ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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NON-LINEAR SCALAR FIELD THEORY

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In the present note a model of the non-linear scalar field theory is treated. From the very beginning this model allows an existence of some kinds of particles, while the vacuum turns out, therefore, to be degenerate.

The vacuum degeneration leads to an appearance of additional vacuum excitations which are represented by particles with very small masses, whereas the presence of potential barriers in the functional space leads to the possibility of the radioactive decay of elementary particles.

If this process is considered as a weak interaction, then in our model it is a direct consequence of strong interaction of particles (cf./1,2/). This is an attractive feature of this model.

However, we are faced here with some difficulties which are characteristic of the non-linear theories: the renormalization of the energy levels does not reduce now to a simple renormalization of the zero field theory; it is necessary to renormalize each level separately (cf./3/).

2. The Problem

The scalar field in our model is described by the Hamiltonian

$$H = \int \{ \tfrac{1}{2} \pi^2(x) + \frac{c^2}{2} \nabla \phi(x)^2 + V(\phi(x)) \} d^3x \tag{1}$$

$\pi(x)$ and $\phi(x)$ obeying the quantum condition

$$[\pi(x), \phi(x')] = i h \delta(x - x') \tag{2}$$

We shall consider, in contrast to the linear field theory, that the potential energy of the field at the point $x - V(\phi(x))$ is limited and has the maxima and the minima (see Fig. 1).

Small oscillations about the equilibrium positions should be regarded as such field states $'a'$, $'b'$, $'c'$..., which correspond to the particles with different masses: the particles $'a'$, $'b'$, $'c'$...

The particles of the same type strongly interact. Indeed, for the non-interacting particles, instead of potential wells there would be parabolic potential curves shown in Figure by the dashed line. It is seen from this Figure that if the field is excited in one of these wells, then it can penetrate through the potential barrier separating them. This penetration would mean the vanishing of the particles of one type and the appearance of other ones. i.e., the radioactive decay of an elementary particle.

Particularly, according to Fig. 1, a particle $'a'$ with a greater mass $(m_a = \frac{h\omega_a}{c^2})$ can decay into particles $'b'$ having smaller masses $(m_b = \frac{h\omega_b}{c^2} < m_a)$. From this figure one can also see the vacuum degeneration which arises when zero energies of the two wells a and b are equal.

3. Approximate Solution with the Aid of the Spatial Lattice

To find the eigenfunctions and eigenvalues of Hamiltonian (1), we introduce, instead of a continuous space, a lattice with cells Δ^3 in volume, and instead of the operators $\pi(x)$ and $\phi(x)$, new operators

$$\Pi_s = \Delta^{-3/2} \int_{\Delta^3} \pi(x)\, d^3x, \qquad \Phi_s = \Delta^{-3/2} \int_{\Delta^3} \phi(x)\, d^3x \tag{3}$$

where the integrals are extended throughout the cell volume Δ^3 near the $x = s$ (s is a cell number).

Then, instead of (1) and (2), we get

$$H = \sum_s \{ \tfrac{1}{2} \Pi_s^2 + \frac{c^2}{4\Delta^2} (\Phi_s - \Phi_{s+1})^2 + \frac{c^2}{4\Delta^2} (\Phi_s - \Phi_{s-1})^2 + V(\Phi_s) \} \tag{4}$$

and

$$[\Pi_s, \Phi_{s'}] = i h \delta_{ss'} \tag{5}$$

Note, that the terms with $(\Phi_s - \Phi_{s\pm 1})$ appear after $\nabla\phi$ was substituted by the finite difference. Further, the function $V(\Phi)$ coincides with $V(\phi)$ if the absolute scale of the field ϕ_0 (which is necessarily present in the non-linear theory) is substituted by the new scale $\Phi = \Delta^{-3/2} \phi_0$

In the following we restrict ourselves to small momenta of particles. In this case, one can take, as a zero approximation, the field oscillations in separate cells of the space, and consider the propagation of oscillations as perturbation. The whole problem becomes very much alike that describing the motion of excitons in a crystal.

In accordance with what has been said, the Hamiltonian of zero approximation becomes

$$H_0 = \sum_s \{ \tfrac{1}{2} \Pi_s^2 + V(\Phi_s) \} \tag{6}$$

and the perturbation energy is $W = W' + W''$, where

$$W' = \frac{c^2}{\Delta^2} \sum_s \Phi_s^2, \qquad W'' = - \frac{c^2}{2\Delta^2} \sum_s (\Phi_s \Phi_{s+1} + \Phi_s \Phi_{s-1}) \tag{7}$$

Consider first the vacuum states. If the oscillations of the field in the well a are described by the wave function $u_0(\Phi)$, and in the well b -by the function $v_0(\Phi)$, then in view of the degeneration of this level the 'true' wave functions are

$$\Psi^+ = \frac{1}{\sqrt{1+\alpha^2}} (u_0 + \alpha v_0), \qquad \Psi^- = \frac{1}{\sqrt{1+\beta^2}} (u_0 + \beta v_0) \tag{8}$$

We shall assume that out of these two functions it is Ψ^{+} which corresponds to a lower level, rather than u_0 and v_0. Therefore, the wave function of the vacuum of the whole field becomes

$$\Psi_0 = \prod_s \Psi^{+}(\Phi_s) \qquad (9)$$

where the product is taken over all the cells of the lattice.

As far as the vacuum is degenerated, there are also other states which will produce the vacuum zone. In particular, the one-particle states in this zone will have the wave function

$$\Psi'_0 = \sum_m c_m \Psi_{0m} \quad , \quad c_m = \frac{e^{im\xi}}{\sqrt{N}} \qquad (10)$$

Here $N \to \infty$ is the number of cells, ξ is the quasi-momentum (the particle momentum is $P = \frac{h\xi}{\Delta}$) and

$$\Psi_{0m} = \prod_s{}' \Psi^{+}(\Phi_s)\, \Psi^{-}(\Phi_m) \qquad (11)$$

Now we calculate the energy of these states. We have

$$E_0 = <\Psi_0 / H / \Psi_0> = <\Psi_0 / H_0 / \Psi_0> + <\Psi'_0 / W' / \Psi_0)^{+} + <\Psi_0 / W'' / \Psi_0> = (\epsilon_0 + c_0) N \qquad (12)$$

where the constant c_0 is

$$c_0 = \frac{c^2}{\Delta^2} \frac{1}{(1+a^2)} [<u_0 | \Phi^2 | u_0> + a^2 < v_0 | (\Phi - \Phi_b)^2 | v_0> + \frac{a^2}{(1+a^2)} \Phi_b^2] \qquad (13)$$

Here Φ_b is the average field in the state v_0 (the well b); the average field in the well a is taken to be zero.

The constant $c_0 \to \infty$ with $\Delta \to 0$. Therefore, it must enter the renormalization E_0.

In a similar manner, for the one-particle vacuum excitation, we get

$$E'_0 = <\Psi'_0 / H / \Psi'_0> = E_0 + c'_0 + \frac{P^2}{2m^{x}} \qquad (14)$$

where the new constant c'_0 has the form which is similar to (13), namely, $c'_0 = \frac{c^2}{\Delta^2} f(a, \beta, \Phi_b)$ and $c'_0 \to \infty$ at $\Delta \to 0$. The effective masses of the particle m^* are determined by

$$m^{x} = ½ \; \frac{(1+a^2)(1+\beta^2)}{a^2\beta^2} \; \frac{h^2}{\Phi_b^2 c^2} \qquad (15)$$

When $\phi_b \to \infty$; $m^{x} \to 0$. It follows from (14), that the excited level should be renormalized so that

$$E_0' = E + m^{*}c^{2} + \frac{P^{2}}{2m^{*}} \qquad (16)$$

In other words, the constant $\Delta \to 0$ infinite at c_0' must be substituted by $m^{*}c^{2}$. This additional renormalization is the peculiarity of the non-linear theory, as has been already mentioned in /3/.

Consider now the state of the field when there is one-particle excitation in the well a, described by the wave function $u_1(\Phi)$ and by the energy $\epsilon_1 = \epsilon_0 + h\omega_0$

The wave function of such a state is

$$\Psi_{a1} = \Sigma A_m \Psi_{m1}, \qquad A_m = \frac{e^{i\xi m}}{\sqrt{N}} \qquad (17)$$

where

$$\Psi_{ma} = \Pi_{a}' \Psi^{+}(\Phi_a)\, u_1(\Phi_m) \qquad (18)$$

The energy corresponding to this state is

$$E_{a1} = E_0 + c_{1a} + h\omega_0 + \frac{P^{2}}{2m_a} \qquad (19)$$

where the constant c_{1a} is given by

$$c_{1a} = c_0 + \frac{c^2}{\Delta^2}\frac{1}{(1+a^2)}\left[(1+a^2) < u_1 |\Phi^2| u_1 > - (u_0 |\Phi^2| u_0 > - a^2 < v_0 |\Delta\Phi^2| v_0 > - \frac{a^2}{1+a^2}\Phi_b^2\right] \qquad (20)$$

and the mass of the particle is m_a:

$$m_a = (1+a^2)\frac{h\omega_0}{c^2} \qquad (21)$$

(since $<u_1/\Phi/u_0> \approx \sqrt{\frac{h}{2\omega_0}}$). At $\Delta \to 0$, $c_{1a} \to -\infty$, and we are to renormalize E_{a1} again, so that

$$E_{a1} = E_0 + m_a c^2 + \frac{P^2}{2m_a} \qquad (22)$$

Analogously the wave function and energy of the two-particle state in the well b can be considered. For this case

$$\Psi_{b11} = \sum_{m,n} c_{mn} \Psi_{bmn}, \qquad c_{mn} = \frac{e^{i(\zeta m + \eta n)}}{N} \qquad (23)$$

$$\Psi_{bmn} = \Pi_{a}'' \Psi^{+}(\Phi_a)\, v_1(\Phi_m)\, v_1(\Phi_n)$$

Just in a similar manner we get the renormalized expression for the energy

$$E_{II} = E_0 + 2m_b c^2 + \frac{P_1^2}{2m_b} + \frac{P_2^2}{2m_b} \qquad (24)$$

provided

$$m_b = \left(\frac{1+a^2}{a^2}\right)\frac{h v_0}{c^2} \qquad (25)$$

Here v_0 is the proper frequency of the well b and $P_1 = \frac{h\zeta}{\Delta}$, $P_2 = \frac{h\eta}{\Delta}$ (let us remind that all these formulae are correct only for small P_1 and P_2).

4. Radioactive Decay of Particles

Let us assume that the mass of the particles is $m_a > 2m_b$. Then the particle a is likely to decay by the mode $a \to b + b$. Let at the moment $t = 0$ there be one particle a with the momentum ξ . The corresponding wave function will be $\Psi_{aI}(\xi)$ (cf. (17)). The transition rate to the state $\Psi_{bII}(\zeta, \eta)$, which describes two particles b with the momenta ζ and η (cf. (23)), is determined by the matrix element

$$\langle \Psi_{bII}(\eta, \zeta) / V / \Psi_{aI}(\xi) \rangle = \frac{1}{N^{3/2}} \sum_m \sum_{rs} A_m c_{rs} \langle m / V / rs \rangle \qquad (26)$$

where the difference V is regarded as a perturbation $V = V(\Phi) - V_\infty(\Phi)$. Here $V_\infty(\Phi)$ is a potential obtained from $V(\Phi)$ if the wells a and b are removed from each other at an infinitely great distance $(\Phi_b \to \infty)$

Note, that the asymptotics of the functions u_1 and v_1 assumes the form

$$u_1(\Phi) \sim e^{-a\Phi} \qquad v_1(\Phi) \sim e^{-a(\Phi_b - \Phi)} \qquad (27)$$

where $a = \frac{1}{h}\sqrt{V_0 - E}$. Here V_0 is the barrier height, and E is the renormalized energy $E = m^* c^2 + \frac{P^2}{2m^*}$, $P = \frac{h\xi}{\Delta}$; the matrix elements $(m / V / z, s \rangle$ different from zero are obtained if $m = r = s$: $\langle m / V / mm \rangle = \langle u_1 / V / v_1 v_1 \rangle$. Then the sum over m in (26) provides the law of the momentum conservation $\xi = \zeta + \eta$.

Therefore,

$$\langle \Psi_{bII} / V / \Psi_{aI} \rangle = \frac{1}{N^{1/2}} \langle u_1 / V / v_1 v_1 \rangle \qquad (26')$$

and in virtue of (27)

$$\langle u_1 / V / v_1 v_1 \rangle \cong e^{-a\Phi_b} \langle \tilde{u}_1 / V / \tilde{v}_1 \tilde{v}_1 \rangle \qquad (28)$$

where $\tilde{u}_1$ and $\tilde{v}_1$ are defined by the formulae

$$u_1(\Phi) = \tilde{u}_1 e^{-a\Phi}, \quad v_1(\Phi) = \tilde{v}_1(\Phi) e^{-a(\Phi_b - \Phi)}$$

Thus, at $\phi_b \to \infty$, as it should be expected, there is a small probability of particle decay which is proportional to

$$P \approx e^{-2a\Phi_b} / < \approx{u}_1 / V / \approx{v}_1 \approx{v}_1 > /^2 \tag{29}$$

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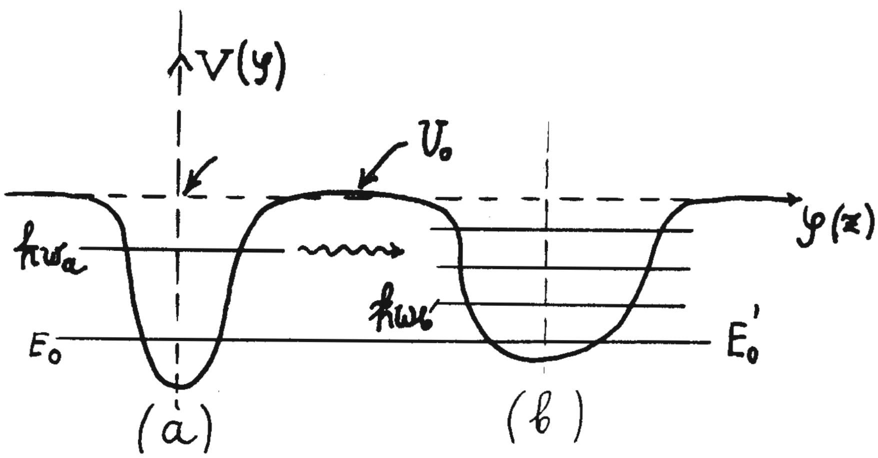

Fig. I The potential barrier in the functional space.