{\gamma}, \quad (20)$$

$$\sigma_r/\bar{r}_o = \alpha_6 + \beta_6\cdot\ln E_{\gamma}, \quad (21)$$

$$\sigma_t/\bar{t} = \alpha_7 + \beta_7\cdot\ln E_{\gamma}. \quad (22)$$

Numerical values of the parameters α_i and β_i (i=4,...,7) are shown in the Table with corresponding chi-square values.

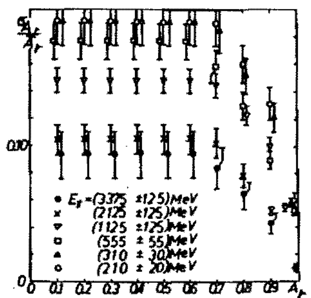

Figure 2

Table

Numerical values of the constants α_i and β_i of corresponding formulae each having number (N) and related chi-square values with n degrees of freedom (r.u. - radiation unit)

Formula number	i	α_i	β_i	χ^2_n/n
(8′)	1	-0.07 ± 0.09	0.24 ± 0.01	1.7/4
(8″)	2	(-7.0 ± 0.7) r.u.	1.7 ± 0.1	6/4
(12)	3	$(4.6 \pm 0.4)10^{-2}$ r.u.	4.5 ± 0.1	17.5/22
(19)	4	(-0.17 ± 0.24) r.u.	0.11 ± 0.04	2.3/4
(20)	5	(-6.2 ± 1.6) r.u.	1.65 ± 0.24	1.5/4
(21)	6	0.53 ± 0.48	0.07 ± 0.07	0.1/4
(22)	7	1.53 ± 0.73	-0.05 ± 0.11	0.02/4

4. CONCLUSION

We have constructed a simple phenomenological model of electromagnetic cascade process initiated by high-energy gamma quanta in heavy media. This model is based on experimental data obtained using pictures from xenon bubble chamber of ITEP (Moscow). Our model is sufficiently compact and convenient from the practical point of view, in particular, for determination of the spatial (i.e., longitudinal and lateral) dimensions of electromagnetic showers as well as fluctuations of these characteristics within the large enough energy interval E_γ = 200-3500 MeV. The parameters of the model reach their asymptotic regime as early as at $E_\gamma \lesssim$ 3500 MeV and no more significant elementary process making up an ECP includes at higher energies. So, one can hope that our model is trustworthy at least qualitatively far beyond investigated region of E_γ, too.

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Статистическое моделирование
в феноменологическом описании электромагнитных
каскадных процессов, вызываемых высокоэнергетическими
гамма-квантами

Работа содержит описание простой феноменологической модели электромагнитных каскадных процессов, вызываемых гамма-квантами больших энергий в плотных поглотителях. В рамках этой модели описана пространственная структура и флуктуации ионизационных потерь ливневых электронов и позитронов. Конкретные формулы получены в результате статистического анализа экспериментальных данных с ксеноновой пузырьковой камеры ИТЭФ /Москва/.

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Statistical Modeling in Phenomenological
Description of Electromagnetic Cascade
Processes Produced by High-Energy Gamma Quanta

The work contains a description of a simple phenomenological model of electromagnetic cascade process (ECP) initiated by high-energy gamma quanta in heavy absorbents. Whithin this model spatial structure and fluctuations of ionization losses are described. Concrete formulae have been obtained as a result of statistical analysis of experimental data from the xenon bubble chamber of ITEP (Moscow).

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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