ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ESTIMATIONS OF THE EFFECTIVE RADIUS OF PARTICLE INTERACTION

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Abstract

Several ways of estimation of the effective radius of strong interactions are discussed: 1) the estimation by the number of the last term in the angular distribution expansion in the Legendre polinomials, 2) the estimation by the known total cross section of the channel and by the value of the angular distribution in one point, 3) the estimation by the total cross section of all channels and by the cross section of the elastic scattering channel, 4) the estimation by the mean square of the transversal momentum by means of the uncertainty relation. The first two of them are generalized to the case of inelastic reactions of a very general form $a + b \rightarrow c + d + e + \dots$ (the particle spins are arbitrary). All the ways are not connected with any model of interaction (potential well, optical model).

The particle interaction effective radius r_0 [1] is an important qualitative characteristic of strong interactions. It is interesting to get information on the radii of the pion-nucleon interaction at different energies, as well as on the radii of the interaction between pions or nucleons and strange particles and the interaction of strange particles between them. The aim of the present paper is to discuss several ways of estimation of r_0 using the experimental data which become available first of all: total and elastic cross sections and angular and momentum distributions (which at first are known only roughly). These ways consist in the determination of the quantity ℓ_0 (see footnote 1), or in the use of the uncertainty relation. Therefore, they are not connected with any model notion of the process. They may be considered as a first step of the phase shift analysis, namely, as the obtaining of information about a minimum number of phase shifts necessary to represent available experimental data. It should be emphasized that any additional data can only increase the necessary number of phase shifts, i.e. increase r_0.

I.

The angular distribution of the reaction of the type $a + b \to c + d$ when the particle spins are arbitrary and the beam a and the target b are unpolarized must have the following form (see, for example, [1] eqs. (4.5) and (4.6) and [2], eq. (5.1)):

$$\sigma(\theta) = \sum_{L=0}^{L_0} B_L P_L(\cos\theta) \; . \qquad (1)$$

Here L_0 must be equal to the lesser of the two numbers $2\ell_0$, $2\ell'_0$ [2], where $\ell_0 \cong r_0 p/h$ and $\ell'_0 \cong r' p'/h$ [2] are maximum orbital angular momenta of the particles a, b and c, d, respectively[3].

Let on the other hand we have the experimental cross section $\sigma(\theta)$. Since it is measured in the finite number of points and with errors, then the coefficients B_L's

$$B_L = \frac{2L+1}{2} \int_0^{\pi} P_L(\cos\theta)\, \sigma(\theta) \sin\theta \, d\theta \qquad (2)$$

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[1] r_0 can be determined, for example, by the classical relation $r_0 p = h \ell_0$, where ℓ_0 is a maximum orbital angular momentum of the relative motion, still essential for the experimental data representation.

[2] ℓ'_0 can be determined as a maximum number of partial waves essential for the channel $c + d \to a + b$ of the reaction of interaction of c and d (the inverse reaction for the appropriate energy of c and d).

[3] If the reaction has a threshold as, for example, $\pi + p \to \Lambda + K$ then owing to the fact that $p' < p$, L_0 will be apparently equal to $2\ell'_0$. Determining L_0 (see below) we obtain the estimation of the radius r'_0 of the $\Lambda - K$ interaction (and an understated estimation for r_0).

in the expansion of $\sigma(\theta)$ in the Legendre polinomials must vanish within the errors starting with a certain number L_e. $L_e \leq L_o$, since with the experimental errors decreasing L_e can increase. Thus, L_e gives apparently an understated value already for the lesser of the quantites $2\ell_o$, $2\ell'_o$.

Appendix *A* shows that (1) holds for any reactions $a + b \to c + d + e + \ldots$ If the number of final particles is larger than two, then $\sigma(\theta)$ must denote the distribution with respect to the angle between the beam direction and the momentum (related to the c.m.s. of the reaction), of any fixed final particle. Over the other variables the integration is made. In particular, directions of emission of other particles must not be detected. For example, for the reaction $\pi + p \to N + n\pi$ $\sigma(\theta)$ may imply the angular distribution of a nucleon.

(1) is true for each channel of the reaction $a + b \to \ldots$ for example, for the channels $\pi^- + p \to p + \pi^-$, $\to n + \pi^0$, $\to p + \pi^- + \pi^0$ etc. After having written down the relations (1) for all possible channels and having summing up them, we see that the result is of the form (1) again. $\sigma(\theta)$ in (2) may imply therefore the angular distribution of a nucleon irrespective of the number of other particles (to say nothing about directions of their emission). I.e. we can obtain information about L_o and further about ℓ_o or ℓ'_o not being sure to pick out any definite channel of the reaction under consideration.

So, the aforementioned way consists in the finding of the number L_e of the last nonzero coefficient in the expansion of the angular distribution of any singled out product of the reaction in the Legendre polynomials. Of course, it is not necessary at all to find beforehand all the previous coefficients. To find that L, which is worth to start with the calculation of integrals (2) we may use more rought estimations of r_o presented below which are based on inequalities giving a lower limit of r_o.

2

Ogievetski and Grishin/3/ have given the inequality for the reactions of the type $a + b \to 1 + 2$ which can turn out to be useful for estimating r_o or r'_o in the cases when the value of the angular distribution in one point θ' and the total cross section of the reaction $a + b \to 1 + 2$ are know. We write it in the form which is valid for arbitrary spins (for the derivation see Appendix B):

$$\frac{d\sigma(\theta')}{d\Omega} < \frac{\sigma}{4\pi} \Sigma(\theta') \qquad (3)$$

where $\Sigma(\theta')$ denotes the largest of the expressions

$$\sum_{J=0}^{J_0} (2J+1)[\, d^{J}_{m_2+m_1,\; m_a+m_b}(\theta')\,]^2 \qquad (4)$$

$$-j_a \le m_a \le +j_a \qquad etc.$$

j - is the spin of the particle a. The function $d^{J}_{mn}(\theta)$ is defined in [10,11]. J_0 denote maximum value of the total angular momentum which is equal to the lesser of the numbers $\ell_0 + j_a + j_b$, $\ell'_0 + j_1 + j_2$. Since the particle spins do not exceed usually the values ½ or 1, then J is almost equal to ℓ, if ℓ is large.

Appendix B shows that (3) is valid for the reactions of the type $a + b \to 1 + 2 + \ldots + N$ too, but with the following changes. $d\sigma(\theta_1)/d\Omega_1$ is the angular distribution of the singled out particle, for example, particle 1, see the previous Section. Instead of m_2 in (3) we must have M_2—projection of the 'spin' of the set (2 ... N) of the other particles. If there are no information about this 'spin' it should be believed that M_2 assumes all values allowed by the condition $|M_2 + m_1| \le J$ (for $|M_2 + m_1| > J$ the function $d^{J}_{M_2+m_1,\; m_a+m_b}$ vanishes). In the case $N > 2$ the value J_0 of the upper limit of summing in (4) which is necessary for satisfying (3) can give estimation only for ℓ_0 but not for any of orbital angular momenta of the reaction products.

We notice the form of the formula (3) is the same for any N. To estimate the radius of pion-proton interaction for example we may apply (3), σ being the total cross section of all channels with a proton in the final state ($\pi + p \to p + \pi$, $p + 2\pi, \ldots$), and $d\sigma(\theta_1)/d\Omega_1$ being the final protons differential cross section at the angle θ_1 (we must not pay any attention to the other particles). Indeed, we can write down the relations (3), for each channel and sum up them. The obtained expression will be of the form (3) with the aforementioned sense of σ and $d\sigma(\theta_1)/d\Omega_1$, if Σ_1 will denote the largest of all partial Σ_1's.

One can see that the inequality (3) is really useful only if $\sigma(\theta')$ differs to a considerable extent from $\sigma/4\pi$. It is only in concrete cases that we may judge whether this way is simpler than the previous one[4].

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[4] Note that the estimation $\ell_0 \sim 16$ for pp scattering at 8.5 BeV[3], basing on the data used in[3] can be made rather simply using the formula (2). Indeed, for $\theta = 0$ all the Legendre polynomials are equal to 1 and further they fall down up to zero so that the first root is at the point $\theta_0 \cong 2{,}4 \times 57^\circ / L$. In particular, the first root of P_{32} is at $4{,}3^\circ$. Since more than the half of all scattered particles is scattered at an angle less than 4° [4], then the coefficient B_{32} must still be different from zero. The estimation of the integral (2) shows in fact that the value B_{32} differs from zero by more than two errors.

Hsien Ding-chang has obtained /5/ the estimation of such a kind (N is arbitrary)

$$\frac{d\sigma(\theta')}{d\Omega} \le \frac{\sigma}{4\pi}(J+1)^2 \qquad (5)$$

which is obtained from (3), (4) by the replacement of all functions d^J by unit. $d^J_{m,n}(\theta) \le 1$, since these functions satisfy the unitarity relation

$$1 = \sum_n d^J_{m,n}(d^{J\dagger})_{n,m} = \sum_n [d^J_{m,n}(\theta)]^2. \qquad (6)$$

Though this estimation is strongly understated, but in return it does not require the calculation of sums such as (4) and can turn out to be useful as a preliminary estimation.

3

Rarita and Schwed/6/ have pointed out an interesting way for estimating the radius of interaction in the elastic reaction $a+b \to a+b$. This way requires knowledge of only the elastic cross section σ^{el} and the total cross section σ^{tot} of the process $a+b \to$ (all channels). As Ogievetski and Grishin/4/ have pointed out, this estimation follows from the formula (3) and the so-called optical theorem (see, for example/7/, § 24). We draw the proof for the case of arbitrary spins.

For $\theta' = 0$, the right hand side of (3) turns into $\sigma^{el}(J_0+1)^2/4\pi$, since $d^J_{m,n}(0^\circ) = \delta_{m,n}$. For the left hand side we have

$$\frac{d\sigma(0^\circ)}{d\Omega}[(2j_a+1)(2j_b+1)] \equiv \sum_{m_a', m_b', m_a, m_b} |\langle m_a' \, m_b' | R(0^\circ) | m_a \, m_b \rangle|^2 \ge$$

$$\ge \sum_{m_a, m_b} |\langle m_a \, m_b | R(0^\circ) | m_a \, m_b \rangle|^2 \ge \sum_{m_a, m_b} [Im \langle m_a m_b | R(0^\circ) | m_a \, m_b \rangle]^2 = \qquad (7)$$

$$= \sum_{m_a, m_b} \left(\frac{p_a}{4\pi h}\right)^2 (\sigma^{tot}_{m_a, m_b})^2 \geqslant \left(\frac{p_a}{4\pi h}\right)^2 (\sigma^{tot})^2 [(2j_a+1)(2j_b+1)]$$

We have used the relation $\sum_{i=1}^{N} (a_i)^2 \geq (\sum_{i=1}^{N} a_i)^2 / N$ and the equation (24.15) from /7/

$$\frac{4\pi h}{p_a} Im < m_a m_b | R(0^\circ) | m_a m_b > = \sigma^{tot}_{m_a, m_b} \qquad (8)$$

(following from the formula (24.14) in(7)); $\sigma^{tot}_{m_a, m_b}$ denotes the total cross section of interaction of particles a and b with the fixed projections of spins m_a and m_b. Finally we get

$$(\frac{p_a}{4\pi h})^2 (\sigma^{tot})^2 \leqslant \frac{\sigma^{el}}{4\pi} (J_0 + 1)^2 . \qquad (9)$$

4

To estimate r_0 we can use also the uncertainty relation $\Delta p_x \, \Delta x \geq h/2$ 5/.

Let the change of the particle state (or creation of new particles) proceeds effectively in the limited volume of relative coordinates with the radius r_0 . This means that before they became free particles of the final state their relative coordinate was fixed with the accuracy up to r_0 . Then, in particular, the component of the relative momentum which is perpendicular to the incident beam must have the uncertainty Δp° of the order of $h/2r_0$ (the uncertainty Δp° is easier detected then that in the component parallel to the beam). The experimental mean value of the square of the transversal momentum $(\Delta p^e)^2_{mv}$ can be only larger than $(\Delta p^\circ)^2$, since a part of the quantity $(\Delta p^e)^2_{mv}$ may be determined by the concrete interaction dynamics, which is not connected with the short-range proper ty of the interaction. For example, the Coulomb interaction has no final radius but it scatters at non- zero angle too, so that $(\Delta p^e)^2 > 0$.

From $h/2 \leqslant \Delta p^\circ r_0 \leqslant [(\Delta p^e)^2_{mv}]^{1/2} r_0$ it follows such an (understated) estimation for r_0 :

$$r_0 \geqslant h/2 \, [(\Delta p^e)^2_{m\nu}]^{-1/2} \qquad (10)$$

(compare with the estimation $r_0 \cong h\ell_0/p$ by the previous ways). To obtain this estimation it is neces-

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5/ This possibility was pointed out, in particular, by Podgoretski M.I. at the seminar of the Laboratory of High Energies of JINR. I takes an opportunity to thank Dr. Podgoretski for the discussion of this problem.

...ary to know only $(\Delta p^{e})^2_{mv}$ for any of the particle - products of the reaction.

Appendix A

In the paper[9] the author has pointed out the way of obtaining the following formula for the transition matrix of the reaction of the type $a + b \to 1 + 2 + \dots + N$:

$$< m_1 m_2 \dots m_N | R(\theta_1, \phi_1, \tilde{p}_1; \theta_2, \phi_2; \dots) | m_a m_b > =$$

$$= \frac{1}{4\pi} \sum_{J_1, J_2, M_2 \dots} \sqrt{\frac{2J_{N-1}+1}{4\pi}} D^{J_{N-1}}_{m_N + m_{N-1}, M_{N-1}} (-\pi, \theta_{N-1}, \pi - \phi_{N-1}) \dots$$

$$\dots \sqrt{\frac{2J_2+1}{4\pi}} D^{J_2}_{M_3 + m_2, M_2} (-\pi, \theta_2, \pi - \phi_2)(2J_1+1) D^{J_1}_{M_2 + m_1, m_a + m_b} (-\pi, \theta_1, \pi - \phi_1) \cdot$$

$$< \dots J_2 M_2 m_2 \tilde{p}_2 m_1 \tilde{p}_1 | R^{J_1} | m_a m_b > =$$

$$= \frac{1}{4\pi} \sum_{J_1 J_2 M_2 a} <2 \dots N | a J_2 M_2 > (2J_1 + 1) \times$$

$$\times D^{J_1}_{M_2 + m_1, m_a + m_b} (-\pi, \theta_1, \pi - \phi_1) < a J_2 M_2 m_1 \tilde{p}_1 | R^{J_1} | m_a m_b > . \qquad (A.1)$$

The notations are somewhat different than in[9]: θ_1, ϕ_1 - are spherical angles of the momentum $\vec{\tilde{p}}_1$ of the particle 1 related to the c.m.s. of the reaction; θ_2, ϕ_2 are spherical angles of the momentum $\vec{\tilde{p}}_2$ of the particle 2 related to the Lorentz system, where the total momentum of the particles 2,3 ... ... N is zero. Etc. In this case θ_1 is the angle between $\vec{\tilde{p}}_1$ and the relative momentum $\vec{p}_a$ of the particles a and b; θ_2 is the angle between the directions $\vec{\tilde{p}}_2$ and $\vec{\tilde{p}}_1$. All spin projections m are projections on the directions of the corresponding momenta.

$$D^{J}_{m,n} (\phi_2, \theta, \phi_1) = e^{-im\phi_2} d^{J}_{m,n} (\theta) e^{-im\phi_1} . \qquad (A.2)$$

The function $d^{J}_{m,n}$ is defined in[10,11]. J_1 denotes the total angular momentum of the system (the total angular momentum of particles a and b or the particles 1,2 ... N), J_2 is the total angular momentum of the system of particles 2,3 ... N etc. For other details see[9,10,12].

Instead of the variables M_2, m_1, J_1, M_1

and $m_a\, m_b\, J_1\, M_1$ (M_1 is the projection of the momentum J_1) we introduce in (A.1) the variables $s'\ell'\, J_1\, M_1$, and $s\,\ell\, J_1\, M_1$ respectively, in order to single out explicitly the relative orbital angular momenta ℓ' (of the particle 1 and the system of particles (2 ... N)) and ℓ (of the particles a and b). s is the summary spin of the particles a and b ; $\vec{s}' = \vec{j}_1 + \vec{J}_2$, where j_1 is the spin of the particle 1. The transformation function from the first set of variables to the second one has been pointed out in/13/ and in/10/, Appendix B. The expression of the transition matrix in terms of the 'phase shifts' R^{J_1}, labelled by new variables is of the form

$$<m_1 \dots m_N |\, R(\theta_1, \phi_1 \dots) \,|\, m_a\, m_b> =$$

$$= \frac{1}{4\pi} \Sigma <2 \dots N | a\, J_2\, M_2 > \sqrt{2\ell'+1}\; C^{s'm'}_{J_2 M_2 j_1 m_1}\; C^{J_1 m'}_{\ell' 0 s' m'}$$

$$D^{J_1}_{m_1' m} (-\pi, \theta_1, \pi-\phi_1) < a\, J_2\, s'\ell'\tilde{p}_1 \,|R^{J_1}|\, s\,\ell > \times \qquad (A.3)$$

$$\times \sqrt{2\ell+1}\; C^{s m}_{j_a m_a j_b m_b}\; C^{J_1 m}_{\ell 0 s m} .$$

Σ denotes the summation over the variables a (determined by the relation (AI)) and over J_2, M_2, m, s', m', ℓ', J_1', ℓ, s, m_a, m_b, m. j_1, j_a, j_b denote spins of particles. $C^{c\gamma}_{a\alpha b\beta}$ is the Clebsch-Gordan coefficient/11/. Integrating the square of the module of (A.3) over all angles and momentum modules[6] besides the angles θ_1, ϕ_1 , summing up or averaging it over all spin variables (a and b are assumed to be unpolarized) and multiplying the result by $(\frac{2\pi h}{p_a})^2$ (see/10/) we get the differential cross section:

$$\frac{d\sigma}{d\Omega_1}(\theta_1) \equiv \sigma(\theta_1) = \frac{h^2}{4p_a^2} [(2j_a+1)(2j_b+1)]^{-1} \int_D \sum_{a, J_2} \sum_{s'm'sm}$$

$$\Big| \sum_{\ell', J, \ell} \sqrt{2\ell'+1}\; C^{J_1, m'}_{\ell' 0 s' m'}\; D^{J_1}_{m', m} (-\pi, \theta_1, \pi-\phi_1) \sqrt{2\ell+1}\; C^{J_1, m}_{\ell 0 s m} \times$$

$$\times < a\, J_2\, s'\ell'\tilde{p}_1 \,|R^{J_1}|\, s\ell > \Big|^2 \qquad (A.4)$$

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[6] Here one may imply the invariant integration $d^3\tilde{p}_1 / E_1$ over the relative momenta $\tilde{p}_1$. The law of conservation of total energy imposes the corresponding restrictions to the integration region D over the modules of relative momenta/12/. For the law of conservation of total momentum in the formalism used see/9/.

We have used the following unitarity properties of the functions D's and the Clebsch-Gordan coefficients:

$$\iint d\cos\theta \, d\phi \, D^{J}_{m,n}(-\pi,\theta,\pi-\phi) \, D^{J'*}_{m,n'}(-\pi,\theta,\pi-\phi) =$$

$$= \frac{4\pi}{2J+1} \delta_{J,J'} \delta_{n,n'} \tag{A.5}$$

$$\sum_{m_1, m_2} C^{sm}_{j_1 m_1 j_2 m_2} C^{\tilde{s}\tilde{m}}_{j_1 m_1 j_2 m_2} = \delta_{s,\tilde{s}} \delta_{m,\tilde{m}} . \tag{A.6}$$

Let us multiply the sum, appearing in (A.4) inside the module sign, by the complex conjugated expression, i.e. write the squared module of this sum in the form of a single but double sum. Use then the relations:

$$D^{*J}_{m,n}(\Phi_2,\Theta,\Phi_1) = (-1)^{m-n} D^{J}_{-m,-n}(\Phi_2,\Theta,\Phi_1) \tag{A.7}$$

$$D^{a}_{\alpha,\alpha'} D^{b}_{\beta,\beta'} = \sum_{c} D^{c}_{\gamma,\gamma'} C^{c\gamma}_{a\alpha b\beta} C^{c\gamma'}_{a\alpha' b\beta'} \tag{A.8}$$

$$D^{L}_{0,0}(\Phi_2,\Theta,\Phi_1) = P_L(\cos\Theta) \tag{A.9}$$

and the formulas (1) and (19) from [14]. From (A.4) we obtain (for similar calculations see [1]):

$$\sigma(\theta_1) = \sum_L P_L(\cos\theta) \{ \frac{h^2}{4p_a^2} [(2j_a+1)(2j_b+1)]^{-1} \times$$

$$\times \int \sum_D (2J_1+1)(2\tilde{J}_1+1)[(2\ell'+1)(2\tilde{\ell}'+1)(2\ell+1)(2\tilde{\ell}+1)]^{1/2}. \tag{A.10}$$

$$\cdot (-1)^{\tilde{\ell}' + \tilde{\ell}} \; W(\ell s L \tilde{J}_1 ; J_1 \tilde{\ell}) \cdot W(\ell' s' l \; \tilde{J}_1 ; J_1 \; \tilde{\ell}') \; \cdot$$

$$\cdot \, C^{LO}_{\ell'_0 \tilde{\ell}'_0} \; C^{LO}_{\ell_0 \tilde{\ell}_0} \; < a \, J_2 \, s' \ell' \, \tilde{p}_1 \, | R^{J_1} | \, s\ell > < a \, J_2 \, s' \; \tilde{\ell}' \; \tilde{p}_1 \; | R^{\tilde{J}_1} | \; s \, \tilde{\ell} >^* \}$$

The Σ denotes the summation over $a, J_2, s', s, \ell', \tilde{\ell}', \ell, \tilde{\ell}, J_1, \tilde{J}_1$. Owing to the presence of the coefficients $C^{LO}_{\ell'_0 \tilde{\ell}'_0}$ and $C^{LO}_{\ell_0 \tilde{\ell}_0}$ the summation over L is performed up to the lesser of the quantities $2\ell'_0$, $2\ell_0$. In this case $\ell'_0 = r'_0 \tilde{p}_{1\,max} / h$ where r'_0 should be interpreted as an effective radius of that volume in which the particle 1 and the system of particles (2 ... N) where created and interact; $\tilde{p}_{1\,max}$ is the maximum value of the module of momentum of the particle I (in the c.m.s. of the reaction), allowed by the law of energy conservation.

Appendix B

By integrating the square of the module of (A1) over all angles and momentum modules, by summing or averaging it over all spin projections we obtain the total cross section of $a + b \to 1 + 2 + ... + N$ in terms of the 'phase shifts' $< a \; J_2 \; M_2 \; m_1 \; \tilde{p}_1 \; | \; R^{J_1} | \; m_a \; m_b >$:

$$\sigma = \frac{\pi h^2}{p_a^2} \left[(2 j_a + 1)(2 j_b + 1) \right]^{-1} \sum_{m_1, m_a, m_b, J_1, J_2, M_2, a} (2 J_1 + 1) \cdot$$

$$\cdot \int_D | < a J_2 \, M_2 \; m_1 \tilde{p}_1 \, | R^{J_1} | \; m_a m_b > |^2 . \qquad (B.1)$$

The formula (A.5) is used. The differential cross section of the particle I obtained by integrating over all variables, besides θ_1, ϕ_1 is of the form:

$$\frac{d\sigma(\theta_1)}{d\Omega_1} = \frac{h^2}{4 p_a^2} \left[(2 j_a + 1)(2 j_b + 1) \right]^{-1} \sum_{m_1, m_a, m_b, J_2, M_2, a} \cdot$$

$$\int_D \left| \sum_{J_1} \sqrt{\frac{2 J_1 + 1}{4\pi}} \; D^{J_1}_{M_2 + m_1, \, m_a + m_b} (-\pi, \theta_1, \pi - \phi_1) \sqrt{2 J_1 + 1} \; < a \; J_2 \; M_2 \; m_1 \; \tilde{p}_1 \, | R^{J_1} | \; m_a \; m_b > \right|^2 \leq \qquad (B.2)$$

$$< \sum_{m_a, m_b, m_1, J_2, M_2, \alpha} \int_D \frac{1}{4\pi} \sum_{J_1} (2J_1+1)[D^{J_1}_{M_2+m_1, m_a+m_b}(0,\theta,0)]^2 \cdot$$

$$\cdot \frac{\pi h^2}{p_a^2} [(2j_a+1)(2j_b+1)]^{-1} \sum_{J_1} (2J_1+1)|R^{J_1}|^2 < \frac{1}{4\pi} \Sigma_1 \cdot \sigma .$$

The Cauchi inequality is used $|\sum_J a_J b_J|^2 < \sum_J |a_J|^2 \sum_J |b_J|^2$. Σ_1 is the largest of the sums

$$\sum_{J_1=0}^{J_0} (2J_1+1)[d^{J_1}_{M_2+m_1, m_a+m_b}(\theta)]^2 \tag{B.3}$$

for the given angle θ_1. $M_2 = -J_2, -J_2+1, \ldots + J_2$; $m_1 = -j_1, -j_1+1 \ldots + j_1$ (j_1 is the spin of the particle 1) etc. Ogievetski V.I. noticed that Christoffel-Darboux formula /15/ can help the calculation of the sums such as (B.3).

The following formulas which can turn out to be useful also can be obtained in a similar way:

$$\frac{d^2\sigma(\theta_2,\phi_2,\theta_1)}{d\Omega_1 d\Omega_2} \leqslant \frac{\sigma}{(4\pi)^2} \times$$

$$\times \max \sum_{J_1, J_2, M_2} (2J_1+1)(2J_2+1) | D^{J_2}_{M_2+m_2, M_2}(0,\theta_2,\pi-\phi_2) D^{J_1}_{M_2+m_1, m_a+m_b}(-\pi,\theta_1,0)|^2 \tag{B.4}$$

$$\frac{d^2\sigma(\theta_2,\phi_2,\theta_1)}{d\Omega_1 d\Omega_2} \leqslant \frac{1}{4\pi} \frac{d\sigma(\theta_1)}{d\Omega_1} (J_{2\,max}+1)^2 . \tag{B.5}$$

To estimate the orbital moment of the particles 2 and 3 in the case of the reaction $a+b \to 1+2+3$ it is better to integrate the last formula over θ_1 :

$$\frac{d\sigma(\theta_2,\phi_2)}{d\Omega_2} \leqslant \frac{\sigma}{4\pi} (J_{2\,max}+1)^2 . \tag{B.6}$$

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