ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

Лаборатория теоретической физики

S. Ciulli and J. Fischer

E –675

INTEGRAL EQUATIONS FOR π-π SCATTERING AND CONVERGENCE PROBLEMS OF THE AMPLITUDE EXPANSION

ЖЭТФ, 1961, т41, в 1, с 256-262.

Дубна 1961

S. Ciulli* and J. Fischer**

1006/6 up.

INTEGRAL EQUATIONS FOR π-π SCATTERING AND CONVERGENCE PROBLEMS OF THE AMPLITUDE EXPANSION

* On leave of absence from the Institute of Atomic Physics, Bucharest, Roumania.

** On leave of absence from the Physical Institute of the Czechoslovak Academy of Science, Prague.

Abstract

The convergence problems connected with the cosine expansions - which are necessary in order to derive integral equations from the Mandelstam representation - are studied on the example of the π-π scattering. A set of equations for low energies is given, in which a good convergence of the real part of the amplitude expansion is achieved with the help of a conformal transformation of the cosine plane. Since each power of the expansion function contains an infinity of partial waves, this approach is convenient in cases in which higher waves are expected to play an important role.

1. Convergence Problems

The problem of obtaining integral equations for the elastic pion-pion scattering amplitude has been studied by many authors in the last time (see /1/-/3/). The common feature of these works is that the singularities of the scattering amplitude are taken from the two-dimensional integral representation of Mandelstam /4,5 ,6/, which exhibits explicitly the analytic properties of the amplitude and permits to write down different one-dimensional dispersion relations.

In the paper of Chew and Mandelstam /1/ dispersion relations are obtained for the partial waves. In the unphysical regions, the imaginary part of the amplitude is obtained as an analytical continuation from the physical region of the crossing reactions, with the help of the Legendre expansion. As shown in /1/, /3/ and /7/, this continuation leads to difficulties due to the fact that in the region of the spectral functions the Legendre expansion is divergent, and even more, in a wide region near the boundary of the spectral functions, the series converges very slowly, so that higher terms cannot be neglected.

Recently Hsien, Ho and Zoellner /3/ proposed another approach, by which this difficulty may be avoided. They write the dispersion relations only for the forward (and backward) scattering, in which cases the integration path never intersects the spectral function regions. The path of integration of the left-hand cut coincides with the boundary of the physical region of the second (or third) reaction, so that no analytical continuation is necessary. Having performed the crossing transformation, they are left only with integrals over positive energies at $\cos^2\theta = 1$. Then, in order to obtain expressions for the partial wave amplitudes they use, besides the dispersion relation for A, also its derivative with respect to t * (see also /8/). For isotopic spin I equal to 0 or 2 only even waves occur, and we have

$$A_\ell^{0,2}(\nu) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n A^{0,2}}{(\partial \cos^2\theta)^n} \right)_{\cos^2\theta = 1} \cdot c_{\ell n} \qquad (\ell \text{ even}) \tag{1}$$

---

* $t = -2q^2/\mu^2 (1-\cos\theta)$ *is the square of the momentum transfer in* μ^2 *units in the center of* mass system of the first reaction. By $s = 4(q^2/\mu^2 + 1)$ we denote the square of the c.m. total energy (in μ^2 units). According to /1/, we also use the symbol $\nu = q^2/\mu^2$.

where

$$c_{ln} = \int_0^1 (\cos^2\theta - 1)^n P_l(\cos\theta)\, d\cos\theta$$

$$A^{0,2}(\nu, -\cos\theta) = A^{0,2}(\nu, \cos\theta)$$

These expressions are put into the unitarity condition in a certain approximation, i.e. after neglecting higher terms. In[3] only the S-wave is taken into account and on the right-hand side of (1) are taken the first two terms. The case of odd l is treated analogously.

The series in (1) converges not only for all elastic energies $\nu < 3$, but for all ν smaller than 4.8. Of course, the convergence rate of the series depends on the distance from the point $\cos^2\theta = 1$ to the nearest singularity. This follows from the fact that the unitarity condition requires information about the amplitude in the whole physical region $0 \leq \cos^2\theta \leq 1$. As the only data known about A are those at $\cos^2\theta = 1$, A can be expressed only in the power series in $(\cos^2\theta - 1)^n$, which has its own radius of convergence. If ν increases, the radius becomes smaller and the convergence slower[9], and for $\nu > \nu_{max.} = 4.8$ the expansion in (1) does not converge at all. From this point of view we can say that the accuracy of the approximation made in[3] is comparatively small at ν greater than $\frac{1}{2}\nu_{max} = 2.4$.

The integral equations given in the present paper differ from those of Hsien, Ho and Zoellner in two essential points. Firstly, in order to achieve a better approximation for the amplitudes we use, instead of the power expansion in $\cos^2\theta - 1$, the following one (in the even case):

$$A^{0,2}(\nu, \cos^2\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n A}{\partial w^n}\right)_{w=0} w^n(\cos^2\theta)$$

$$w(\cos^2\theta = 1) = 0$$

where the function $w(\cos^2\theta)$ is chosen so as to make the expansion converge as quickly as possible. In other words, the function $w(\cos^2\theta)$ represents by itself a conformal transformation of the complex $\cos^2\theta$ — plane, by which ν_{max}. can be shifted arbitrarily far from the region under consideration. Due to this fact our equations are expected to take into account the contribution of higher energy region $\nu > 2$ more exactly, because the new expresion for the partial waves

$$A_\ell^{0,2}(\nu) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n A}{\partial w^n}\right)_{w=0} \cdot \int_0^1 w^n(\cos^2\theta)\, P_\ell(\cos\theta)\, d\cos\theta \qquad (2)$$

converges also for infinite energies. This could be especially important in the case in which the expected resonant state would appear at $\nu > 2$.

Secondly, we note that our $w(\cos^2\theta)$ contains an infinity of waves (see below) so that, by using expansion in w^n, we are able to estimate also contributions from higher waves. In order not to loose this advantage we use everywhere only power series in w. In particular, in the unitarity condition we prefer the expansion in w^n because transition to the partial waves would lead to reordering of two infinite series and, consequently, to some losses of accuracy.

It was shown in /9/ that the best convergence is achieved when $w(\cos^2\theta)$ transforms the cut $\cos^2\theta$ plane into the interior of the unimodular circle. The cut itself transforms into the boundary of the circle. Consequently, the power expansion in w converges in the whole $\cos^2\theta$ plane (without the cut) and, a fortiori, in the whole physical region $0 \leqslant \cos^2\theta \leqslant 1$.

The explicit form of this optimal w is

$$w_M(\cos^2\theta,\nu) = 1 + 2(\tau^2-1)^{1/2} \frac{(\tau^2-1)^{1/2} - (\tau^2-\cos^2\theta)^{1/2}}{1-\cos^2\theta} . \qquad (3)$$

Here τ is the cosine of the nearest singularity. It depends on ν, and will be determined in Section 4. Because of the complicated form of w_M, the calculations are to be performed by electronic computers. In some cases, however, for rough estimations or calculations which are of no influence on the final result, we replace w_M by a simpler function

$$w_p(\cos^2\theta,\nu) = \frac{1-\cos^2\theta}{\alpha^2-\cos^2\theta} , \qquad \alpha^2 = 2\tau^2-1 . \qquad (3')$$

This function transforms the left half-plane $\mathrm{Re}(\cos^2\theta) < \tau^2$ of the $\cos^2\theta$ — plane into the unity circle.

2. The Unitarity Condition

From (3) and (3') it follows that both w_M and w_p contain contributions from all partial waves. Therefore we write the unitarity condition without using the partial wave expansion:

$$\mathrm{Im}\, A(\nu,\cos\theta) = \frac{1}{4\pi}\sqrt{\frac{\nu}{\nu+1}} \int_0^{2\pi}\int_{-1}^{1} A^{*}(\nu,\cos\theta_1)\, A(\nu,\cos\theta_2)\, d\cos\theta_1\, d\varphi \tag{4}$$

$$\cos\theta_2 = \cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos\varphi$$

Next we expand A in powers of w:

$$A^{0,2}(\nu,\cos\theta) = A_0^{0,2}(\nu) + A_1^{0,2}(\nu)\cdot w + A_2^{0,2}(\nu)\, w^2 + \dots$$

$$A^{1}(\nu,\cos\theta) = \cos\theta\left(A_0^{1}(\nu) + A_1^{1}(\nu)\, w + A_2^{1}(\nu)\, w^2 + \dots\right) \tag{5}$$

Now, we differentiate (4) with respect to $t = -2\nu(1-\cos\theta)$ and put $t=0^*$. We get

$$\mathrm{Im}\, A^{I}(\nu,\cos\theta=1) = \frac{1}{4\pi}\sqrt{\frac{\nu}{\nu+1}} \sum_{n,m=0}^{\infty} A_n^{I*}(\nu)\, A_m^{I}(\nu)\, K_{nm}^{I}(\nu) \tag{6}$$

$$\left(\frac{\partial^i\, \mathrm{Im}\, A^I}{\partial t^i}\right)_{\cos\theta=1} = \frac{1}{4\pi(2\nu)^i}\sqrt{\frac{\nu}{\nu+1}} \sum_{n,m=0}^{\infty} A_n^{I*}(\nu)\, A_m^{I}(\nu)\, K_{nm}^{I(i)}(\nu)$$

where

$$K_{nm}^{I(i)} = \frac{\partial^i}{\partial(\cos\theta)^i} \int_0^{2\pi}\int_{-1}^{1} w^n(\cos\theta_1)\cdot w^m(\cos\theta_2)\cdot d\cos\theta_1\, d\varphi \Big|_{\cos\theta=1}, \quad \text{for } I=0 \text{ or } 2.$$

$$K_{nm}^{1(i)} = \frac{\partial^i}{\partial(\cos\theta)^i} \int_0^{2\pi}\int_{-1}^{1} \cos\theta_1\cdot\cos\theta_2\cdot w^n(\cos\theta_1)\, w^m(\cos\theta_2)\cdot d\cos\theta_1\, d\varphi \Big|_{\cos\theta=1} \quad \text{for } I=1. \tag{7}$$

---

* The dispersion integrals are written for $t=$ const rather than for $\cos\theta =$ const. It is obvious that for $t=0$, both formulations are equivalent, but the former is preferable because the differentiation of the dispersion integrals for $t=$ const with respect to t, corresponds to the <u>subtracted</u> dispersion relation differentiated with respect to $\cos\theta$.

In this paper we use the first-order approximation, i.e. 0 and 1 in (6). Owing to the form of $w_M(\cos\theta)$ (see (3)), the explicit calculation of the $K^{(i)}_{nm}$ - integrals is awkward, so that it is convenient to perform it by digital computers. For some purposes, however, it is necessary to have closed explicit expressions for $K^{(i)}_{nm}$. In these cases one must renounce to the optimal w_M - functions and adopt the w_p - ones. Then we have, for $I = 0$ or 2,

$$
\begin{aligned}
K^{I}_{00} &= 4\pi \\
K^{I}_{10} = K^{I}_{01} &= 4\pi - 2\pi \frac{\alpha^2-1}{\alpha} \ln \frac{\alpha+1}{\alpha-1} \\
K^{I}_{11} &= 2\pi \frac{3\alpha^2-1}{\alpha^2} - \pi \frac{(3\alpha^2+1)(\alpha^2-1)}{\alpha^3} \ln \frac{\alpha+1}{\alpha-1} \\
K^{I'}_{00} = K^{I'}_{01} &= K^{I'}_{10} = 0 \\
K^{I'}_{11} &= \frac{\pi}{6\alpha^4}(3\alpha^4 - 2\alpha^2 + 3) - \frac{\pi}{4\alpha^5}(\alpha^2+1)(\alpha^2-1)^2 \ln \frac{\alpha+1}{\alpha-1}
\end{aligned}
\qquad (8a)
$$

and for $I = 1$:

$$
\begin{aligned}
K^{1}_{00} &= \frac{4\pi}{3} \\
K^{1}_{01} = K^{1}_{10} &= \frac{4\pi}{3} + 4\pi(\alpha^2-1) - 2\pi(\alpha^2-1)\alpha \ln \frac{\alpha+1}{\alpha-1} \\
K^{1}_{11} &= \frac{4\pi}{3} + 10\pi(\alpha^2-1) - \pi \frac{(\alpha^2-1)(5\alpha^2-1)}{\alpha} \ln \frac{\alpha+1}{\alpha-1} \\
K^{1'}_{00} = K^{1}_{00} &, \quad K^{1'}_{01} = K^{1'}_{10} = K^{1}_{01} \\
K^{1'}_{11} &= 2\pi + \pi \frac{(\alpha^2-1)(15\alpha^2+1)}{2\alpha^2} - \pi \frac{(\alpha^2-1)(15\alpha^4+1)}{4\alpha^3} \ln \frac{\alpha+1}{\alpha-1}
\end{aligned}
\qquad (8b)
$$

3. The Integral Equations

We start from the constant t dispersion relations :

$$A^I(\nu,t) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\, \mathcal{J}m\, A^I(\nu',t) \quad + \sum_J \frac{1}{2}\tilde{\alpha}_{IJ}\frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'+t/4}\, \mathcal{J}m\, A^J(\nu',t) \tag{9}$$

where $\frac{1}{2}\tilde{\alpha}_{IJ}$ is the isotopic matrix of the crossing transformation

$$\tilde{\alpha}_{IJ} = \begin{pmatrix} 2/3 & -2 & 10/3 \\ -2/3 & 1 & 5/3 \\ 2/3 & 1 & 1/3 \end{pmatrix} \tag{10}$$

(α_{IJ} from /1/ is $=(-1)^{I+J}\tilde{\alpha}_{IJ}$).

Limiting ourselves only to two terms of the expansions (5), we have to differentiate (9) with respect to t and put $t = 0$. Using (6) we obtain the (unsubtracted) integral equations for $A^I_0(\nu)$ and $A^I_1(\nu)$:

$$A^I_0(\nu) = \frac{1}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\sqrt{\frac{\nu'}{\nu'+1}} \sum_{n,m=0}^{1} A^{I*}_n A^I_m K^I_{nm} + \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{1}{8\pi^2}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\sqrt{\frac{\nu'}{\nu'+1}} \sum_{n,m=0}^{1} A^J_n A^J_n K^J_{nm} \tag{11a}$$

$$\left(\frac{\partial w}{\partial\cos\theta}\right)_{\cos\theta=1} A^I_1(\nu) = -\delta_{I1}A^I_0(\nu) + \frac{\nu}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\frac{1}{\sqrt{\nu'}\sqrt{\nu'+1}} \sum_{n,m=0}^{1} A^{I*}_n A^I_m K^{I'}_{nm} + \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{\nu}{8\pi^2}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\frac{1}{\sqrt{\nu'}\sqrt{\nu'+1}} \sum_{n,m=0}^{1} A^{J*}_n A^J_m K^{J'}_{nm} - \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{\nu}{16\pi^2}\int_0^\infty \frac{d\nu'}{(1+\nu+\nu')^2}\sqrt{\frac{\nu'}{\nu'+1}} \sum_{n,m=0}^{1} A^{J*}_n A^J_m K^J_{nm} \tag{11 b}$$

The factor $\left.\frac{\partial w}{\partial\cos\theta}\right|_{\cos\theta=1}$ equals to

$$-\tfrac{1}{2}\,\frac{1}{\tau^2-1} \qquad \text{for} \qquad w \equiv w_M(\cos\theta)$$

and

$$-\frac{1}{\tau^2-1} \qquad \text{for} \qquad w \equiv w_p(\cos\theta)$$

As it was shown (see the footnote in Section 2) Eq. (11b) is already subtracted. It is therefore sufficient to perform a subtraction only on Eq. (11 a). In contrast with the papers /1/ and /3/, in which the subtraction is performed in the points $s=\bar{s}=t=4/3$ and $t=0$, $s=\bar{s}=2$ respectively, we choose the threshold of the first reaction , $s=4$, $\bar{s}=t=0$. We define

$$a^0 = A^0(\nu=0,t=0)$$
$$a^2 = A^2(\nu=0,\,t=0) \qquad \left(A^1(\nu=0,t=0) = 0 \right)$$

The two scattering lengths a^0 and a^2 are not independent. The relation between them is *

$$a^0 = \tfrac{5}{2}\,a^2 + \frac{1}{8\pi^2}\int_0^\infty \frac{d\nu'}{\nu'^{1/2}(\nu'+1)^{3/2}} \sum_{n,m=0}^{1}\left(2A_n^{0*}A_m^0K_{nm}^0 + 3A_n^{1*}A_m^1K_{nm}^1 - 5A_n^{2*}A_m^2K_{nm}^2\right) \tag{12}$$

After the subtraction, Eq. (11a) takes the form

$$\begin{aligned} A_0^I(\nu) = a^I &+ \frac{\nu}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\,\frac{1}{\sqrt{\nu'}\sqrt{\nu'+1}}\sum_{n,m=0}^{1} A_n^{I*}A_m^I K_{nm}^I - \\ &- \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\,\frac{\nu}{8\pi^2}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\,\frac{\sqrt{\nu'}}{(1+\nu')^{3/2}}\sum_{n,m=0}^{1} A_n^{J*}A_m^J K_{nm}^J \,. \end{aligned} \tag{11 a'}$$

4. Estimation of the Errors in the Unitarity Condition

The convergence rate of the expansion depends on the position of the first singularity of $A(\nu,\cos\theta)$. This is located at $t=4$ which represents the first line $t=$ const. intersecting (asymptoticaly) the spectral functions A_{13} and A_{23} . (In the cosine complex plane this corresponds to a cut beginning at $\tau_\infty = 1+\frac{2}{\nu}$). However, according to the general philosophy of the effective range theory one can expect that the influence of very remote regions of the spectral function is negligible at small energies. Therefore it is sufficient to take for the cut of $\mathrm{Re}\,A$ for instance the limit $t=44/7$ which corresponds to the line $t=$ const. which intersects the boundary $s=\frac{16t}{t-4}$ of the spectral function at $\nu=10$ ($s=44$). In the cosine plane this corresponds to a branching point at

$$\tau_{10} = 1+\frac{22}{7\nu}$$

---

* Zollner and Wolf (private comunication) obtained a good fit to the τ - decay data with $a^0 = -0.3$ and $a^2 = 0.2$

The estimations which are following were performed at the threshold of the first inelastic process, i.e. $\nu = 3$, where τ_{00} and τ_{10} are respectively 5/3 and 2.0476. In order to take the upper limit of the errors, we have placed us in the worse case taking the trial function $A^I = \frac{1}{\tau - \cos\theta} + (-1)^I \frac{1}{\tau + \cos\theta}$ whose singularities are concentrated at the very begining of the former cut. For comparison of the convergence of the w^n expansion and the $(\cos^2\theta - 1)^n$ one, we refer the reader to Table 1 of the paper/9/, where some partial waves are calculated (for $I = 0$ and 2) in both approximations. We shall now concern ourselves with the errors brought in the unitarity condition by limiting the amplitude expansion to linear terms in $w(\cos\theta)$. We shall take $w \equiv w_p(\cos\theta)$ (see (3')) with $\tau = 2.047$ (i.e. we shall use the (8) expressions for the K^I_{nm} integrals).

In the even case (isotopic spin 0 or 2) the errors of $\sum_{n,m=0}^{1} A_n^{I*} A_m^I K_{nm}^I$ and of its derivative $\sum_{n,m=0}^{1} A_n^{I*} A_m^I K_{nm}^{I\prime}$ relative to their exact values $\sum_{n,m=0}^{\infty} A_n^{I*} A_m^I K_{nm}^I$ and $\sum_{n,m=1}^{\infty} A_n^{I*} A_m^I K_{nm}^{I\prime}$ are respectively

$$-4.08\% \quad \text{and} \quad +33.2\%$$

Although the latter seems to be large, its value relative to the other terms of the sum is nevertheless very small. This is a consequence of the fact that in the equation (11 b) the $\sum A_n^{I*} A_m^I K_{nm}^{I\prime}$ terms occur together with the $\sum A_n^{I*} A_m^I K_{nm}^I$ terms which (for $I = 0$ and 2) are very large in comparison with the formers. So the error of the derivative terms reported to $\sum A_n^{I*} A_m^I K_{nm}^I$ is only of 0.67%.

In the odd case ($I = 1$) both terms are of the same order of magnitude, but the errors are in both cases small. They are respectively -1.74% and -1.046%.

5. Conclusion

The information about the analyticity of the amplitude contained in the Mandelstam representation forms a basis which together with the unitarity property of the S - matrix can be used to write down integral equations for A.

Of course, such equations are not in any way solvable without making some approximations. Among them the most important one are the two-particle approximation in the unitarity condition, and the fact that only a few coefficients of the cosine expansion are taken into the equations.

The analytic properties of A contained in the Mandelstam representation are often used for writting down the dispersion relation only in one variable (energy, say) which has a simpler form that the two dimensional one. The dependence of A on the second variable (momentum transfer or cosine) is then expanded in a series. In principle, Legendre expansions could be used, but troubles arise due to their not converging in the unphysical region. This fact was taken into account in /3/ but, again, such an approach needs several derivatives $\left(\frac{\partial^n A}{\partial \cos\theta^n}\right)_{\cos\theta = \pm 1}$ in terms of which the waves are expressed. This means that together with the Legendre expansion (now only for positive energies) also the Taylor one is necessarily used, which has its own radius of convergence.

In the present paper we attempted to expand the cosine dependence of $A(\nu, \cos\theta)$ also around $\cos\theta = \pm 1$ but into the powers of a certain function $w(\cos\theta, \nu)$ which has the same location of the singularities as A has. This leads, firstly, to the 'most rapidly' convergent power expansion for the amplitude (see Appendix 1 of /9/). The estimations made above in Section 4 about the errors introduced by taking into account only the first two terms of the expansion lead to the same conclusion. Secondly, we removed at least in part the asymmetry in the treating of the energy and momentum transfer in the theory.

We are especially indebted to Prof. H. Y. Tzu for valuable sugestions made during this work. It is also a pleasure to express our thanks to all the participants of Prof. Bogoliubov's seminar for fruitful discussions.

References

1. G. Chew and S. Mandelstam. Phys.Rev. 119, 467 (1960).
2. G. Chew. Annual Rev. of Nucl. Science. 9, (1959).
3. Ho Tso-hsiu, Hsien Ding-chang, W. Zoellner. JETF, 39, 1660 (Dec. 1960).
4. S. Mandelstam. Phys.Rev. 112,1344 (1958).
5. S. Mandelstam. Phys.Rev. 115, 1741 (1959).
6. S.Mandelstam. Phys.Rev. 115, 1752 (1959).
7. A. Efremov, V. Meshcheryakov, H. Y. Tzu and D. Shirkov. On the Derivation of Integral Equations from the Mandelstam Representation. Nucl. Phys. in print.
8. G. Chew, M. Goldberger, F. Low and Y. Nambu. Phys.Rev. 106, 1337 (1957).
9. S. Ciulli and J. Fischer. A Convergent Set of Integral Equations for the Singlet Proton-Proton Scattering Nucl.Phys. 23 (1961).

Received by Publishing Department on February 15, 1961.