ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON A REGULARIZATION METHOD IN THE FIELD THEORY

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The S -matrix is constructed by using the functions which go over into the plane waves at $t \to \pm \infty$ instead of the plane waves.

The notion of the 'free' field quantum corresponds to that of the free particle only at $t \to \pm \infty$.

The formalism is constructed as follows: an ordinary free field $\phi^{\circ}(x)$ is supplemented with a certain 'counter-field' which is dampling with time, quantized according to the indefinite metric (with an 'opposite sign') and makes the regularization of the $\Delta^{()}$ functions (§§ 2, 3).

The quanta of the 'counter-field' $\phi'(x)$ vanish at $t \to \pm \infty$ (§§ 1,6). The counter-field does not make any contribution to the probabilities and the cross sections of the observable effects (§ 7). The theory is causal (§ 4), unitary (§ 5) and relativistic invariant' (§ 7).

Though the Pauli-Villars regularization method has rather a formal, even an illogical character, this procedure is widely applied in the modern field theory[1]. This is likely to be accounted for the fact that may be not incidentally, (from the point of view of the future consistent theory)the procedure in question makes it possible to assign the meaning of the finite expressions to the divergent integrals of the modern theory without arriving to a contradiction either with the causality principle, or with the unitarity requirement. It seems attractive to give to this procedure the meaning of the consistent theory, having in view that in such a theory the regularization we are discussing would arise automatically.

As is well-known, the verbal meaning of the regularization by Pauli-Villars method consists in the formal introduction at a later stage of the calculations of some counter-fields which alter the form of the propagation functions only but remain (what is essential) unchanged the state vector's describing the fields in the modern theory.

A question arises as to how, by introducing new formulations of the basic concepts of the modern field theory, one can get automatically the regularization under discussion? As is well-known it is easy to introduce the real 'counter-fields' which regularize the propagation functions but it is still impossible to rule out the peculiar difficulties which arise in so doing.[2]

In the present paper are also introduced 'counter-fields' which regularize the propagation functions. In contrast to other attempts, the counter-fields introduced here are damping with time, and the quanta of these 'counter-fields' are absent in the initial and final states of the systems (at $t \to \pm \infty$). The division of the total field into the 'field' and 'counter-field' is artificial-this is only a method of exposition convenient for comparison of the developed theory with the ordinary one. The physical content of this theory is that the quantum of the field becomes a free particle only if $t \to \pm \infty$.

In the modern theory the process of particle production is considered (apart from the uncertainty principle $\Delta t \Delta E \approx \hbar$) as a momentary appearance of a free particle with a definite rest mass. In the scheme at small times of the arising particle its proper mass is indefinite, indefinite is also its energy. The components distorting the usual plane wave are dampling with time, and the state becomes that of a free particle as it is understood in the ordinary theory.

The scheme is symmetrical with respect to the production and absorption of particles, symmetrical with respect to the past and the future.

1. 'Counter- Field'

As an example, consider first a scalar field. Each plane wave which in the ordinary theory describes a particle with a momentum $\vec{k}$ and an energy $\sqrt{\vec{k}^2 + m^2}$

$$\sim \frac{e^{\pm i[\vec{k}\vec{x} - (\vec{k}^2 + M^2)^{1/2} t]}}{(2\sqrt{\vec{k}^2 + M^2})^{1/2}} \qquad (1)$$

is added by a certain accompanying wave of a 'counter-field', characterized by the same value of the vector $\vec{k}$, but represented by a certain integral over the mass parameter m

$$\sim e^{\pm i\vec{k}\vec{x}} \int \frac{e^{\pm i(\vec{k}^2 + m^2)^{1/2} t}}{(2\sqrt{\vec{k}^2 + m^2})^{1/2}} \Phi\left(\frac{m}{\ell}\right) \frac{dm}{\ell} \qquad (2)$$

The quantum of the ordinary field is associated with the plane wave (1), while the quantum of the 'counter-field' is associated with a still more complicated mathematical expression of form (2). In the following it is the functions of type (2) that the 'counter-field' will be quantized over. In other words, the corresponding creation and annihilation operators of the counter-field quanta are the amplitudes of the functions of type (2). The ordinary free field $\phi^\circ(x)$ which is emplbyed in the modern theory to construct the S-matrix is written down as

$$\phi^{\circ\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{k}}{\sqrt{2k_\circ}} e^{\pm ikx} \phi^{\pm}(\vec{k}) \qquad (3)$$

$$\phi^\circ(x) = \phi^{\circ +}(x) + \phi^{\circ -}(x) \qquad (4)$$

where (+) and (−) designate positive and negative frequency functions.

In what follows we will start from an explicit form of the operator functions of the counter fields

$$\phi'^{+}(x) = \frac{1}{(2\pi)^{3/2}} \sum_n \int \frac{d\vec{k}}{\sqrt{2k_\circ}} e^{ikx} \frac{dm}{\ell} \phi^{+}_n(\vec{k}) f\left(\frac{m^2}{\ell^2}\right) \cdot \sqrt{\frac{m}{\ell}} H_n\left(\frac{m}{\ell}\right) \qquad (4)$$

$$\phi'(x) = \frac{1}{(2\pi)^{3/2}} \sum_n \int \frac{d\vec{k}}{\sqrt{2k_0}} e^{-ikx} \frac{dm}{\ell} \phi_n^-(\vec{k})\, f\left(\frac{m^2}{\ell^2}\right) \sqrt{\frac{|m|}{\ell}}\, H_n\left(\frac{m}{\ell}\right). \qquad (5)$$

Here $f\left(\frac{m^2}{\ell^2}\right)$ is a certain function rapidly falling off with the increasing m.

$H_n\left(\frac{m}{\ell}\right)$ is a certain complete system of orthogonal and normalized functions. Therefore,

$$\sum_n H_n\left(\frac{m}{\ell}\right) H_n\left(\frac{m'}{\ell}\right) = \delta\left(\frac{m-m'}{\ell}\right) \qquad (6)$$

ℓ is the constant adopted in the theory.

For the sake of definiteness, the Hermitian functions* are chosen further as particular functions H_n. To carry out the same consideration the Bessel's functions or any other system of functions may be used. Thus, functions (2) describing the quantum of the added field in the mass state n assume an explicit form

$$\phi'^{+}_{n\vec{k}} = \approx e^{i\vec{k}\vec{x}} \int_a^b \frac{e^{-i\sqrt{\vec{k}^2+m^2}\,t}}{(2\sqrt{\vec{k}^2+m^2})^{1/2}}\, f\left(\frac{m^2}{\ell^2}\right) \sqrt{\frac{|m|}{\ell}}\, H_n\left(\frac{m}{\ell}\right) \frac{dm}{\ell}. \qquad (7)$$

The integration limits are yet indefinite, but, in particular, it is possible to take $\int_{-\infty}^{+\infty}$.

For the future analysis it is essential that the functions $\phi'^{\pm}_{n\vec{k}}$ are taken to damp with time. Owing to the oscillatory time dependence of intergrand (7) the damping of $\phi^{\pm}_{n\vec{k}}$ functions with time may be realised by a wide scope of functions $f\left(\frac{m^2}{\ell^2}\right)$. So, choosing in (7) $H_n = H_0 \approx e^{-m^2/2\ell^2}$, $f = 1$, in the rest system $(\vec{k} = 0)$ we get

$$\phi^{+}_{n,\vec{k}=0} \sim \int_{-\infty}^{+\infty} e^{-imt - \frac{m^2}{2\ell^2}} \frac{dm}{\ell} \to e^{-\frac{\ell^2 t^2}{2}}, \qquad (7')$$

that is $\phi'^{+}_{0,k=0}$ is, indeed, damping with time. This damping can be made to proceed even stronger by an appropriate choice of the function $f\left(\frac{m}{\ell}\right)$.

The damping of function (7 ') is symmetrical in time. The function (7') has a privileged time point $t = 0$ - it corresponds to the time of creation or annihilation of a particle.

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* To be more exact, the Hermitian polynomials with the normalizing factor, e.g., $H_0 = 1 \cdot e^{-\frac{m^2}{2\ell^2}}$.

The counter field arises only in the interacting systems. The 'really' free field ($t \to \pm\infty$) coincides with the notion of the free field adopted in the ordinary theory.

The field ϕ° satisfies usual equations for the free field. The counter field is defined, according to (4) and (5), as an accompanying field. The total field is not assumed to satisfy any equation.

2. Quantization

The functions $\phi_{\circ}(x)$ are quantized in the usual manner. The corresponding quantum brackets for the counter field ϕ' are taken with an 'opposite sign':

$$[\phi'^{-}_{n}(\vec{k});\ \phi'^{+}_{n'}(\vec{k}')] = -\delta(\vec{k} - \vec{k}')\,\delta_{nn'}$$

$$[\phi'^{-}_{n}(\vec{k});\ \phi'^{-}_{n'}(\vec{k}')] = [\phi'^{+}_{n}(\vec{k});\ \phi'^{+}_{n'}(\vec{k}')] = 0 \qquad (8)$$

$$[\phi^{\circ}(x),\ \phi'(y)] = 0 .$$

According to (8)

$$[\phi'^{-}(x);\phi'^{+}(y)] = \frac{1}{(2\pi)^3}\int d\vec{k}\ d\vec{k}'\sum_n\sum_n \frac{e^{i(k'y-kx)}}{\sqrt{2k_o}\ \sqrt{2k'_o}}\ .$$

$$[\phi^{-}_{n}(\vec{k});\ \phi^{+}_{n'}(\vec{k}')]\ f(\frac{m^2}{\ell^2})\ f(\frac{m'^2}{\ell^2})\sqrt{\frac{/m/}{\ell}}\ \sqrt{\frac{/m'/}{\ell}}\ H_n(\frac{m}{\ell})\ H_n(\frac{m'}{\ell})\ \cdot\frac{dm}{\ell}\ \frac{dm'}{\ell'} =$$

$$= -\frac{1}{(2\pi)^3}\int d\vec{k}\ \sum_n \frac{e^{i(k_m y - k_{m'} x)}}{2\sqrt{k^{\circ}_m\ k^{\circ}_{m'}}}\ f(\frac{m^2}{\ell^2})\ f(\frac{m'^2}{\ell^2})\ \cdot \qquad (9)$$

$$\sqrt{\frac{/m//m'/}{\ell^2}}\quad H_n(\frac{m}{\ell})\ H_n(\frac{m'}{\ell})\quad \frac{dm\,dm'}{\ell^2}\ ;$$

$$k^{\circ}_{m} = \sqrt{\vec{k}^2 + m^2}\ ; \qquad k^{\circ}_{m'} = \sqrt{\vec{k}^2 + m'^2}\ .$$

According to (6)

(9')

$$[\phi'(x)\phi'^{+}(y)] = -\frac{1}{(2\pi)^3}\int d\vec{k}\ \frac{e^{ik(y-x)}}{2k_o} f^2(\frac{m^2}{\ell^2})\ \frac{/m/}{\ell}\ \frac{dm}{\ell}$$

$$= \frac{-1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dm}{\ell}\int dk\,\delta(k^2-m^2)\,\theta(k_o)\,e^{ik(y-x)}\ f^2(\frac{m^2}{\ell^2})\frac{/m/}{\ell} =$$

$$= -i\int_{-\infty}^{+\infty}\frac{dm}{\ell}\ \Delta^{+}(y-x,m^2)\ f^2(\frac{m^2}{\ell^2})\frac{/m/}{\ell} = -\Delta'^{+}(y-x) \qquad (10)$$

where

$$\int_{o}^{\infty} f^2(\frac{m^2}{\ell^2})\ \frac{dm^2}{\ell^2} = 1. \qquad (11)$$

All other $\Delta'^{()}$ functions of the counterfield have the same structure

$$-\Delta'^{()} \sim \int_{-\infty}^{+\infty}\Delta^{()}(x,m^2)\ f^2(\frac{m^2}{\ell^2})\ \frac{/m/}{\ell}\ \frac{dm}{\ell}\ . \qquad (12)$$

Using the adopted properties of the operators $\phi_n(k)$

$$<\phi_n^{-}(\vec{k})\ \phi_n^{+}(\vec{k})>_o = \delta(\vec{k}-\vec{k}'),\quad <\phi_n^{-}(\vec{k})\ \phi_n^{-}(\vec{k})>_o = 0$$

$$<\phi_n^{+}(\vec{k})\ \phi_n^{+}(\vec{k})>_o = 0,\qquad <\phi^{o}(\vec{k})^{\pm}\phi_n^{\pm}(\vec{k})>_o = 0 \qquad (13)$$

one can show that

$$<T(\phi'(x)\phi'(y))>_o = \begin{cases} <\phi'(x)\phi'(y)>_o = \frac{1}{i}\Delta'^{(-)}_{x^o>y^o} \\ <\phi'(y)\phi'(x)>_o = i\,\Delta'^{(+)}_{x^o<y^o} \end{cases}$$

and $$\Delta'^{c} = i < T\, \phi(x)\, \phi(y) >_{\circ} .$$ / 14 /

Here formula (14) is extrapolated in usual manner for $x_\circ = y_\circ$. Thus, by means of the functions $H_n(\frac{m}{\ell})$ having the properties (6) it becomes possible to quantize the field ϕ' over the functions of type (2) or (7) which are damping in time.

3. Singularities of the Propagation Functions

The propagation functions of the field $\overset{\circ}{\phi} + \phi'$ are the differencies of the propagation functions. For instance,

$$< \Delta^{c}_{res} = \Delta^{c}_{m} - \Delta'^{c} . \qquad (15)$$

In the vicinity of the light cone

$$\Delta^{c}_{M} \sim \frac{1}{4\pi} \delta(s) + \frac{1}{4\pi^2 i s} + i \frac{M^2}{8\pi^2} \ell n \frac{s^{1/2} M}{2} - \frac{M^2}{16 s T} \theta(s) . \qquad (16)$$

Hence

$$\Delta^{c}_{res} = i \frac{M^2}{8\pi^2} \ell n\, s^{1/2} \frac{M}{2} - \frac{M^2}{16\pi} \theta(s) - \frac{i}{8\pi^2} \int m^2 \ln m \;\times \qquad (17)$$

$$\times \frac{s^{1/2}}{2} f(\frac{m^2}{\ell^2}) /\frac{m}{\ell}/ \frac{dm}{\ell} + \frac{\theta(s)}{16\pi} \int m^2 f^2(\frac{m^2}{\ell^2}) /\frac{m}{\ell}/ \frac{dm}{\ell} .$$

According to (17), the main singularities of the Δ^{c}_{M} function ($\delta(s)$ and $\frac{1}{s}$) are removed by introducing the counter field, irrespective of a particular form of the f^2 -function. To be more precise, the f^2 -function must be only the normalized function, i.e., satisfy condition (11) and even condition (18)

$$\int_b^a f^2 (\frac{m^2}{\ell^2}) /\frac{m}{\ell}/ \frac{dm}{\ell} = 1 . \qquad (18)$$

The removal of the logarithmic divergencies in $\Delta^{()}$ functions and of the ruptures at the points $s = 0$ require that the form of the f functions should be specified.

The $\Delta^{()}$ functions of the total field ($\Delta^{()}_{res}$) can be written as

$$\Delta^{()}_{res} \sim \int \Delta^{()} (s, m^2)\, \rho\, (m^2)\, dm^2 . \qquad (19)$$

where

$$\rho\,(m^2) = \delta\,(m^2 - M^2) - f^2(\frac{m^2}{\ell^2}) . \qquad (20)$$

Choosing f^2 e.g., in the form

$$f^2\left(\frac{m^2}{\ell^2}\right) = \frac{1}{2} a^2 \, Re \, \frac{H_1'(\ell m)}{\ell m} \qquad (21)$$

we get the $\Delta^{(\,)}$ functions we considered in /3/. These $\Delta^{(\,)}$ functions have the singularities not on a cone but on a hyperholoid. In the functions of type (21) f^2 are signvariable what leads to the appearance of the imaginaries in $\sqrt{f^2}$, and, hence, changes the Hermition properties of the field operators ϕ'. More complicated situations, will be considered elsewhere. Here we restrict ourselves only to the cases of the real f functions. It should be noted, that there is hardly any sense in trying to get the 'complete' regularization of the Λ function i.e., the equalities $\int m^{2n} \rho(m^2)\, dm^2 = 0$ for any integer n. To construct a theory which would not involve the diverging values of the observable quantities it is quite sufficient to remove the strongest singularities from the $\Delta^{(\,)}$ functions.

Indeed, in the momentum representation Δ^c_{res} is written (with account of (11)) as

$$\Delta^c_{res} = \int \frac{(m^2 - M^2)\, f^2\left(\frac{m^2}{\ell^2}\right) \frac{/m/}{\ell} \frac{dm}{\ell}}{(M^2 - p^2 - i\epsilon)\,(m^2 - p^2 - i\epsilon)} \,. \qquad (22)$$

Thus, in calculating the degree of the convergence of the corresponding integrals, each internal line of the Feynmann diagram in the theory under study brings two degrees of the momentum (p^2) more into the dominator of the integral than it occurs in the usual theory. This implies that all the diverging integrals in any usual case of the field theory (up to the four-fermion interaction) in the given theory turn into convergent ones.

4. Causality

The commutator of the operators $\phi'(x)$ and $\phi'(y)$ are written down in the form

$$'\Delta = \int_{-\infty}^{+\infty} \Delta\left((x-y), m^2\right) f^2\left(\frac{m^2}{\ell^2}\right) \frac{/m/}{\ell} \frac{dm}{\ell} \,. \qquad (23)$$

It can be easily seen that if m does not take on the imaginary values, to be more precise, if m^2 is positive everywhere, then $'\Delta$ vanishes outside the light cone, i.e., in the region

$$s = c^2 (t - t')^2 - (\vec{r} - \vec{r}')^2 < 0 \,.$$

Indeed, the integrand in (23) can be put as

$$\Delta(x-y, m^2) = \frac{1}{2\pi}\Sigma(x^0)\,\delta(s^2) - \frac{\sqrt{m^2}}{4\pi\sqrt{s}}\Sigma(x^0)\,\theta(s)\,I_1(m\sqrt{s}) \qquad (24)$$

$$\theta(s) = \begin{cases} 1, & s > 0 \\ 0, & s < 0 . \end{cases}$$

As $\theta(s)$ is independent of m, then, substituting (24) into (23) we see that the property of the commutator in the usual theory to vanish at $s < 0$ holds true also in our case.

The restriction is imposed by the condition $m^2 > 0$ only, for which the Δ function may be written down in the form of (24).

Thus, the condition of the locality of the operators ϕ

$$[\,\phi(x), \phi(y)\,] = 0 \qquad (I)$$

with $s < 0$, provides the fulfilment of the causality principle in the S -matrix constructed on the basis of the field operators ϕ (§ 5).

5. The S- Matrix

For the sake of simplicity, we consider two interacting scalar fields. One of them describes the field whose quantum has the mass M, the field ϕ_M, while the quantum of the other scalar field ψ_o characterizes the particle with zero the rest mass. The field ϕ_M consists of the field ϕ^o_M and the 'counter field' ϕ'_M which is defined according to (1), (5), (11)

$$\phi_M = \phi^o_M + \phi'_M .$$

For the sake of simplicity, we take the field ψ^o in the form of a usual field (without the counterfield) which would satisfy the D'Alembert equation.

The field operators ϕ_M and ψ^o are Hermitian and local.

$$\phi^+_M = \phi_M , \qquad \psi^{+o} = \psi^o$$

$$[\,\phi_M(x), \phi^+_M(y)\,] = 0, \quad [\,\psi^o(x), \psi^o(y)\,] = 0$$

outside the light cone.

The interaction Lagrangian is written down as follows

$$L(x) = g\,\phi_M(x) \quad \psi^{\circ}(x) \tag{25}$$

$$L^{+}(x) = L(x) \tag{26}$$

$$[\,L(x),\, L(y)\,] = 0, \quad if \quad x \sim y\,. \tag{27}$$

Like in the ordinary theory, the n-th term of the S -matrix is put as

$$S_n(x_1 \ldots x_n) = i^n\, T\,(L(x_1) \ldots L(x_n)) \tag{28}$$

and the whole S -matrix

$$S = 1 + \sum_{n=1} \frac{i^n}{n!} \int T\,(L(x_1) \ldots L(x_n)\, dx_1 \ldots dx_n\,. \tag{29}$$

It can be easily verified that the condition of the unitarity is fulfilled. It is given in the form of the relation (I).

$$S_n(x_1 \ldots x_n) + S_n^{+}(x_1 \ldots x_n) + \sum_{1<k<n-1} P\left(\frac{x_1 \ldots x_k}{x_{k+1} \ldots x_n}\right) S_k(x_1 \ldots x_k)\, S^{+}(x_{k+1} \ldots x_n) = 0 \tag{II}$$

where the symbol $P\left(\frac{x_1 \ldots x_k}{x_{k+1} \ldots x_n}\right)$ designates the sum over all the $n!\,/\,k!\,(n-k)!$ ways of breaking up the set of points $x_1 \ldots x_n$ into two sets of k and $n-k$ points. Here, due to the symmetry of S_k by arguments (28) the permutations inside each of these sets are not taken into account. Like in the usual theory, the condition II is fulfilled because the operators L are Hermitian, while the very possibility of writing S_n as the T -product (28) arises as a result of the locality of the operators $L(x)$ which, in its turn, is the consequence of the locality of operators ϕ and ψ .

6. Norm, Vacuum, Damping States

In the scheme under study the amplitude of e.g., single-particle states consists of the two components

$$\phi_1 = \phi_1^{\circ} + \phi'_{1,n} \tag{30}$$

ϕ_1° is identical with the states described by the usual field theory [1]

$$\Phi^{o}_{1} = \int c(\vec{k})\ \overset{+}{\phi}(\vec{k})\, dk\ \Phi^{I}_{o}\ \Phi^{II}_{o} = \Phi^{I}_{1}\ \Phi^{II}_{o}$$

$$k_o = \sqrt{\vec{k}^2 + M^2} \qquad (31)$$

$\Phi'_{1,n}$ describes the state of the counter-field, with the same momentum distribution but in the mass state n

$$\Phi'_{1,n} = \int c(\vec{k})\,\overset{+}{\phi}_n(\vec{k})\ e^{i\overset{o}{k}_m t}\ f(\frac{m^2}{\ell^2})\ H_n(\frac{m}{\ell})\ .$$

$$\sqrt{\frac{m/}{\ell}}\ \frac{dm}{\ell}\ \Phi^{II}_{o}\ \Phi^{I}_{o} = \Phi^{II}_{1,n}\ \Phi^{I}_{o}. \qquad (32)$$

Here just as in the well-known papers of Heisenberg, the so-called ''second vacuum' is introduced: Φ^{II}_{o}. All the Hilbert space is divided thereby into two spaces. One of them (the excited states Φ^{I}_{o}) coincides with the Hilbert space of the modern theory of elementary particles. The second Hilbert space (the states Φ^{II}_{o}) has an auxilliary character. All the excited states Φ^{II}_{o} are damped with time and vanish at $t \to \pm\infty$.

The states of the first and the second vacua are normalized

$$\Phi^{*I}_{o}\ \Phi^{I}_{o} = 1, \qquad \Phi^{*II}_{o}\ \Phi^{II}_{o} = 1.$$

The total vacuum Φ_o is described by the product of the function Φ^{I}_{o} and Φ^{II}_{o}

$$\Phi_o = \Phi^{I}_{o}\ \Phi^{II}_{o}.$$

If the properties of $\overset{+}{\phi}{}^{-}(\vec{k})$ and $\overset{+}{\phi}{}^{-}_{n}(\vec{k})$ are taken into account the norm of the state (30) takes on the form

$$\Phi^{*}_{1}\ \Phi_{1} = \int c^{*}(\vec{k})\ c(\vec{k})\, d\vec{k} - \int \frac{dm}{\ell}\ \frac{dm'}{\ell} \int c^{*}(\vec{k})\ c(\vec{k}) \quad \times \qquad (33)$$

$$\times \quad d\vec{k}\ e^{it(\overset{o}{k}_m - \overset{o}{k}_{m'})}\ f(\frac{m^2}{\ell^2})\ \sqrt{\frac{m/}{\ell}}\ H_n(\frac{m}{\ell})\ f(\frac{m'^2}{\ell^2})\ \sqrt{\frac{m'/}{\ell}}\ H_n(\frac{m'}{\ell})\ .$$

The functions $H_n(\frac{m}{\ell})$ and $f(\frac{m}{\ell^2})$ are chosen so that after the integration over m there arises a strong damping of states (32) with time. Therefore,

$$\Phi^{*}_{1}\ \Phi_{1} \to \int c^{*}(\vec{k})\,c(\vec{k})\ d\vec{k} \qquad t \to \pm\infty\ . \qquad (34)$$

Thus, the normalization of the physical states at $t \to \pm\infty$ coincides with the usual one. It is possible to arrive at the same conclusion in the more general case without restricting to the definite mass state n.

A particular case $(\sum_{n=0}^{\infty} \Phi'_{1,n} a_n)$ with equal coefficients a_n requires a more detailed consideration and some other arguments in favour of the same normalization (34). These arguments are connected with the analysis of the expressions for the probabilities of the corresponding transitions which, as appears, (8 7) are realized only between the functions of the class (31).

7. Probabilities and Cross Sections

The essential feature of the theory we are developing here is that the counter-fields do not introduce the contribution to the probabilities and the cross sections of the physical processes. To be more exact, their role is to regularize the $\Delta^{()}$ functions only.

Consider an example of the same two interacting scalar fields ϕ and ψ^{o}. The matrix element of the scattering in the second order has the form

$$< >_{t} \sim \Phi \ldots \vec{k}'(\ldots \int : \phi_M(x_1)\, \psi_o(x_1)\, \Delta^{c}_{res}(x_1 - x_2)\, \phi_M(x_2) \times \psi_o(x_2) : dx_1\, dx_2\, \Phi \ldots \vec{k} \ldots \qquad (35)$$

where Δ^{c}_{res} is set by (15).

Since ϕ_M consists of two terms $\phi_M = \phi^{o}_M + \phi'_M$ (36), then the matrix element (36) is divided into the sum of the matrix elements

$$= \Phi^{*} \ldots \int : \phi^{o}_M(x_1)\, \psi^{o}(x_1)\, \Delta^{c}_{res}(x_1 - x_2)\, \phi^{o}_M(x_2)\, \psi^{o}(x_2) : dx_1\, dx_2\, \Phi \ldots +$$

$$+ \Phi^{*} \ldots \int : \phi'_n(x_1)\, \psi^{o}(x_1)\, \Delta^{c}_{res}(x_1 - x_2)\, \phi'_n(x_2)\, \psi^{o}(x_2) : dx_1\, dx_2\, \Phi \ldots = \qquad (37)$$

$$= \; < M_1 > + < M_2 >$$

ϕ^{o}_M and ψ^{o} functions describe free particles, ϕ'_n describes the additional counter-field of the free particle in the mass state n. The state $\Phi \ldots$ in (37) is written down in more detail as

$$\Phi_{\ldots} = \Phi^{I}_{1}\, \Phi^{II}_{o} + \Phi^{I}_{o}\, \Phi^{II}_{1,n} \qquad (38)$$

where Φ^{I}_{1} and $\Phi^{II}_{1,n}$ are the amplitudes of the singleparticle states of the field and the counter-field which are given by (31) and (32).

For the sake of simplicity, the amplitude of the field state ψ^{o} is not written down here.

Since, according to the meaning of the S -matrix and to the rule of constructing the matrix elements the states are referred to $t = \pm \infty$,and

$$\Phi^{II}_{1,n} \to 0 \qquad t \to \pm \infty \tag{39}$$

then in (38)

$$\Phi_{\dots} = \Phi^{II}_{1} \; \Phi^{II}_{o} \qquad t \to \infty . \tag{40}$$

Since the vacuum (Φ^{II}_{o}) average of the normal product of the operators ϕ'_{n} vanishes, out of the whole expression (35) remains only the matrix element which has the structure of the matrix element of the usual theory, the Δ^{c} function, however, being substituted for the regularized function (Δ^{c}_{res}) .

The previous consideration shows that the transitions take place between the states which are realized by the excitation of vacuum Φ^{I}_{o} . The transitions between the field and counter-field states must not occur because of the very meaning of introducing the added field what, as we have seen is brought into effect in the developed formalism.

The consideration made holds for the matrix element of any order: in the scheme under study the rule is valid everywhere according to which the calculations are reduced to the matrix element in the usual theory, but with the regularized Δ^{c} functions.

8. Fermi Field

In the case of the Fermi field the counter-field $\psi'^{\pm}(x)$ is written down as follows

$$\psi'^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{n} \int d\vec{k}\, e^{\pm ikx} \; \psi^{\pm}_{n}(mk)\, f\left(\frac{m^2}{\ell^2}\right) H_{n}\left(\frac{m}{\ell}\right) \frac{dm}{\ell} \tag{41}$$

where

$$\psi^{+}_{n}(\vec{k}m) = \frac{\theta(k_o)\,\psi(mk)}{2k_o} ; \qquad \psi^{-}(\vec{k},m) = -\frac{\theta(k_o)\,\psi(-k,m)}{2k_o} \qquad k_o = \sqrt{\vec{k}^2 + m^2} .$$

Let

$$\psi^{\pm}_{n\sigma}(\vec{k}m) = \sum a^{\pm}_{\nu n}(\vec{k})\, V^{\nu\pm}_{\sigma}(\vec{k}m) . \tag{42}$$

The counter-field $\psi^{\pm\prime}(x)$ is postulated in an explicit form of expression (41), where the operators $a^{\pm}_{\nu n}(\vec{k})$ are independent of the parameter m . It is postulated that

$$[\, a^{+}_{in}(\vec{k}),\ \bar{a}_{jn'}(\vec{k}')\,] = -\,\delta(\vec{k}-\vec{k}')\,\delta_{ij}\,\delta_{nn'} . \qquad (43)$$

In contrast to usual theory here an opposite sign is chosen in the quantization of the counter-field.

Let

$$\sum_{\nu=1,2} V^{\nu-}_{\alpha}(\vec{k})\, V^{\nu+}_{\beta}(\vec{k}) = \left(\frac{\hat{k}+(m+M)}{2k_o}\right)_{\alpha\beta} \qquad (44)$$

where M is the particle mass.

The corresponding commutator is written down as

$$[\, \bar{\psi}_{\alpha}(x),\ \bar{\psi}^{+}_{\beta}(y)\,]_{+} = \frac{1}{(2\pi)^3} \sum_{n} \sum_{n'} \int d\vec{k}\, d\vec{k}'\, e^{i(k'y-kx)} \times$$

$$\times \sum V^{-\nu}_{\alpha}(\vec{k}\,m)\, V^{M+}_{\beta}(\vec{k}'\,m)\,[\, \bar{a}_{\nu n}(\vec{k})\, a^{*+}_{\mu n}(\vec{k}')\,] +$$

$$+ f\left(\frac{m^2}{\ell^2}\right) H_n\left(\frac{m}{\ell}\right) H_n\left(\frac{m'}{\ell}\right) \sqrt{\frac{/m/\,/m'/}{\ell^2}}\, \frac{dm}{\ell}\, \frac{dm'}{\ell} .$$

According to (42)

$$[\,]_{+} = -\frac{1}{(2\pi)^3} \int d\vec{k}\, \exp\, \{\, i\,[\vec{k}(\vec{y}-\vec{x}) - \sqrt{\vec{k}^2+m'^2}\, y_t +$$

$$+ \sqrt{\vec{k}^2+m^2}\, x_t\, \}\, \sum_{\nu} V^{-\nu}_{\alpha}(\vec{k}\,m)\, V^{+\nu}_{\mu}(\vec{k}\,\bar{m}).$$

$$f\left(\frac{m^2}{\ell^2}\right) f\left(\frac{m^2}{\ell^2}\right) \sqrt{\frac{/m/\,/m'/}{\ell}} \sum_{n} H_n\left(\frac{m}{\ell}\right) H_n\left(\frac{m'}{\ell}\right) \frac{dm}{\ell}\, \frac{dm'}{\ell} . \qquad (45)$$

According to (6) and (44)

$$[\,]_{-} = -\frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dm}{\ell} \int d\vec{k}\, e^{ik(y-x)}\, f^2\left(\frac{m^2}{\ell^2}\right) \left(\frac{\hat{k}+m+m}{2k_o}\right)_{\alpha\beta} \frac{/m/}{\ell} \qquad (46)$$

where $k_o = \sqrt{\vec{k}^2+m^2}$

or

$$[\,]_+ = -\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dm}{\ell}\int d\vec{k}\,(\hat{k}+M)_{\alpha\mu}\,\delta(k^2-m^2)\,\theta(k_o)\,e^{ik(y-x)}\,f^2\left(\frac{m^2}{\ell^2}\right)\frac{/m/}{\ell} \qquad (47)$$

or

$$[\,]_+ = -\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dm}{\ell}\,f^2\left(\frac{m^2}{\ell^2}\right)\frac{/m/}{\ell}\left(i\gamma_\mu\frac{\partial}{\partial x_\mu}+M\right)i\Delta^+(y-x,m) \qquad (48)$$

or

$$[\,]_+ = -\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dm}{\ell}\,f^2\left(\frac{m^2}{\ell^2}\right)\frac{/m/}{\ell}\left\{iS^+(y-x,m)+iM\Delta^+(y-x,m)\right\} \qquad (49)$$

the equivalence of expressions (48) and (49) is based on the fact that

$$\int_{-\infty}^{+\infty} m\;F\left(\frac{m^2}{\ell^2}\right)\frac{dm}{\ell} = 0$$

due to the parity of the integrand.

In deriving the commutator in the form (49) it was possible not to introduce a somewhat unusual normalization (44), but to represent the counter-field $\psi'(x)$ as two fields

$$\psi'(x) = \psi_1'(x) + \sqrt{M}\;\psi_2' \qquad (50)$$

where the normalization for V_1 instead of (44) is

$$\sum_{\nu=1,2} V_{1\alpha}^{\nu-}(\vec{k})\,V_{1\beta}^{\nu+}(\vec{k}) = \left(\frac{\hat{k}+m}{2k_o}\right)_{\alpha\beta} \qquad (51)$$

while for V_2 of the field ψ_2' is

$$\sum V_{2\alpha}^{\nu'}(\vec{k})\,V_{2\beta}^{\nu+}(\vec{k}) = \left(\frac{1}{2k_o}\right)_{\alpha\beta}. \qquad (52)$$

In the theory we are developing there is no description of the events in time. The theory does not contain the equations of motion for the so-called 'free' field which forms the basis for constructing the S-matrix.

In the present theory the time, in difference to space coordinates is considered to be not an operator but a C-number

$$[\vec{p}\ \vec{x}] \neq 0; \qquad [E\ t] = 0. \qquad (53)$$

This asymmetry between time and space arises due to the Dirac equation (equation of motion) which chooses just such a representation of the Lorentz group /4/. In other words, owing to the equation of motion the energy is expressed in terms of the momentum ($E^2 = p^2 + m^2$) which commutates with time.

In our case there is no equation of motion which 'makes' time by the parameter, i.e. by the c -number. Therefore, it is reasonable that at small t, particles produced have no definite energy, to be more exact, have no definite mass (since all the ambiguity in $\sqrt{\vec{k}^2 + m^2}$ of the function (2) with the fixed $\vec{k}$ is connected just with the ambiguity in m).

The character of the ambiguity (the spread) is defined by the functions f and H_n. As we have seen, the results of the calculations are independent of the choice of the concrete functions H_n either in the initial or in the final functions of the counter-field. This circumstance corresponds to the fact that within the framework of the developed theory, there appears to be no possibility of choosing any particular H_n function or their superposition.

In this theory are not formulated such notions as the energy and momentum tensors the conservation laws in the form of divergence. These are too detailed descriptions for such a theory. But at $t \to \pm\infty$ the energy, momentum can be regarded as the quantities of the ordinary theory, and the δ functions appearing in the matrix elements provide for the corresponding conservation laws.

We were not concerned with the gauge invariance problems in the electrodynamics. Generally speaking, the gauge invariance is provided for the regularization of the products of the Δ^c function as a whole, but not of the Δ^c functions, taken separately, what occurs in the given theory. To tell the bruth, here an arbitrary function $f\left(\frac{m^2}{\ell^2}\right)$ in available, the momenta of which one can still make use of. In general, there can hardly exist a consistent theory describing only one field (for instance, electrodynamics) isolated from other fields. Therefore, a more consistent scheme of the field theory seems to us that developed in Heisenberg's papers. It is the future investigations, in particular – four fermion investigations, that we have developed present formalism for.

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