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THE DESIGN OF SPECIFYNG AND DISCRIMINATING NN - SCATTERING EXPERIMENTS

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THE DESIGN OF SPECIFYNG AND DISCRIMINATING NN - SCATTERING EXPERIMENTS

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The phase-shift analysis is one of the most convenient methods of reconstructing the NN - scattering matrix from pp- and np -scattering experimental data. A single set of phase shifts is now determined for the range of energies from 23 to 310 MeV.

Outside this range the amount of obtained experimental data is not sufficient to determine a single set of phase shifts.

Thus it is necessary to use both regression /1/ and discriminating experiments.

An example of the necessity of the first type of experiments is the need of a precise estimation of the np -scattering mixing parameter ϵ_1 which allowed to define the role of tensor forces in NN -interaction /2/. Discriminating experiments are needed for the elimination of ambiguities in phase shifts which occur within a broad region of energies.

The purpose of this paper is to develop further the planning methods (see, e.g. /1,3,4,6/) of experiments of the above types. The two below discussed methods essentially rest on the formula (5).

1. The Design of Experiments for the Specification of Parameters

Let $\eta(x)$ be the measured quantity:

$$\eta(x) = \sum_{\alpha=1}^{m} \theta_\alpha f_\alpha(x) = \vec{\theta}'\vec{f}(x), \qquad (1)$$

where $\vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$ are given functions,

$\vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix}$ - are unknown **parameters**, the prime denotes the transposition of a vector.

If $\eta(x,\theta)$ is a nonlinear function of the parameters, it is necessary to use a linear approximation (/1/, chapter II, 9):

$$\eta(x,\vec{\theta}) = \eta(x,\vec{\theta}^{(0)}) + \sum_{\alpha=1}^{m} \frac{\partial\eta(x,\theta)}{\partial\theta_\alpha}\Big|_{\vec{\theta}=\hat{\vec{\theta}}^{(0)}} (\theta_\alpha - \hat{\theta}_\alpha^{(0)}) \qquad (2)$$

$$f_\alpha(x) = \frac{\partial\eta(x,\theta)}{\partial\theta_\alpha}\Big|_{\vec{\theta}=\hat{\vec{\theta}}^{(0)}}, \qquad (3)$$

where $\hat{\vec{\theta}}^{(0)}$ is the estimate of the parameters at the time $t = 0$.

Let the dispersion matrix (the error matrix) of the parameters $\vec{\theta}$ at the beginning of the planned experiment have the value $D(0)$. After making the measurement during the time T the dispersion matrix will take on the value $D(T,x)$. We shall consider the measurement made at the point x_1 more efficient than the analogous one at the point x_2 if

$$|D^{kk}(T,x_1)| < |D^{kk}(T,x_2)|, \qquad (4)$$

where $D^{kk}(T,x)$ is the submatrix of the matrix $D(T,x)$ which corresponds to those parameters $\theta_1, \theta_2, \ldots, \theta_k$ which are of interest to the experimenter ($k \leq m$).

The design of an additional experiment which specifies the parameters $\theta_1, \theta_2, \ldots, \theta_k$ consists in finding such a point x_0 for which $(D^{kk}(T,x))$ takes the minimal value. It should be noted that the minimum $|D^{kk}|$ corresponds to the minimal value of the dispersion ellipsoid (see[5], chapter 22) in the space of the parameters $\theta_1, \theta_2, \ldots, \theta_k$.

It may be shown (see Appendix I) that after the measurement made at the point x during the time T the dispersion matrix takes the value:

$$D(T,x) = \left(I_m - \frac{\lambda(x) T D(0) \vec{f}(x) \vec{f}'(x)}{1 + \lambda(x) T \vec{f}'(x) D(0) f(x)}\right) D(0), \tag{5}$$

where I_m is a $m \times m$ matrix, $\lambda(x)$ is the efficiency[1] of the experiment measuring $\eta(x,\theta)$ The optimal position x_0 of the measurement instrument is defined by the equation:

$$|D^{kk}(T,x_0)| = \min_x |D^{kk}(T,x)|, \tag{6}$$

where $D^{kk}(T,x)$ is the submatrix of the dispersion matrix $D(T,x)$ which corresponds to (5). The analytical expression for $(D^{kk}(T,x))$ for an arbitrary k is given in the appendix I. Simple and obvious results are obtained for $k=m$ and $k=1$. In the first case

$$|D(T,x)| = \frac{|D(0)|}{1 + \sigma^2(x)\lambda(x) T}, \tag{7}$$

where $\sigma(x)$ is the corridor of the errors of the curve $\eta(x, \vec{\theta}^{(0)})$. It follows from (7) that if it is necessary to specify the whole set of the parameters $\vec{\theta}$ the measuring instrument should be placed in the point x in which $\sigma^2(x)\lambda(x)$ is maximal.

If it is necessary to specify a single parameter $\theta_\alpha (k=1)$ the position of the measuring instrument will be defined by the function

$$D_{\alpha\alpha}(x,T) = D_{\alpha\alpha}(0) - \frac{\lambda(x)\,T\left(\sum_{\beta=1}^{m} D_{\alpha\beta}(0)\,f_{\beta}(x)\right)^2}{1+\sigma^2(x)\,\lambda(x)\,T} \qquad (8)$$

It follows from (8), that it is not possible, generally speaking, to specify the parameter θ_{α} to an arbitrary accuracy.

Actually,

$$\lim_{T\to\infty} D_{\alpha\alpha}(x,T) = D_{\alpha\alpha}(0) - \frac{\left(\sum_{\beta=1}^{m} D_{\alpha\beta}\,f_{\beta}(x)\right)^2}{\sigma^2(x)} \; . \qquad (9)$$

Using (5) it is possible to solve the problem that is in some sense contrary to (6): to find the point x_0 for which the time T required for a N-fold decrease of $|D^{kk}|$ should be minimal.

In the case $k=m$ and $k=1$ for $T(x)$ the following expressions are obtained:

$$T(x) = \frac{N-1}{\lambda(x)\,\sigma^2(x)} \;, \; k = m \;, \qquad (10)$$

$$T(x) = \frac{(N-1)\,D_{\alpha\alpha}(0)\,\lambda^{-1}(x)}{N\left[\sum_{\beta=1}^{m} D_{\alpha\beta}(0)\,f_{\beta}(x)\right]^2 - \sigma^2(x)\,D_{\alpha\alpha}(0)\,(N-1)} \; . \qquad (11)$$

For an arbitrary set of parameters $\theta_1, \theta_2, \ldots, \theta_k$ $T(x)$ can be obtained directly from (1) in appendix 1.

The Design of Discriminating Experiments

Let $\hat{\theta}_1^{(0)}$ and $\hat{\vec{\theta}}_2^{(0)}$ be two least square estimates of the parameters $\vec{\theta}$ which are obtained from the analysis of experiments in the points

$x_1, \dots, x_n$ with the weights $w_1, \dots, w_n$. Let the statistics we have be insufficient to give preference to either of the estimates. That means that the sums:

$$S_1(0) = \sum_{i=1}^{n} w_i [y_i - \eta(\vec{\theta}_1^{(0)}, x_i)]^2 \qquad (12)$$

and

$$S_2(0) = \sum_{i=1}^{n} w_i [y_i - \eta(\vec{\theta}_2^{(0)}, x_i)]^2$$

do not differ essentially. Here y_i is a result of the measurement at x_i.

The purpose of the planning is to find such a point x_0 in which a measurement will give the maximum difference increment $S_j(T,x) - S_k(T,x)$ supposing that the k-th hypothesis is true. (See also /7/). (The hypothesis H_k means that the true vector lies near $\vec{\theta}_k^{(0)}$ or, more exactly, that $\vec{\theta}_k^{(0)}$ is an unbiased estimate of $\vec{\theta}$). Here:

$$S(T,x) = \sum_{i=1}^{n} w_i [y_i - \eta(\hat{\vec{\theta}}_j, x)]^2 + \lambda(x) T [y - \eta(\hat{\vec{\theta}}_j, x)]^2, \qquad (13)$$

where y is the result of the additional measurement made at x during the time T, $\hat{\vec{\theta}}_j$ ($j = 1,2$) are estimates including the additional measurement. The value of $S_j(T,x)$ depends on the result of the measurement at x. Apriori the value of y is unknown, so that we cannot determine the exact value of $S_j(T,x)$ before the experiment. However we can give its mean value. Actually, assuming that $\hat{\vec{\theta}}_j^{(0)}$ is unbiased (see (1), (2) and appendix 2 in /1/) the random variable

$$u_j = \frac{y - \eta(\hat{\vec{\theta}}_j^{(0)}, x)}{s_j(x)}, \qquad (14)$$

where $s_j(x) = \sqrt{(\lambda(x)T)^{-1} + \sigma_j^2(x)}$ has a normal distribution with parameters 0,1:

$$\phi(u)\,du = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u_j^2}\, du\ . \tag{15}$$

Let the hypothesis H_1 be true. From (5), (13) and (15) follows that the mean value of $S_2(T,x) - S_1(T,x)$ with respect to u_1 is equal (see appendix 2) to:

$$E_1[S_2(T,x) - S_1(T,x)] = S_2(0) - S_1(0) + \frac{[\eta(\hat{\theta}_2^{(0)},x) - \eta(\hat{\theta}_1^{(0)},x)]^2 + [\sigma_2^2(x) - \sigma_1^2(x)]}{s_2^2(x)}$$

Evidently, the optimal experimental point x_0 is the one where $E_1[S_2(T,x) - S_1(T,x)]$ attains its maximum. The disign of a discriminating experiment consists in a search for the

$$\max_x \frac{[\eta(\hat{\theta}_2^{(0)},x) - \eta(\hat{\theta}_1^{(0)},x)] + \sigma_2^2(x) - \sigma_1^2(x)}{s_2^2(x)} \tag{17}$$

If H_2 is true, the measurement must be made at the point which is given by:

$$\max_x \frac{[\eta(\hat{\theta}_1^{(0)},x) - \eta(\hat{\theta}_2^{(0)},x)]^2 + \sigma_1^2(x) - \sigma_2^2(x)}{s_1^2(x)} \tag{18}$$

If the points obtained by (17) and (18) don't coincide, then the measurement must be made in the point obtained from:

$$\max_x \{ W_1 E_1[S_2(T,x) - S_1(T,x)] + W_2 E_2[S_1(T,x) - S_2(T,x)] \}, \tag{19}$$

where the weights W_1 and W_2 generally speaking, depend on the aim of the experiment. If the loss occuring when the false hypothesis is accepted is equal to the loss occuring when the true hypothesis is rejected, the weights are defined as:

$$W_1 \approx e^{-\frac{1}{2} S_1(0)}, \quad W_2 \approx e^{-\frac{1}{2} S_2(0)} \tag{20}$$

In some cases the aim of the experiments cannot be expressed in a form which allows to find the ratio of the loss due to the false acceptation of the first hypothesis to the loss due to the false acceptation of the second one. In such a case the measurement must be made at the point obtained from:

$$\max_x \min_{\substack{j=1,2 \\ k \neq j}} E_j [S_k (T,x) - S_j (T,x)] . \tag{21}$$

Let us compare the above method of designing discriminating experiments with the method proposed in [4]. The main difference of these methods is due to the original formulation of the problem of planning.

In the present paper the criterion for optimality is the requirement for the maximum increment of the difference $S_j(T,x) - S_k(T,x)$ supposing that the k-th hypothesis is correct (T is fixed). Thus two estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ obtained following a certain rule are compared.

In [4] two complex hypotheses ([5], chapter 35) with the parameters $\eta(\hat{\theta}_1, x), \sigma_1(x)$ and $\eta(\hat{\theta}_2, x), \sigma_2(x)$ are considered. The experiment (for fixed $\bar{T}$) as a result of which the probability of taking the false hypothesis will be minimal is optimal. In principle in [4] two possibilities are compared: either the true vector $\vec{\theta}$ belongs to the set with the parameter $\vec{\theta}_{12}^{(0)}, D_1$ or to that with the parameter $\hat{\theta}_2^{(0)}, D_2$. (The values $\eta(\hat{\theta}_j, x), \sigma_j(x)$ and $\hat{\theta}_j, D_j$ are supposed to be one-to-one related, see (1) and (1.9)).

It may be shown that if $\sigma_1(x) = \sigma_2(x)$ the position of the optimal measuring point x_0 is the same for both methods and follows from the equation:

$$\frac{[\eta(\hat{\theta}_2^{(0)}, x_0) - \eta(\hat{\theta}_1^{(0)}, x_0)]^2}{s^2(x_0)} = \max_x \frac{[\eta(\hat{\theta}_2^{(0)}, x) - \eta(\hat{\theta}_1^{(0)}, x)]^2}{s^2(x)} \tag{22}$$

If $\sigma_1(x) \neq \sigma_2(x)$ then the position of the optimal points differs the more the larger is the term

$$[1 - \frac{s_1^2(x)}{s_2^2(x)}]^2 .$$

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APPENDIX I

Theorem I-I

If the quantity $\eta(\vec{\theta}, x) = \vec{\theta}' \vec{f}(x)$ is measured at the point x during the time T, then the dispersion matrix of the parameters is expressed as:

$$D(T, x) = (I_m - \frac{\lambda(x) T D(0) \vec{f}(x) \vec{f}'(x)}{1 + \lambda(x) T \vec{f}'(x) D(0) f(x)}) D(0), \qquad (I.I.)$$

where $\lambda(x)$ is the efficiency of the experiment and $D(0)$ is the dispersion matrix at the time $T = 0$.

Proof.

As is known (see [1], chapter II)

$$D^{-1}(T) = D^{-1}(0) + \lambda(x) T \vec{f}(x) \vec{f}'(x) \qquad (1.2)$$

or

$$D(T) = [D^{-1}(0) + \lambda(x) T \vec{f}(x) \vec{f}'(x)]^{-1} =$$

$$= \{D^{-1}(0)[I + \lambda(x) T D(0) \vec{f}(x) \vec{f}'(x)]\}^{-1} = [I_m + \lambda(x) T D(0) \vec{f}(x) \vec{f}'(x)]^{-1} D(0). \qquad (1.3)$$

We shall use the matrix formula:

$$(I_p + AB)^{-1} = I_p - A(I_q + BA)^{-1} B, \qquad (1.4)$$

where I_p and I_q are unit matrices of the rank p and q, A is a $p \times q$ matrix and B is a $p \times q$ matrix. Let us denote

$$A=\lambda(x)TD(0)\vec{f}(x) \quad \text{and} \quad B=\vec{f}'(x) \quad . \qquad (1.5)$$

From (1.3), (1.4) and (1.5) we obtain:

$$D(T,x)=(I_m-\frac{\lambda(x)TD(0)\vec{f}(x)\vec{f}'(x)}{1+\lambda(x)T\vec{f}'(x)D(0)\vec{f}(x)})D(0). \qquad (1.6)$$

The theorem is proved.

An analogical result was first obtained by Box and Hunter/8/ for discrete measurements with equal weights.

Lemma I-I

If A and B are two $p\times p$ matrices with the ranks $r(A)=p$ and $r(B)=1$ then

$$|A+B|=|A|(1+\sum_{\alpha,\beta=1}^{p} A^{-1}_{\alpha\beta}B_{\beta\alpha}). \qquad (1.7)$$

Proof.

Let us denote $\alpha_i(\beta_i)$ the i-th column of $A(B)$. Then:

$$|A+B|=|\alpha_1+\beta_1,\dots,\alpha_p+\beta_p|=|\alpha_1,\alpha_2,\dots,\alpha_p|+|\beta_1,\alpha_2,\dots,\alpha_p|+\dots+|\alpha_1,\alpha_2,\dots,\beta_p| . \qquad (1.8)$$

In (1.8) we drop all determinants containing two or more columns of the matrix B. They are equal to zero, since $r(B)=1$. We decompose the determinants in (1.8) in terms of the elements of the β_i $(i=1,\dots,p)$ and attain (1.8). The lemma is proved.

Theorem 1-2

Using the assumptions of theorem I-1 the determinant of the dispersion matrix of the "useful" parameters is equal to:

$$|D^{kk}(T,x)|=|D^{kk}(0)|(1-\frac{\lambda(x)T}{1+\sigma^2(x)\lambda(x)T}\sum_{\alpha\beta=1}^{k}[D^{kk}(0)]^{-1}_{\alpha\beta}C_{\alpha\beta}(x). \qquad (1.9)$$

Here $\alpha,\beta=1,\dots,k$, $\sigma^2(x)=\vec{f}'(x)D(0)\vec{f}(x)$ is the corridor of errors of the curve $\eta(\hat{\vec{\theta}}^{(0)},x)$ and $C=D(0)\vec{f}(x)\vec{f}'(x)D(0)$.

Proof.

It follows from (1.1) that

$$|D^{kk}(T,x)| = |\{ \sum_{\gamma=1}^{m} I_{\alpha\gamma} D_{\gamma\beta}(0) - \frac{\sum_{\delta,\gamma=1}^{m} D_{\alpha\delta}(0) f_\delta(x) f_\gamma(x) D_{\gamma\beta}(0)}{1 + \lambda(x) T \vec{f}'(x) D(0) \vec{f}(x)} \lambda(x) T \}_{\alpha\beta}| \quad (1.10)$$

$$= |\{ D_{\alpha\beta}(0) - \frac{\lambda(x)T}{1+\sigma^2(x)\lambda(x)T} C_{\alpha\beta} \}_{\alpha\beta}|.$$

Since the rank of C is equal to I, we shall use the lemma I-I. From (1.7) and (1.10) follows:

$$|D^{kk}(T,x)| = |D^{kk}(0)| \left(1 - \frac{\lambda(x)T}{1+\sigma^2(x)\lambda(x)T} \sum_{\alpha\beta=1}^{k} [D^{kk}(0)]^{-1}_{\alpha\beta} C_{\alpha\beta}(x)\right). \quad (1.11)$$

The theorem is proved.

From (1.9) the formulas (6), (7), (10) and (11) are obtained.

APPENDIX II

Theorem II-1

If $\eta(\hat{\theta}^{(0)}, x')$ is the estimate of $\eta(x)$ at x' for $t = 0$ and if T is the time of the measurement at x then for $t = T$ the estimate of $\eta(x)$ at x' is expressed as:

$$\eta(\hat{\theta}, x') = \eta(\hat{\theta}^{(0)}, x') + \frac{\sigma^2(x', x)(y - \eta(\hat{\theta}^{(0)}, x))}{(\lambda(x)T)^{-1} + \sigma^2(x)}, \quad (II.1)$$

where $\sigma^2(x', x) = \vec{f}'(x') D(0) \vec{f}(x)$ and the other notations are as in appendix I.

Proof.

As is known (see /1/, chapter II):

$$\hat{\theta} = D(T,x) \vec{Y}(T,x). \quad (II.2)$$

Here $D(T,x)$ is given by (I.1) and the vector $Y(T,x)$ is equal to:

$$\vec{Y}(T,x) = \sum_{i=1}^{n} y_i w_i \vec{f}(x_i) + \lambda(x) T y \vec{f}(x). \qquad (II.3)$$

Substituting (I.1), (II.2) and (II.3) into (I) we obtain after simple calculations:

$$\eta(\hat{\vec{\theta}}, x') = \vec{f}'(x')\left[I_m - \frac{\lambda(x) T D(0) \vec{f}(x) \vec{f}'(x)}{1 + \lambda(x) T \vec{f}'(x) D(0) \vec{f}(x)} \right] \times \qquad (II.4)$$

$$\times D(0)\left[\sum_{i=1}^{n} y_i w_i \vec{f}(x_i) + \lambda(x) T y \vec{f}(x) \right] =$$

$$= \vec{f}'(x') \hat{\vec{\theta}}^{(0)} + \frac{\lambda(x) T \vec{f}'(x') D(0) \vec{f}(x) [y - \vec{f}'(x) \hat{\vec{\theta}}^{(0)}]}{1 + \lambda(x) T \vec{f}'(x) D(0) \vec{f}(x)} .$$

Using the notations given above we rewrite (II.4) as to obtain (II.1). The theorem is proved.

Theorem II-2

If the measurement is made at x during the time T and if $\hat{\vec{\theta}}_k^{(0)}$ is an unbiased estimate, then the mean value of the sum of the weighted squared deviations $S_j - S_k$ $(j \neq k, j = 1.2)$ is given as:

$$E_{u_k}(S_j - S_k) = S_j(0) - S_k(0) + \frac{[\eta(\vec{\theta}_j^{(0)}, x) - \eta(\vec{\theta}_k^{(0)}, x)] + \sigma_j^2(x) + \sigma_k^2(x)}{s_j^2(x)}, \qquad (II.5)$$

where

$$s_j^2(x) = (\lambda(x) T)^{-1} + \sigma_j^2(x).$$

Proof.

After the measurement at x the sum of the weighted squared deviations S_j $(j = 1,2)$

$$S_j(T,x) = \sum_{i=0}^{n} (\eta(\hat{\vec{\theta}}_j, x_i) - y_i)^2 w_i . \qquad (II.6)$$

The index $i = 0$ corresponds to the measurement at the point $x, w_0 = \lambda(x) T$.

Using (II.1) we transform (11.6) to the form:

$$S_j(T,x) = S_j(0) + \sum_{i=1}^{n} w_i \Delta_j^2(x_i, x) + \lambda(x) T [y - \eta(\hat{\vec{\theta}}_j^{(0)}, x) - \Delta_j(x,x)]^2 \tag{II.7}$$
$$- 2 \sum_{i=1}^{n} w_i [y_i - \eta(\hat{\vec{\theta}}_j^{(0)}, x_i)] \Delta_j(x_i, x),$$

where

$$\Delta_j(x_i, x) = \frac{\sigma_j^2(x_i, x)(y - \eta(\hat{\vec{\theta}}^{(0)}, x))}{s_j^2(x)} \quad . \tag{II.8}$$

Let $\hat{\vec{\theta}}_1^{(0)}$ be the unbiased estimate. Then from (14) and (15) follows that:

$$E_{u_1} \Delta_1(x_i, x) = 0$$

$$E_{u_1} \Delta_1(x_i, x) = \frac{\sigma_1^4(x_i, x)}{s_1^2(x)}$$

$$E_{u_1} \Delta_2(x_i, x) = \frac{\sigma_2^2(x_i, x)}{s_2^2(x)} [\eta(\hat{\vec{\theta}}_1^{(0)}, x) - \eta(\hat{\vec{\theta}}_2^{(0)}, x)] \tag{II.9}$$

$$E_{u_1} \Delta_2^2(x_i, x) = \frac{\sigma_2^4(x_i, x)}{s_2^4(x)} [s_1^2(x) + (\eta(\hat{\vec{\theta}}_1^{(0)}, x) - \eta(\hat{\vec{\theta}}_2^{(0)}, x))^2] .$$

From (II.7) and (II.9) follows:

$$E_{u_1}[S_2(T,x) - S_1(T,x)] = S_2(0) - S_1(0) + s_1^2(x) \sum_{i=1}^{n} w_i \left[\frac{\sigma_2^4(x_i, x)}{s_2^4(x)} - \right.$$
$$\left. - \frac{\sigma_1^4(x_i, x)}{s_1^4(x)} \right] + s_1^2(x) w_0 \left[\frac{\sigma_2^4(x)}{s_2^4(x)} - \frac{\sigma_1^4(x)}{s_1^4(x)} \right] +$$
$$+ (\eta(\hat{\vec{\theta}}_1^{(0)}, x) - \eta(\hat{\vec{\theta}}_2^{(0)}, x))^2 \left[\sum_{i=1}^{n} w_i \frac{\sigma_2^4(x_i, x)}{s_2^4(x)} + \frac{1}{w s_2^4(x)} \right] - \tag{II.10}$$
$$- \frac{\eta(\hat{\vec{\theta}}_1^{(0)}, x) - \eta(\hat{\vec{\theta}}_2^{(0)}, x)}{s_2^2(x)} \sum_{i=1}^{n} w_i (y_i - \eta(\hat{\vec{\theta}}_2^{(0)}, x_i)) \sigma_2^2(x_i, x)).$$

Simplifying the obtained expression using the definition of $\sigma^2(x_i, x)$ we get:

$$\sum_i w_i \frac{\sigma_2^4(x_i, x)}{s_2^4(x)} = \frac{1}{s_2^4(x)} \sum_i w_i \vec{f}_2'(x_i) D_2(0) \vec{f}_2(x) \vec{f}_2(x_i) D_2(0) \vec{f}_2(x) \quad (II.11)$$

Using $\vec{f}'(x_i) D(0) \vec{f}(x) = \vec{f}'(x) D(0) \vec{f}(x_i)$ we rewrite (II.11) in the form:

$$\sum_i w_i \frac{\sigma_2^4(x_i, x)}{s_2^4(x)} = \frac{1}{s_2^4(x)} \vec{f}_2'(x) D_2(0) [\sum_{i=1}^n w_i \vec{f}_2(x_i) \vec{f}_2'(x_i)] D_2(0) \vec{f}_2(x) =$$

$$= \frac{1}{s_2^4(x)} \vec{f}_2(x) D_2(0) D_2^{-1}(0) D_2(0) \vec{f}_2(x).$$

Finally:

$$\sum_i w_i \frac{\sigma_2^4(x_i, x)}{s_2^4(x)} = \frac{\sigma_2^2(x)}{s_2^4(x)}. \quad (II.12)$$

Analogical transformations are used for other terms in (II.10):

$$\sum_{i=1}^n w_i \frac{\sigma_1^4(x_i, x)}{s_1^4(x)} = \frac{\sigma_1^2(x)}{s_1^4(x)};$$

$$\sum_{i=1}^n w_i y_i \sigma_2^2(x_i, x) = \sum_{i=1}^n w_i y_i \vec{f}_2'(x_i) D_2(0) \vec{f}_2(x) = \quad (II.13)$$

$$= \vec{f}_2'(x) D_2(0) \sum_{i=1}^n w_i y_i \vec{f}_2(x) = \vec{f}_2'(x) D_2(0) \vec{Y}_2 =$$

$$= \vec{f}_2'(x) \vec{\theta}_2^{(0)} = \eta(\hat{\vec{\theta}}_2^{(0)}, x);$$

$$\sum_{i=1}^n w_i \eta_2(\vec{\theta}_2^{(0)}, x_i) \sigma_2^2(x_i, x) =$$

$$= \vec{f}'_2(x) D_2(0) D_2^{-1}(0) \hat{\vec{\theta}}_2^{(0)} = \eta(\vec{\theta}_2^{(0)}, x). \qquad (\text{П}.15)$$

Substituting (П.12) - (П.15) into (П.10), we obtain in a straightforward way:

$$E_{u_1}[S_2(T,x) - S_1(T,x)] = S_2(0) - S_1(0) +$$

$$+ \frac{[\eta(\hat{\vec{\theta}}_2^{(0)}, x) - \eta(\hat{\vec{\theta}}_1^{(0)}, x)]^2 + \sigma_2^2(x) - \sigma_2^2(x)}{s_2^2(x)}.$$

The theorem is proved.

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