ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

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QUANTUM ELECTRODYNAMICS IN TERMS OF ELECTROMAGNETIC FIELD STRENGTHS

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Abstract

The Lorentz-invariant formulation of quantum electrodynamics is given, which does not involve potentials but electromagnetic field strengths only.

Till now nobody succeded in writing down equations of quantum electrodynamics in terms of electromagnetic field strengths only, in contrast with classical Maxwell electrodynamics. Quantum equations of motion contain usually 4-vector-potential A_μ .

This caused difficulties. So, it is well known that the Maxwell equations for A_μ can not be quantized. In fact they determined A_μ on principle up to the 4-gradient of an arbitrary function. This means A_μ have the component, for which the equation of motion does not exist/1/. Therefore it is not possible to write down the commutation relation for A_μ . In conventional approach one restricts the gauge in some degree and obtains, therefore, the equation of motion which defined A_μ more strictly. Because of this they permit quantization. The disadvantages of such formulations are well-known. The Coulomb gauge formulation (Dirac) is most consistent and clear one. However it does not explicitly covariant and in contrast to Maxwell theory one needs to consider interaction due to transversal quanta and Coulomb interaction separately. In a Fermi's formulation one had to introduce nonphysical indefinite metric (even this can not be done in an explicitly covariant form). Therefore many authors are seeking to overcome the quantization difficulty of a theory, which explores Maxwell equations, by means of giving commutation relaion not for A_μ but for gauge independent quantities only/2-8/, e.g. for field strengths. In such theories, however either the vector-potential excluded not entirely and it is difficult to operate with it or explicit Lorentz-invariance is absent.

Some authors are inclined to consider these difficulties as an indication that vector potential in quantum theory has an independent significance in contrast to the classical theory (/9/ see also/8/).

In this paper it is shown that quantum electrodynamics can be built from the very outset in terms of electromagnetic field strengths only in an explicitly covariant manner. This formulation is based on our previous paper/1/. The interaction of a charged field with photons possesses seeming nonlocality and can be written down in many equivalent forms. The consideration is carried out in interaction picture. It's aim is to demonstrate the possibility of carrying out all the calculations without any references to the vector-potential. In this respect there are no differences between classical and quantum theories.

2. In an interaction picture free field operators obey Maxwell and Dirac equations

$$\partial_\mu F_{\mu\nu} = 0 \tag{1}$$

$$\partial_\mu \overset{\vee}{F}_{\mu\nu} = 0 \qquad /\, \overset{\vee}{F}_{\mu\nu} = \tfrac{1}{2}\, \epsilon_{\mu\nu\lambda\rho} F_{\lambda\rho} / \tag{2}$$

$$F_{\mu\nu} = - F_{\nu\mu} \tag{3}$$

$$(\gamma_\mu \partial_\mu + m)\, \psi = 0 \tag{4}$$

where $F_{\mu\nu}$ is the electromagnetic field strength operator and ψ is spinor field operator. These equations are compatible with the well-known commutation relations*.

* We have already note the known fact that for potentials there is not possible to write down the commutation relations which would be compatible with Maxwell equations.

$$[F_{\mu\nu}(x), F_{\lambda\rho}(y)] =$$

$$[\delta_{\nu\rho}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\lambda} - \delta_{\mu\rho}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\lambda} - \delta_{\nu\lambda}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\rho} + \delta_{\mu\lambda}\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial y_\rho}] \cdot i\Delta(x-y) \qquad (5)$$

$$\{\psi(x), \psi(y)\} = 0; \quad \{\psi(x), \bar{\psi}(y)\} = -iS(x-y) \qquad (6)$$

The equation (1) can be derived by varying the lagrangian $-\frac{1}{4}\int d^4x\, F_{\mu\nu}(x) F_{\mu\nu}(x)$ provided equation (2) is satisfied. The commutation relation (5) can be obtained according to the Peierls procedure /10/.

Fourier-expansion for $F_{\mu\nu}(x)$ and the commutation relations in p -representation are given in Appendix.

3. Let us take an interaction lagrangian in the form

$$L_I(x) = ie: \bar{\psi}(x)\gamma_\nu\psi(x)\,\partial_\mu \Box^{-1} F_{\mu\nu}(x): \qquad (7)$$

The operator $\Box^{-1}$ can be understood as an integral operator of the convolution with any Green function of D'Alembert equation

$$\Box^{-1} f(x) = \int dy\, G(x-y)\, f(y)$$

$$\Box\, G(x-y) = \delta(x-y). \qquad (8)$$

It is not necessary to remove an arbitrariness in a choice of Green function G. As we shall see below any choice will lead to the same results.

Let us note also that from equation (1) it follows vanishing of $\Box\,\partial_\mu \Box^{-1} F_{\mu\nu}$ but not of $\partial_\mu \Box^{-1} F_{\mu\nu}$. We shall take the convention to apply firstly the operator $\Box^{-1}$.

As to $S-$ matrix we shall require only, that it must satisfy conditions of Lorentz-invariance, unitarity and causality in the Stueckelberg-Bogolubov's spirit /11/. Let us take it in the form

$$S = T^* \exp[\, i\int dx\, L_I(x)\,] \qquad (9)$$

where notation T^* imply only that in normal form of S -matrix the following contractions for quantities $\partial_\mu \Box^{-1} F_{\mu\nu}(x)$ must be taken

$$\frac{\partial}{\partial x_\mu}\Box_x^{-1} F_{\mu\nu}(x)\,\frac{\partial}{\partial y_\lambda}\Box_y^{-1} F_{\lambda\rho}(y) = (\delta_{\nu\rho} - \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\rho}\Box^{-1})(-i)\Delta^c(x-y) \qquad (10)$$

and the usual contraction must be taken for spinor field. Propagators (10) correspond to electrodynamics in the Landau-Khalatnikov's gauge. The term with derivatives in (10) is not essential because of the current conservation and of vanishing of the equal-time current's commutator. Thus, all the coefficient functions turns out to be the same as in conventional electrodynamics. It is worthwhile to note that contraction (10) can be considered as a result of the action of differential and integral operators on contraction $F_{\mu\nu}(x)$ defined by Hori[/12/].

$$\overbrace{F_{\mu\nu}(x)\, F_{\lambda\rho}}(y) = \Big(\delta_{\nu\rho} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\lambda} - \delta_{\mu\rho} \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial y_\lambda} - \delta_{\nu\lambda} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} + \delta_{\mu\lambda} \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial y_\rho} \Big)(-i)\, \Delta^c(x-y)$$

Instead of postulating contraction (10) it might be more consistent to build S -matrix for nonlocal lagrangian (7) by Kirgnitz method[/15/] involving $\tilde{T}_g$ ordering.

4. The only nonusual constituent in the normal form of S -matrix are N -products

$$: \frac{\partial}{\partial x^1_{\mu_1}} \Box^{-1} F_{\mu_1\nu_1}(x^1) \,\ldots\ldots\ldots\, \frac{\partial}{\partial x^n_{\mu_n}} \Box^{-1} F_{\mu_n\nu_n}(x^n) : \qquad (11)$$

It can be shown that the calculation of matrix elements of such N -products don't lead to difficulties. For this sake first of all we define the proton state as a result of action on vacuum of the negative frequency part of field strength tensor $F_{\mu\nu}$. The one-particle state with definite momentum will have the form*

$$\frac{1}{q_0} \mathfrak{f}^{+}_{\mu\nu}(\vec{q})\, \Psi_0 \qquad (12)$$

With ortonormalizability condition

$$\Big(\frac{1}{p_0} \mathfrak{f}^{+}_{\mu\nu}(\vec{p})\, \Psi_0\, \frac{1}{q_0} \mathfrak{f}^{+}_{\lambda\rho}(\vec{q})\Psi_0\Big) =$$

$$= \frac{1}{p_0^2}\{ p_\mu p_\lambda \delta_{\nu\rho} - p_\mu p_\rho \delta_{\nu\lambda} - p_\nu p_\lambda \delta_{\mu\rho} + p_\nu p_\rho \delta_{\mu\lambda}\}\, \delta(\vec{p} - \vec{q}).$$

A photon state will be defined if the strength of its electric field $(E_m = i F_{m4})$ or magnetic field $H_m = i \overset{\nu}{F}_{m4}$ or $F_{\mu\nu} \pm \overset{\nu}{F}_{\mu\nu}$ (i. e. $\vec{E} \pm i\vec{H}$)** is given. If we consider the first possibility (the other possibilities can be considered similarly) then a normalized state vector will have the form

$$\frac{-i}{q_0} \mathfrak{f}^{+}_{\mu 4}(\vec{q})\, \Psi_0 = \frac{1}{q_0} \mathbf{E}^{+}_{m}(\vec{q})\, \Psi_0. \qquad (13)$$

The normalization condition for a state vector is

$$\Big(\frac{1}{p_0} \mathbf{E}^{+}_{m}(\vec{p})\, \Psi_0,\ \frac{1}{q_0} \mathbf{E}^{+}_{n}(\vec{q})\Psi_0 \Big) = \Big(\delta_{mn} - \frac{p_m p_n}{p_0^2}\Big)\, \delta(\vec{p} - \vec{q}). \qquad (14)$$

The convolution of indices m and n (sum over spin states) give $2\,\delta(\vec{p} - \vec{q})$, where factor 2 corresponds to a number of independent spin states.

By means of the relation

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* Operators $\mathfrak{f}_{\mu\nu}(\vec{q})$ and their properties are defined in an Appendix.

** Last quantities transform by irreducible representations (0,1) and (1,0) of the homogeneous Lorentz group.

$$[\partial_\mu \Box^{-1} F_{\mu\nu}(x), \mathfrak{L}^{+}_{\lambda\rho}(\vec{q})] = i(\delta_{\nu\lambda} q_\rho - \delta_{\nu\rho} q_\lambda) \frac{\exp(iqx)}{\sqrt{(2\pi)^3 2q_0}}$$

(which is the consequence of Eqs. (A2) and (A5) of Appendix) we obtain

$$(\Psi_0, \partial_\mu \Box^{-1} F_{\mu\nu}(x) \frac{-i}{q_0} \mathfrak{L}_{\lambda 4}(\vec{q}) \Psi_0) = \tag{15}$$

$$\frac{1}{q_0}(\delta_{\nu\lambda} q_4 - \delta_{\nu 4} q_\lambda) \frac{\exp(iqx)}{\sqrt{(2\pi)^3 2q_0}} .$$

Therefore in the Feynman graph the factor $\frac{1}{q_0}(\delta_{\nu\lambda} q_4 - \delta_{\nu 4} q_\lambda)$ corresponds to an external photon, which is connected with a current j_ν and which spin state is characterized by λ. This solves the problem of the matrix elements calculation of the normal product (11). The summation of the squared matrix element over photon spin states gives

$$-\frac{1}{q_0^2}(\delta_{\nu\nu 1} q_4^2 - \delta_{\nu 4} q_{\nu 1} q_4 - \delta_{\nu 4} q_\nu q_4). \tag{16}$$

Taking into account the conservation of current and vanishing of the equal-time commutator of currents we see that only first term gives contribution. Therefore we obtain the Feynman' rule for summation over photon spin states. If photon state vectors are constructed by application of creation operators $\mathfrak{C}^{+}(\vec{p}, s)$ (see Appendix) then the S- matrix element for n-photon processes will be written down in the following visual form

$$\langle f | S | i \rangle \sim \tag{17}$$

$$\sim \bar{u} \ldots \gamma_{\mu_1} \ldots \gamma_{\mu_n} \ldots u F_{\mu_1 4}(\vec{q}_1 s_1) \ldots\ldots F_{\mu_n 4}(\vec{q}_n s_n).$$

5. Thus, the formulation of quantum electrodynamics has been given in which one needs not make any reference to a vector potential. We have used experimentally measurable and uniquelly defined by Maxwell equations field's strength's $\vec{E}$ and $\vec{H}$ only.

This theory is based on our previous paper/1/. In /1/ decomposition of a potential into gauge-independent and gauge-dependent parts has been given

$$A_\mu = (A_\mu - \partial_\mu \partial_\nu \Box^{-1} A_\nu) + \partial_\mu \partial_\nu \Box^{-1} A_\nu = \partial_\nu \Box^{-1} F_{\nu\mu} + \partial_\mu \partial_\nu \Box^{-1} A_\nu$$

The gauge-dependent part of A_μ can be removed by the transformation $\psi \to \exp(i \Box^{-1} \partial_\nu A_\nu)\psi$ after which an interaction lagrangian takes form (7).

Having been defined nonuniquely (up to 4-gradient) the potential A_μ is nonmeasurable. Our definition of an gauge independent part is also arbitrary in some degree. (Any Green function of the D'Alembert equation can be taken as the kernel of an integral operator $\Box^{-1}$). Such an arbitrariness is also leaved in the given formulation of quantum electrodynamics. However it does not concern the photon operator $(F_{\mu\nu})$ and means a freedom in the choice of one of many equivalent forms for writing down S -matrix in terms of uniquely defined photon operator.

Appendix

3- and 4- dimensional Fourier expansions for strength operators $F_{\mu\nu}(x)$ may be written in the form

$$F_{\mu\nu}(x) = \int \frac{d\vec{p}}{\sqrt{(2\pi)^3}\, 2p_0} \{ \mathfrak{f}_{\mu\nu}(\vec{p})\, \ell^{ipx} + \overset{+}{\mathfrak{f}}_{\mu\nu}(\vec{p})\, \ell^{-ipx} \} \tag{A.1}$$

(here $p_0 = |\vec{p}\,|$) and

$$F_{\mu\nu}(x) = \int \frac{d^4 p}{\sqrt{(2\pi)^3}} \{ f_{\mu\nu}(p)\, \ell^{ipx} + \overset{+}{f}_{\mu\nu}(p)\, \ell^{-ipx} \} . \tag{A.2}$$

Operators $\mathfrak{f}_{\mu\nu}$ and $f_{\mu\nu}$ satisfy Maxwell equations

$$p_\mu f_{\mu\nu} = 0 , \qquad p_\mu \check{f}_{\mu\nu} = 0 .$$

and are connected by relations

$$f_{\mu\nu}(p) = \delta(p^2)\, \theta(p_0) \sqrt{2p_0}\; \mathfrak{f}_{\mu\nu}(\vec{p}) ; \qquad \mathfrak{f}_{\mu\nu}(\vec{p}) = \int dp_0\, f_{\mu\nu}(p) \sqrt{2p_0}$$

Following commutation relations take place for them

$$[\mathfrak{f}_{\mu\nu}(\vec{p}), \overset{+}{\mathfrak{f}}_{\lambda\rho}(\vec{q})\,] =$$

$$= [p_\mu p_\lambda \delta_{\nu\rho} - p_\mu p_\rho \delta_{\nu\lambda} - p_\nu p_\lambda \delta_{\mu\rho} + p_\nu p_\rho \delta_{\mu\lambda}]_{(p_0 = |\vec{p}\,|)}\, \delta(\vec{p} - \vec{q}\,) \tag{A.3}$$

$$[\, f_{\mu\nu}(p), \overset{+}{f}_{\lambda\rho}(q)\,] =$$

$$= \{ p_\mu p_\lambda \delta_{\nu\rho} - p_\mu p_\rho \delta_{\nu\lambda} - p_\nu p_\lambda \delta_{\mu\rho} + p_\nu p_\rho \delta_{\mu\lambda} \}\, \delta(p-q)\, \theta(p_0)\, \delta(p^2) \tag{A.4}$$

$$[\, f_{\mu\nu}(p), \mathfrak{f}^{+}_{\lambda\rho}(\vec{q})\,] =$$

$$= \{ p_{\mu}p_{\lambda}\,\delta_{\nu\rho} - p_{\mu}p_{\rho}\,\delta_{\nu\lambda} - p_{\nu}p_{\lambda}\,\delta_{\mu\rho} + p_{\nu}p_{\rho}\delta_{\mu\lambda}\}\,\delta(\vec{p}-\vec{q})\,\theta(p_0)\,\delta(p^2)\,\sqrt{2p_0} \qquad (A.5)$$

Another commutators vanish. When applying creation and annihilation operators $\mathfrak{f}^{+}_{\mu\nu}(f^{+}_{\mu\nu})$ and $\mathfrak{f}_{\mu\nu}(f_{\mu\nu})$ we imply that a photon spin state is characterized by indices (μ, ν).

We can pass also to the expansion

$$\mathfrak{f}_{\mu\nu}(\vec{p}) = \sum_{s=1}^{2} \mathfrak{c}(\vec{p}, s)\, F_{\mu\nu}(\vec{p}, s) \qquad (A.6)$$

where s is spin index, $F_{\mu\nu}(\vec{p}, s)$ are c-number solutions of Maxwell equations (photon wave functions[14]) and $\mathfrak{c}(\vec{p}, s)$ are annihilation operators with commutation relations

$$[\,\mathfrak{c}(\vec{p}, s), \mathfrak{c}^{+}(\vec{q}, s')\,] = \delta_{ss'}\,\delta(\vec{p}-\vec{q}). \qquad (A.7)$$

Fourier expansion (A.1) will be written down in these terms as

$$F_{\mu\nu}(x) = \sum_{s=1}^{2} \int \frac{d\vec{p}}{\sqrt{(2\pi)^3 2p_0}} \{\mathfrak{c}(\vec{p}, s)\, F_{\mu\nu}(\vec{p}, s)\exp(ipx) + \mathfrak{c}^{+}(\vec{p}, s)\, F_{\mu\nu}(\vec{p}, s)\exp(-ipx)\} . \qquad (A.8)$$

In conclusion we give an useful formula for the sum over spin states (in an unnormalized form)

$$\sum_{s=1}^{2} F_{\mu\nu}(\vec{p}, s)\, F^{+}_{\lambda\rho}(\vec{p}, s) = \delta_{\mu\lambda}\, p_{\nu}p_{\rho} - \delta_{\nu\lambda}\, p_{\mu}p_{\rho} - \delta_{\mu\rho}p_{\nu}p_{\lambda} + \delta_{\nu\rho}p_{\mu}p_{\lambda} . \qquad (A.9)$$

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