ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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PROPERTIES OF THE SOLUTION OF THE EQUATION FOR ONE MODEL OF THE LOCAL FIELD THEORY

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Introduction

The problem of the nonuniques of the solution of the Low equation for the scattering amplitude has been investigated in numerous papers[1,2,3]. Using, as an example, exactly soluble models, it was established that to a whole class of Hamiltonians differing one from another in the set of states in which the scatterer can be, there corresponds one Low equation with many value solution. Besides, it has been shown[1] that for charged mesons with a fixed nucleon the solutions of the Low equation contain restrictions on the coupling constant g_r . Recently an attention was payed to the fact that similar restrictions on the coupling constant can arise not only for individual models but also in the exact theory following from the analysis of dispersion relations for the π-N -scattering[4]. In this connection an interesting question arises whether these restrictions follow from the general principles used in deriving dispersion relations and the Low equation or they follow from the approximations which being made in this case. These approximations are:

1. The replacement of the exact unitarity relation by an approximate one with account of the two-particle intermediate state only (the two-particle unitarity), and

2. The restriction on the finite number of partial waves in the scattering amplitude.

Khalfin[5] asserts that the second assumption together with the assumption that the coupling constant g_r is arbitrary, can lead to the incompatibility of the dispersion relations and the unitarity relation. In the solved model where as usual the scattering involves only one wave (S- or P-wave), the restrictions on arise from the exact solution of the Low equation. Such a case is just considered in the present paper. The role on many particle contributions to the scattering amplitude is then evaluated on the basis of the solution of the Schrődinger equation for the giving model. It is shown that for the energies less than the inelastic scattering threshold $\omega < 2\mu$ the contribution of the lowes states does not exceed 15 per cent.

1. Model and equation for the scattering amplitude

Here we shall consider the Low equation for the scattering amplitude in one simple model of the field theory with a fixed nucleon[6]. The Hamiltonian of the system is of the form

$$H = m_0 (\psi^+ \psi) + \tfrac{1}{2} \int d\vec{x} \, [\pi^2(\vec{x}) + (\vec{\nabla} \phi(\vec{x}))^2 + \mu^2 \phi^2(\vec{x})] + \tag{1}$$
$$+ g (\psi^+ \tau_1 \psi) \int d\vec{x} \, \phi(\vec{x}) \delta(\vec{x}) + \Delta m_0 (\psi^+ \tau_3 \psi)$$

The rest nucleon can be in the two states (proton, neutron) which differ in mass $m_{op} = m_0 + \Delta m_0$, $m_{on} = m_0 - \Delta m_0$ (m_0 denotes the mass of the bare nucleon). The processes of elastic and inelastic scattering of mesons by a nucleon take place due to these degrees of freedom the nucleon. The authors of the paper/6/ have obtained the expression for the elastic scattering amplitude starting from the Hamiltonian formalism.

In the following we shall use this expression to compare it with the amplitude obtained from the Low equation. In deriving this equation we shall start with the dispersion relation for the scattering amplitude (notes that in this model only the S-scattering is possible).

$$M_N(\omega) = \frac{\delta_N}{(2\pi)^3} \frac{g_r^2}{2\omega} \left[\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta} \right] + \tag{2}$$
$$+ \frac{1}{\pi\omega} \int_\mu^\infty d\omega' \, \omega' \, \mathrm{Im}\, M_N(\omega') \left[\frac{1}{\omega' - \omega - i\epsilon} + \frac{1}{\omega' + \omega} \right]$$

$M_N(\omega)$ is the amplitude of the scattering of the meson with the energy $\omega = \sqrt{\mu^2 + k^2}$ on the nucleon ($N = p,n$) ; Δ is the difference of the observable masses of 'proton' and 'neutron'; $\Delta = m_p - m_n$ which locates the one-nucleon pole; g_r is the observable (renormalized) coupling constant. The one-nucleon terms has different sign for proton $N = p$, $\delta_p = 1$ and neutron $N = n$, $\delta_n = -1$. We assume $\Delta < \mu$, since otherwise the nucleon would have an unstable state which would decay into 'neutron' and meson.

Using the unitarity condition *

$$\mathrm{Im}\, M_N(\omega) = (2\pi)^2 k \omega \, | M_N(\omega) |^2 + a_N(\omega) \tag{4}$$

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* The total elastic cross section is expressed in terms of $M_N(\omega)$ as follows:

$$\sigma_N(\omega) = \frac{\omega^2 | M_N(\omega) |^2}{\pi}$$

If we introduce the amplitude $f_N(\omega) = \frac{\omega}{(2\pi)^4} M_N(\omega)$, then the relation (3) takes more usual form:

$$\mathrm{Im}\, f_N(\omega) = \frac{k}{4\pi} \sigma^N_{el}(\omega) + \frac{k}{4\pi} \sigma^N_{in}(\omega)$$

where $a_N(\omega)$ are the contributions from inelastic processes, we obtain, by inserting (3) into (2), the Low equation for the amplitude $M_N(\omega)$, if many particle contributions $a_N(\omega)$ are neglected. Here it should be noted that the two-particle unitarity relation obtained in such a way which valids for $\omega < 2\mu$ in the range $\mu < \omega < \infty$, imposes on the amplitude $M_N(\omega)$ rather rigorous restrictions; $M_N(\omega)$ at infinity cannot decrease more slowly than $\frac{1}{\omega^2}$. From (2) and (3) we obtain the Low equation

$$M_N(\omega) = \frac{\delta_N \, g_r^2}{(2\pi)^3 \, 2\omega} \left[\frac{1}{\omega - \Delta} - \frac{1}{\omega + \Delta} \right] + \frac{4\pi}{\omega} \int_\mu^\infty d\omega' k' \omega'^2 |M_N(\omega')|^2 \left[\frac{1}{\omega' - \omega - i\epsilon} + \frac{1}{\omega + \omega'} \right] \tag{4}$$

This equation can be solved by the well-known method of Castillego, Dalitz, Dyson /1/. Here we do not give the calculations but we give at once the result:

$$M_N(\omega) = \frac{\frac{2\delta_N g_r^2}{(2\pi)^3} \frac{\Delta}{\omega}}{(\omega^2 - \Delta^2)\{1 - \frac{\delta_N g_r^2}{4\pi} \frac{\Delta}{\sqrt{\mu^2 - \Delta^2}} \cdot \frac{\sqrt{\mu^2 - \Delta^2} - \sqrt{\mu^2 - \omega^2}}{\sqrt{\mu^2 - \Delta^2} + \sqrt{\mu^2 - \omega^2}}} \tag{5}$$

where $\sqrt{\mu^2 - \omega^2}$ is taken to be positive for $-\mu < \omega < \mu$.

Note only the following facts essential constracting the solution. First, in (5) the function $S(\omega) = \sum_i R_i \left[\frac{1}{\omega_i - \omega} + \frac{1}{\omega_i + \omega} \right]$ is omit in the denominator. This function, as Dyson /3/ has shown, describes the contributions to the scattering amplitude which are due to unstable states of the scatterer. However, in the considered model nucleon and meson can not form a coupling system /6/. Secondly, the undetermined constants which appear in solving equations by the method giving in /1/ are determined exactly in our case since the location of the one-nucleon pole and its residue are known.

2. Properties of the solution of the integral equation (4) and restrictions on the coupling constant

From (5) it is obvious that $M_N(\omega)$ has pole in the points $\pm\Delta$ and the branch cut $(-\infty, -\mu]$ and $[\mu, \infty)$, but besides, this function can have one more pole in the interval $[-\mu, \mu]$ when

$$-\frac{\delta_N g_r^2}{4\pi}\,\frac{\Delta}{\sqrt{\mu^2-\Delta^2}}\,\frac{\sqrt{\mu^2-\Delta^2}-\sqrt{\mu^2-\omega^2}}{\sqrt{\mu^2-\Delta^2}+\sqrt{\mu^2-\omega^2}}=1 \tag{6}$$

An additional pole in the amplitude $M_N(\omega)$ would contradict the analytical properties of $M_N(\omega)$ assumed earlier, by means of which the Eg. (4) has been derived. This pole can be excluded by restricting the observable coupling constant g_r. With this aim we rewrite the equality (6) in another form

$$\sqrt{\mu^2-\omega^2}=-\sqrt{\mu^2-\Delta^2}\;\frac{1-\delta_N\dfrac{g_r^2}{4\pi}\dfrac{\Delta}{\sqrt{\mu^2-\Delta^2}}}{1+\delta_N\dfrac{g_r^2}{4\pi}\dfrac{\Delta}{\sqrt{\mu^2-\Delta^2}}} \tag{7}$$

If now we put*

$$\frac{g_r^2}{4\pi}<\frac{\sqrt{\mu^2-\Delta^2}}{\Delta} \tag{8}$$

then the root of the Eq. (7) will lay on the other Riman sheet since $\sqrt{\mu^2-\omega^2}<0$. Thus, the restriction on the coupling constant (7) arises from the solution of the Low equation (4) and the one-particle unitarity relation. A similar situation has been discussed in/4,7/. Khalfin /5/ has shown in the general case that such restrictions will arise when a finite number of scattering phase shifts remain in the amplitude; in our model we have only the S-phase shift in the scattering.

The assumptions about the arbitraniness of the magnitude of g_r, the finite number of scattering phase shifts and about the unrestricted energy ω turn out in fact to be nonselfconsistent. Besides the arguments of the paper/5/ the following considerations may confirm this point of view. Introducing in our model a cutoff of the momentum $v(k)=\frac{L^2}{k^2+L^2}$, we obtain by the aforementioned method the scattering amplitude $M_N(\omega, L)$ which will represent a function of the cutoff momentum L. Instead of (6), the location of the additional pole will be determined by the equality

$$-\delta_N\frac{g_r^2}{4\pi}\,\frac{\Delta}{\sqrt{\mu^2-\Delta^2}}\,\frac{\sqrt{\mu^2-\Delta^2}-\sqrt{\mu^2-\omega^2}}{\sqrt{\mu^2-\Delta^2}+\sqrt{\mu^2-\omega^2}}\;\frac{L^3\{(\sqrt{\mu^2-\omega^2}+\sqrt{\mu^2-\Delta^2}+L)^2+L\sqrt{\mu^2-\Delta^2}\}}{(\sqrt{\mu^2-\omega^2}+L)^2(\sqrt{\mu^2-\Delta^2}+L)^3}=1 \tag{9}$$

It is seen now that the restriction of the energy (the introduction of the cutoff of the momentum L) extends the region of the allowed g_r.

* The inequality (8) is similar outwardly to the restriction on the constant at the presence of the bound state/8/.

It is interesting to treat the problem of the existence of a resonance in the solutions of the Low equation $\mathrm{Re}\, M_N(\omega_{res}) = 0$, $\mathrm{Im}\, M_N(\omega_{res}) \neq 0$. From (5) it follows that the condition $\mathrm{Re}\, M_N = 0$ is fulfilled for

$$k^2_{res} = (\mu^2 - \Delta^2)\,\frac{\delta_N \dfrac{g_r^2 \Delta}{4\pi\sqrt{\mu^2-\Delta^2}} - 1}{\delta_N \dfrac{g_r^2 \Delta}{4\pi\sqrt{\mu^2-\Delta^2}} + 1}\,.$$

Therefore, if g_r is restricted to the inequality (8) (the condition of the absence of the nonphysical pole in the amplitude), then there is nonresonance, since in this case k_{res} becomes an imaginary quantity. Thus, in our case the resonance exists when the amplitude have a nonphysical pole. The same situation takes place in the other exactly soluble model[1] (charged scalar mesons, fixed nucleon). The solution valids there for $\frac{g_r^2}{2\pi} < 1$ (the absence of the nonphysical pole), and the resonance energy is $\omega_{res} = \mu\,\frac{g_r^2}{2\pi}$ consequently, in this case too the resonance exists only when the amplitude has a nonphysical pole.

As is well known, in analysing their equation for the $\pi - N$ scattering (pseudoscalar mesons, fixed nucleon) chew and Low[9] have found a resonance in the P-wave. Following the paper of Khalfin[5], there must exist a restriction on g_r^2 in this model too, as in the two previous ones. However, Chew and Low don't point to a such restriction. The question therefore arises whether the resonance in the Chew-Low equation does not depend, as was the case of the aforementioned exampl es, on the existence of a nonphysical pole in the amplitude for the 'resonance' values of g_r^2 and for the cutoff momentum L. But it is difficult to answer this question not knowing an exact solution.

3. Comparison with the solution of the Schrodinger equation

Starting from the Schrodinger equation with the Hamiltonian (1) the amplitude of the scattering of meson on a nucleon has been obtained [6] as a power series in the parameter $\Delta m = \Delta m_0 \exp\{-g^2 \sum_K \omega_K^{-3}\}$

$$\widetilde{M}_N(\omega) = \frac{\delta_N g^2}{(2\pi)^3 \omega^2}\,\frac{2\Delta m}{\omega}\,\{1 - i\delta_N \Delta m \int_0^\infty dx\,(1 - \mathrm{Cos}\,\omega x)[\exp\{2g^2 \sum_{\vec K} \frac{e^{-i\omega_K x}}{\omega_K^3}\} - 1] + \ldots\} \quad (10)$$

g is the nonrenormalizable (bare) coupling constant. In the same paper it has been shown that the observable constant g_r and the difference of the masses of 'proton' and 'neutron' $\Delta = m_p - m_n$ is also presented as power series in Δ m, each term of these expansion being finite

$$\frac{g_r}{g} = 1 - 2\Delta m^2 \int_0^\infty dx\, x\, [\exp\{2g^2 \sum_{\vec{K}} \omega_K^{-3}\, e^{-\omega_K x}\} - 1] - \ldots \tag{11}$$

$$\Delta = 2\Delta m \{ 1 + \Delta m^2 \int_0^\infty dx_1 \int_0^\infty dx_2 x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} \exp\{2g^2 \sum_{\vec{K}} \omega_K^{-3} [e^{-\omega_K x_1} - e^{-\omega_K (x_1 + x_2)} + e^{-\omega_K x_2}]\} + \ldots \} \tag{12}$$

(For details see paper/6/, formula (11)). To compare the amplitudes M_N and $\tilde{M}_N$ it is necessary to express the renormalized qualities of (5) g_r and Δ with the help of (11) and (12), in term of g and Δm. We shall assume further $\Delta m << \mu$ and we restrict ourselves everywhere to the expansion in $\frac{\Delta m}{\mu}$ up to the second order. As a result we have the expression for the amplitude

$$M_N(\omega) = \frac{\delta_N g^2}{(2\pi)^3 \omega^2} \frac{2\Delta m}{\omega} \{ 1 + \frac{\delta_N g^2}{\pi} \frac{\Delta m}{\mu} (\frac{\mu^2 - k^2}{2\omega^2} + \frac{ik\mu}{\omega^2}) \} \tag{13}$$

Now to compare (10) and (13) we have only to single out the real and the imaginary parts of (10). For the energy $\omega < 2\mu$ (2μ is the inelastic scattering threshold) we have

$$\tilde{M}_N(\omega) = \frac{\delta_N g^2}{(2\pi)^3 \omega^2} \frac{2\Delta m}{\omega} \{ 1 - \delta_N \frac{\Delta m}{\mu} I_1(\omega, g) + \frac{\delta_N g^2}{\pi} \frac{\Delta m}{\mu} (\frac{\mu^2 - k^2}{2\omega^2} + \frac{ik\mu}{\omega^2}) \} \tag{14}$$

where

$$I_1(\omega, g) = 2\mu \int_0^\infty dx (1 - \mathrm{Ch}\, \omega x) [\exp\{2g^2 \sum_{\vec{K}} \frac{e^{-\omega_K x}}{\omega_K^3}\} - 1 - 2g^2 \sum_{\vec{K}} \frac{e^{-\omega_K x}}{\omega_K^3}] \tag{15}$$

Thus in the second order in $\frac{\Delta m}{\mu}$ the exact amplitude $\tilde{M}_N$ (14) differ from the Low amplitude M_N (13) by the term $-\delta_N \frac{\Delta m}{\mu} I_1(\omega, g)$, which takes into account contributions from the higher states to the real part of the amplitude; the imaginary parts of M_N and $\tilde{M}_N$ coincide in that range of energies. When $\mu < \omega < 2\mu$, then $0 \lesssim I_1(\omega, g) \lesssim 0{,}13$ and $1{,}5 \gtrsim \frac{g^2}{\pi} \frac{\mu^2 - k^2}{2\omega^2} \gtrsim -0{,}78$. (Here it is assumed $g^2/\pi^2 = 1$, since $I_1(\omega, g)$ for this value is maximum and the value $g^2/\pi^2 > 1$ has no sence in this model, as has been shown in [6]). It is seen that the contribution of many particle states to the real part of the amplitude does not exceed 15 per cent, and is quite satisfactory from the point of view of the influence of many particle states of the low energy processes.

We consider now the region $2\mu < \omega < 3\mu$ in this case the exact amplitude is equal to

$$\widetilde{M}_N(\omega) = \frac{\delta_N g^2}{(2\pi)^3\omega^2} \frac{2\Delta m}{\omega} \{ 1 + \frac{\delta_N g^2}{\pi} \frac{\Delta m}{\mu} \left(\frac{\mu^2 - k^2}{2\omega^2} + \frac{ik\mu}{\omega^2} \right) - \tag{16}$$

$$- \delta_N \frac{\Delta m}{\mu} I_2 (\omega, g) + \frac{\delta_N g^4}{2\pi^2} \frac{\Delta m}{\mu} \, i \int_{\omega-\mu}^{\omega^2/4\mu} \frac{dx}{x^2} \sqrt{\frac{(x+\mu)^2 - \omega^2}{\omega^2 - 4\mu x}}$$

$I_2(\omega,g)$ is the real function which is analogical to (15), but of more complicated form. From the comparison of the one-meson amplitude (13) and the exact one (16) it follows that the real and imaginary parts of the amplitudes are different in this energy range. When $2\mu \leqslant \omega \leqslant 3\mu$, then

$$0{,}13 \leqslant I_2(\omega,g) \leqslant 0{,}92 \quad , \quad -0{,}8 \geqslant \frac{g^2}{\pi} \frac{\mu^2 - k^2}{2\omega^2} > -0{,}98$$

i.e. the real parts differ anyway by 100 per cent, and the imaginary ones

$$0 \leqslant \frac{g^4}{2\pi^2} \int_{\omega-\mu}^{\omega^2/4\mu} \frac{dx}{x^2} \sqrt{\frac{(x+\mu)^2 - \omega^2}{\omega^2 - 4\mu x}} \leqslant 0{,}15,$$

$$1{,}3 \geqslant \frac{g^2}{\pi} \frac{k\mu}{\omega^2} \geqslant 0{,}3$$

differ by 20 per cent. Thus in the interval $2\mu < \omega < 3\mu$ the contribution from the higher states turns out to be essential and the one-particle amplitude M_N is half the exact one.

This model confirms well the assumption which underlies the dispersion approach in the modern field theory that for low energies (up to the inelastic processes threshold) contributions from many particle states to the scattering amplitude are unessential. The given example cannot, of course, point to a situation in the real case of the relativistic particle scattering.

Further we would like to discuss the problem whether the relation (8) holds for the coupling constant and for Δ arising from the solution of the Eq. (4) and for g_r and Δ obtained by the renormalization procedures [6] and satisfied by (11) and (12). The paper [6] shows that a necessary condition of the convergence of (11) and (12) is

$$\frac{\Delta m}{1 - \frac{g^2}{\pi^2}} < 1 \tag{17}$$

If we substitute in (8) g_r and Δ by their expansions, then in the first orders in Δm the inequality (8) is fulfilled for the quantities g and Δm under the condition (17). However, we may not speak of a rigorous fulfilment of (8) for the renormalized quantities g_r and Δ obtained by solving the Hamiltonian (1), if we don't know exact sums of the expansions (11) and (12).

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