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THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF MANDELSTAM's EQUATIONS FOR PION-PION SCATTERING

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Abstract

It is shown, using the strip approximation, that there are two possible asymptotic forms of the scattering amplitude. The first type of solutions can not give a constant total cross section, the second type asymptotic behaviour exactly the same as given by Regge/1/: $A_s(s, t) \approx f(t)\, s^{L(t)}$, if $s \to \infty$.

Аннотация

Показано, что с помощью полосового приближения получены две возможные асимптотические формы амплитуды рассеяния. Первый тип решений не может дать постоянного значения полного сечения, второй тип асимптотического поведения совершенно идентичен полученному Редже/1/.

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I. Introduction

A great effort has recently been done to determine the scattering amplitude at very high energies. According to Regge[1] the asymptotic form of the imaginary part of the scattering amplitude for nonrelativistic quantum mechanics, for a given class of potentials has the form:

$$A_t(t, s) = \Sigma f_i(s)\, t^{L_i(s)} , \quad \text{if} \quad t \to \infty ; \qquad (1.1)$$

where s is the total energy squared in the c.m.s. system, t is the momentum transfer.

Chew and Frautschi[2], using the unitarity condition in the third chanel obtained an expression for Mandelstam's spectral function in the strip approximation (here under 'strip' we understand the interval $4 < t < 16$):

$$\rho(s, t) = \frac{1}{16\pi^2} \int\int ds_1\, ds_2 \frac{A^*_s(s_1, t)\, A_s(s_2, t)}{\sqrt{\bar{t}}\, \sqrt{t(t-t_o)}} , \qquad (1.2)$$

where

$$\bar{t} = s^2 - 2s\,(s_1 + s_2) + (s_1 - s_2)^2 ,$$

and

$$t_o = 4 + \frac{4 s s_1 s_2}{\bar{t}} .$$

The upper limit of the integrations is given by the following inequality:

$$s > s_1 + s_2 + \frac{2 s_1 s_2}{t-4} + \sqrt{\left(s_1 + s_2 + \frac{2 s_1 s_2}{t-4}\right)^2 - (s_1 - s_2)^2} \qquad (1.3)$$

where $s_1 + s_2$ is always less than s .

Amati and others[3], using this fact, obtained an iterative solution of equation (1.2), using the dispersion relation for $A_s(s, t)$:

$$A_s(s, t) = \frac{1}{\pi} \int \frac{\rho(s, t')}{t' - t}\, dt' . \qquad (1.4)$$

At finite values of t the left hand cut does not appear, if $s \to \infty$. They succeeded in geting the first approximation analytically and it was in good agreement with experimental data in the BeV region.

Independently of them Domokos[4] obtained a similar amplitude by an other method.

However, all this papers give the amplitude in a limited region.

In an other paper Domokos[5] assumed, that the imaginary part of the amplitude in the crossed chanel has a form given by Regge[1]. He found it satisfies equation (1.2) and he was able to obtain an equation for the exponent $L(t)$.

There arises a very interesting question, which are the possible asymptotic behaviours of amplitudes satisfying (1.2) and (1.4).

2. Integral Equation for the Amplitude

We try to give an other form to equation (1.2). Therefore we divide the region of integration into three parts (as may be seen on Fig. 1). λ is chosen so as to satisfy the inequalities:

$$\lambda \gg 1, \qquad \lambda \ll s. \tag{2.1}$$

The expressions for a and b on the figures and their asymptotic forms are:

$$a = \lambda + s + \frac{2\lambda s}{t-4} - \sqrt{\left(\lambda + s + \frac{2\lambda s}{t-4}\right)^2 - (\lambda - s)^2} \approx \frac{s(t-4)}{4\lambda} \equiv s\, h(\lambda, t) \ll s$$

$$b = 4 + s + \frac{8s}{t-4} - \sqrt{\left(4 + s + \frac{8s}{t-4}\right)^2 - (4-s)^2} \approx s\left(1 + \frac{8}{t-4} - \sqrt{\left(1+\frac{8}{t-4}\right)^2 - 1}\right) \equiv s\,h(4, t). \tag{2.2}$$

We denote the integrals for the appropriate regions by I_1 , I_2 , I_3 respectively.

In the integrals I_1 and I_2 we have always $s_2 < \lambda \ll s$ (see Fig. 1) and therefore asymptoticaly the expression of f will be simply $f = (s - s_1)^2$.

In the integral I_3 $s_1 < s \ll s$ and so the asymptotic form of f will be: $f = (s - s_2)^2$.

So after inserting asymptotic expressions, the integrals I_1 , I_2 and I_3 take on the following form:

$$I_1 = \int_4^a \frac{ds_1}{s_1} A_s^*(s_1, t)\, F_1\left(\frac{s}{s_1}, t\right), \tag{2.3}$$

$$I_2 = \int_a^b \frac{ds_1}{s_1} A_s^*(s_1, t)\, F_2\left(\frac{s}{s_1}, t\right),$$

$$I_3 = \int_\lambda^b \frac{ds_2}{s_2} A_s(s_2, t)\, F_2^*\left(\frac{s}{s_2}, t\right),$$

where

$$F_1(z, t) = \frac{1}{16\pi^2}\, \frac{1}{z-1} \int_4^\lambda ds_2 \frac{A_s(s_2, t)}{\sqrt{t(t-t_o)}} ,$$

$$F_2(z, t) = \frac{1}{16\pi^2}\, \frac{1}{z-1} \int_4^{\frac{(z-1)^2(t-4)}{4z}} ds_2 \frac{A_s(s_2, t)}{\sqrt{t(t-t_o)}} .$$

Here
$$t_o = 4 + \frac{4 z s_2}{(z-1)^2} .$$

For sake of simplicity we omit the contribution of the integral:

$$\int_4^\lambda \frac{ds_1}{s_1} A_s^*(s_1, t)\, F_1\left(\frac{s}{s_1}, t\right) \sim 0(1)$$

from integral I_1 . We hope the spectral function goes to infinity if $s \to \infty$, and in this case this integral gives only a small contribution.

We know the spectral function $\rho(s, t)$ is real, so we may write:

$$\rho(s,t) = \frac{1}{2}[I_1 + I_2 + I_3 + I^*_1 + I^*_2 + I^*_3].$$

Using the well known connection between $\rho(s, t)$ and $A_s(s,t)$ we obtain the integral equation:

$$A_s(s,t) - \int_\lambda^\infty \frac{ds_1}{s_1} A_s(s_1,t)\, F\left(\frac{s}{s_1}, t\right) =$$

$$= A^*_s(s,t) - \int_\lambda^\infty \frac{ds_1}{s_1} A^*_s(s_1,t)\, F^*\left(\frac{s}{s_1}, t\right), \tag{2.4}$$

where

$$F(z,t) = -i\,\Theta\left(z - \frac{1}{h(\lambda,t)}\right) F^*_1(z,t) + i\,\Theta\left(z - \frac{1}{h(4,t)}\right)\Theta\left(\frac{1}{h(\lambda,t)} - z\right) F^*_2(z,t) +$$

$$+ \; i\,\Theta\left(z - \frac{1}{h(4,t)}\right) F^*_2(z,t).$$

We look for the solution of equation (2.4) satisfying dispersion relation (1.4).

3. The Method of Solving of the Integral Equation

The form of the solutions of equation (2.4) may be obtained by Mellin transformation. The transforms of the functions of equation (2.4) are the following:

$$\Phi_+(L) = \int_\lambda^\infty A_s(s,t)\, s^{L-1}\, ds, \qquad \overset{+}{\Phi}_+(L) = \int_\lambda^\infty A_s^*(s,t)\, s^{L-1}\, ds,$$

$$\Phi_-(L) = \int_0^\lambda A_s(s,t)\, s^{L-1}\, ds, \qquad \overset{+}{\Phi}_-(L) = \int_0^\lambda A_s^*(s,t)\, s^{L-1}\, ds, \tag{3.1}$$

$$V(L) = \int_0^\infty F(z,t)\, z^{L-1}\, dz, \qquad V^+(L) = \int_0^\infty F^*(z,t)\, z^{L-1}\, dz,$$

respectively.

The following equalities are satisfied:

$$\Phi^*_+(L^*) = \overset{+}{\Phi}_+(L), \quad \Phi^*_-(L^*) = \overset{+}{\Phi}_-(L), \quad V^*(L^*) = V^+(L). \tag{3.2}$$

The inversion formulas of transformations (3.1) are:

$$A_s(s,t) = \int_{-i\infty+\sigma'_1}^{i\infty+\sigma'_1} \frac{\Phi_+(L)}{s^L}\, dL + \int_{-i\infty+\sigma'_0}^{i\infty+\sigma'_0} \frac{\Phi_-(L)}{s^L}\, dL,$$

$$A^*_s(s,t) = \int_{-i\infty+\sigma'_1}^{i\infty+\sigma'_1} \frac{\overset{+}{\Phi}_+(L)}{s^L}\, dL + \int_{-i\infty+\sigma'_0}^{i\infty+\sigma'_0} \frac{\overset{+}{\Phi}_-(L)}{s^L}\, dL, \tag{3.3}$$

$$F(z,t) = \int_{-i\infty+r'}^{i\infty+r'} \frac{V(L)}{z^L}\, dL, \qquad F^*(z,t) = \int_{-i\infty+r'}^{i\infty+r'} \frac{V^+(L)}{z^L}\, dL$$

the functions $\Phi_+(L)$, $\Phi_-(L)$ and $V(L)$ are defined in a domain bounded by the straight lines ReL equal to σ_1, σ_o and r respectively.

$V(L)$ is defined from $-\infty$ to r, because as it may be seen from (2.4), $F(z,t) = 0$ for small values of z.

If $A_s(s,t)$ has no essential singularity at infinity, there exists such $\sigma'_1 < r$, for which $\Phi_+(L)$ and $\Phi^+_+(L)$ have no singularity for L values $L < \sigma'_1$ (for sufficiently high values of λ).

Then if follows from equation (2.4)[6]:

$$\Phi_+(L)(1 - V(L)) - \Phi^+_+(L)(1 - V^+(L)) = C(L)$$
$$\Phi_-(L) - \Phi^+_-(L) = -\hat{C}(L) \qquad (3.4)$$

where $C(L)$ is an arbitrary function holomorphic in the strip $\sigma'_1 < ReL < r'_o$.

4. The First Type of Solutions of Equation (3.4)

First we examine the analytic properties of $\Phi_-(L)$. (3.4) shows, that in our approximation for small energies $(s < \lambda)$ Mandelstam's spectral function equals to zero in the interval $4 < t < 16$ [1].

The singularities of $\Phi_-(L)$ are given by the elastic part (in the s chanel) of the spectral function. But we know the analytic properties of $A_s(s,t)$ as a function of s.

Oehme[7] has shown, that $A_s(s,t)$ has two branching points in $s = 0$ and $s = 4$, and the types of the singularities are $\sqrt{s}$ and $\sqrt{s-4}$ [2]. Expanding into asymptotic series easy to show that the singularities of $\Phi_-(L)$ are at the points $L = 0, \pm 1/2, \pm 1, \ldots$ [3].

In the equation (3.4) $V(L)$ and $V^+(L)$ still have dependence on high energetic values of $A^*_s(s,t)$ (see formulas (2.3), (2.4) and (3.1)). But it is easy to show that for sufficiently large λ values the third term of expression (2.4) has the following form:

$$i\,r(L)\Phi^+_+(L) + \frac{i}{16\pi^2}\int_0^{\frac{t-4}{4}} \frac{du}{\sqrt{t(t-4-4u)}} \int_0^{\lambda/u} dz\,(z^{L-1} - z^{L-2})\,\Theta\left(z - \frac{1}{h(4,t)}\right) \times \qquad (4.1)$$
$$\times A^*_s\left(\frac{(z-1)^2}{z}u, t\right),$$

where $r(L) = \frac{1}{16\pi^2}\int_0^{\frac{t-4}{4}} \frac{u^{-L}\,du}{\sqrt{t(t-4-4u)}}$ [4].

1/ This is the effect of omitting the finite contribution from I_I. It is obvious that $\Phi_-(L) - \Phi^+_-(L) = \int_4^\lambda s^{L-1} ds(A_s(s,t) - A^*_s(s,t))$ is holomorphic everywhere, only if $A_s(s,t) - A^*_s(s,t) = 0$.

2/ Exactly speaking Oehme has shown this form to be correct for the partial waves. We assume this properties to hold for the whole function $A_s(s,t)$.

3/ A simple example: $A_s(s,t) = \frac{f(t)}{\sqrt{s(s-4)}}$. The singularities of $\Phi_-(L)$ are at the same places where the singularities of $\Phi'(L) = \int_4^\infty \frac{f(t) s^{L-1}\,ds}{\sqrt{s(s-4)}}$ are. But $\Phi'(L) = C\frac{4^{L-1}}{2}\frac{\Gamma(1-L)\Gamma(\frac{1}{2})}{\Gamma(3/2-L)}f(t)$ have poles at $L = 1, 2, \ldots$ where C is a constant.

4/ $r(L)$ is the Mellin-transforms of $\frac{1}{\sqrt{t.s[(t-4)s-4]}}$.

From equation (3.4) we obtain a new one:

$$\Phi_+ (L) = \frac{C(L) + \Phi_+^+(L)(1 - V'^+(L))}{1 - 2ir(L)\Phi_+^+(L) - V'(L)} , \qquad (4.2)$$

where $V'(L) = V(L) - ir(L)\Phi_+^+(L)$.

Differentiating the denominator of eq. (4.2) with respect to λ we can prove that it is independent of λ (for sufficiently high values of λ)[5].

Using this property $V'(L)$ may be written in the following form:

$$V'(L) = \frac{2i}{16\pi^2} \int_0^{\frac{t-4}{4}} \frac{du}{\sqrt{t(t-4-4u)}} \int_0^{\lambda u} dz \left(z^{L-1} - z^{L-2} \right) \Theta \left(z - \frac{1}{h(4,t)} \right) A^* \left(\frac{(z-1)^2}{z} u, t \right). \qquad (4.3)$$

After substituting $v = \frac{(z-1)^2}{z}$ and expanding according to v and $\frac{1}{v}$ we see that $V'(L)$ has singularities at points L_Φ where $\Phi_-(L)$ has them, and in points $L_{V'} = L_\Phi + n$ (n integer) too. So the singularities of $V'(L)$ are at the points $L = 0, \pm ½, \pm 1, \dots$

Where does $\Phi_+(L)$ have singularities? First, $\Phi_+(L)$ may have a singularity at the points where $V'(L)$ and $V'^+(L)$ have them, in any case at points, where $Im\, L = 0$ and $\frac{dL}{dt} = 0$ are satisfied. We call these singularities 'fixed singularities'.

According to equation (4.2) $\Phi_+(L)$ may have a singularity if $\Phi_+^+(L)$ has one. We show for this type of singularities that they may be 'fixed' only.

First, we assume L_x , the point of singularity to be complex. Then from equations (3.2) and (3.3) it follows:

$$A_s(s,t) = f(\log s, t)\, s^{-L_x} + g^*(\log s, t)\, s^{-L_x^*} + \dots \quad .$$

L_x and L_x^* must be boundary values of an analytic function on the real axis according to the dispersion relation (1.4). These analytic functions may have only cuts, from $t = 4$ to $t = \infty$ but no other singularities. Now, it is easy to see that L^*_x is the boundary value of the same function on the second Riemann sheet. But the function $A_s(s,t)$ and so $L(t)$ has only two Riemann sheets[7] (in our case there is an elastic approximation in the t chanel), so $Im\, L_x$ satisfies the very simple singular integral equation of first order:

$$\frac{P}{\pi} \int_4^\infty \frac{Im\, L(x')}{x' - x} dx' = 0, \quad \text{if} \quad x > 4. \qquad (4.5)$$

With the help of the general solution of this integral equation we get the following form for $L(t)$ (boundary condition $L(t)$ is finite if $t \to \infty$):

$$L(t) = c_1 + \frac{c_2}{\sqrt{t-4}} \qquad (4.6)$$

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[5] This is necessary, because as we shall see, the zeros of this denominator give very important singularities of $\Phi_+(L)$ and the place of these singularities may not depend on λ .

But if $c_2 \neq 0$, $A_s(s,t)$ has an essential singularity at the point $t = 4$. Of course c_1 real.

At last we examine the case of $L(t)$ without any cut, that is to say $L(t)$ is an entire function. However, an entire function which has no singularity at infinity, is a constant.

All the solutions found as far are of the form:

$$A_s(s,t) = f(\log s, t)\, s^{-L} + \dots , \tag{4.7}$$

where L is a real constant and $f(x,t)$ has no essential singularity at infinity. The solutions of the type (4.7) we call solutions of first type.

These solutions can not give a constant total cross section at high energies, only the form $A_s(s,t) = f(t).s$ of this type have this behaviour. This function, however, doesn't satisfy (1.2)[9].

5. The Second Type of Solutions of Equation (3.4)

We get the most interesting solutions, if there are points where the denominator of equation (4.4) vanishes. Than $\Phi_+(L)$ has a pole, and $\Phi^+_+(L)$ has some finite value. In these points the following equation is satisfied:

$$1 - 2\, i\, r(L)\, \Phi^+_+(L) - V'(L) = 0 . \tag{5.1}$$

Of course, the roots of (5.1) are always complex and they depend on t [6].

As we mentioned equation (5.1) does not depend on λ . This means, that $\Phi^+_+(L)$ and $V'(L)$ must have same order of magnitude in λ .

If equation (5.1) have a root, then in the expression there appears a term of type: $(\log s)^r s^{-L_1}$ where r is the multiplicity of the root L_1 , and L_1 is the smallest root of equation (5.1). Then $\Phi^+_+(L)$ have the following terms:

$$\Phi^+_+(L) = \int_\lambda^\infty ds\, .\, s^{L-1}\left[(\log s)^r s^{-L^*_1} + \dots\right] =$$

$$= \frac{(\log\lambda)^r \lambda^{L-L^*_1}}{L - L^*_1} - \frac{(\log\lambda)^{r-1}\lambda^{L-L^*_1} . 2\,!}{(L-L^*_1)^2} + \dots \tag{5.2}$$

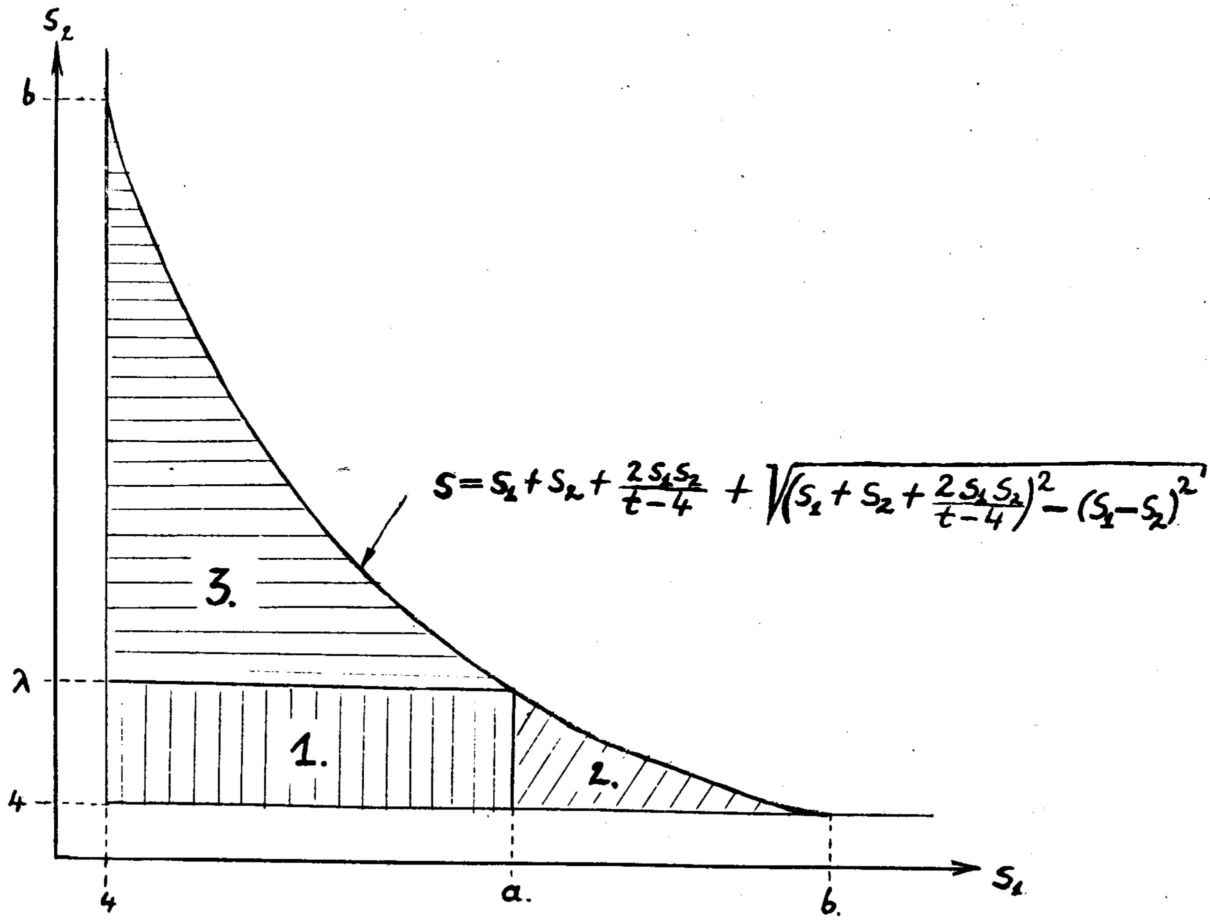
$\Phi^+_+(L_1)$ increases as: $\Phi^+_+(L_1) \sim 0\left((\log \lambda)^r\right) + \dots$.

As we mentioned in the expansion of $A_s(s,t)$ for $s < \lambda$ there appear only terms of type s^k , where k is an entire or a half of an entire number. This means that the terms of $V'(L)$ have the following structure at L_1 :

$$V'(L_1) = \sum_{k_i} \left(C_{k_i} \frac{\lambda^{L_1 - k_i}}{L_1 - k_i} + \dots \right) = \lambda^{L_1} \sum_{k_i} C_{k_i} \frac{\lambda^{-k_i}}{L - k_i} +$$

+ terms in dependent of λ .

6/ L is complex, because in this point. $\Phi_+(L)$ has a singularity and $\Phi^+_+(L)$ has none, but if L real $\Phi^+_+(L) = \Phi^*_+(L)$. If in the expression s^{-L} L may be complex anywhere, then it must depend on t , because for $t < 4$. $A_s(s,t)$ and $L(t)$ are real.



F i g. 1.

The region of integration in formula (1.2) and the division of this region in formula (2.3). The region of integration in the integral I_1 we denote by 1 in the integral I_2 by 2 and in the integral I_3 by 3 .

The sum of series of this type can not generally give asymptoticaly a function like $(\log \lambda)^r$, only in the case when $r = 0$ and all the k_i satisfy $k_i > L$, that is to say the poles of $V'(L)$ are on the right from L_1 .

In this way at last we obtained that two sorts of asymptotic behaviours may exist in the strip approximation: either

$$A_s(s,t) \sim f(\log s, t)\, s^{-L}$$

where L is real constant (if equation (5.1) has no root), or $A_s(s,t) \sim f(t)\, s^{-L(t)}$

The properties of the exponent $L(t)$ we shall examine in a forthcoming paper.

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