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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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ON SOME PROBLEMS OF THE THEORY OF SUPERCONDUCTIVITY

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БИБЛИОТЕКА

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In this report I shall give an account of my recent work 'Model hamiltonian in the theory of superconductivity' just published as a preprint by the Dubna Institute.

Here the simplest dynamical system, caracterised by the well known reduced hamiltonian was considered and it was treated by methods which are completely rigorous from the mathematical point of view.

In conclusion of this report we intend to discuss the general situation in the theory of superconductivity and superfluidity.

So let us first consider the simplest model system in the theory of superconductivity caracterised by the reduced hamiltonian of the form:

$$H = \sum_{(f)} T(f) \overset{+}{a}_f a_f - \frac{1}{2V} \sum_{(f,f')} \lambda(f) \lambda(f') \overset{+}{a}_f \overset{+}{a}_{-f} a_{-f'} a_{f'} \qquad (1)$$

The second-quantized operators a_f and $\overset{+}{a}_f$ destroy and create, respectively, free particle states of momentum p and spin s, and satisfy the usnal commutation rules for fermions.

We have adopted the following notations:

$$f = (p, s), \; -f = (-p, -s), \; T(f) = \frac{p^2}{2m} - \mu, \; \mu > 0$$

$$\lambda(f) = \begin{cases} J\epsilon(s), & \left|\frac{p^2}{2m} - \mu\right| < \Delta \\ 0, & \left|\frac{p^2}{2m} - \mu\right| > \Delta \end{cases} \qquad (2)$$

The application of the BCS method and of our method of the compensation of 'dangerous diagrams' leads in this case to the same results.

About two years ago Zubalev, Tserkovnikov and myself have drawn attention to the fact that the case of this model system is one of that very rare problems of statistical mechanics where the assymptotically exact solution can be found.

In our note we have obtained the assymptotically exact (for $V \to \infty$) expression for the free energy.

This result was established there in the following way: The hamiltonian H was in a special way devided into two parts H_0 and H_1. The problem with the hamiltonian H_0 was solved exactly by means of a u - V - transformation. In order to take into account the influence of H_1 perturbation theory was used. It was shown that any n^{th} term of the corresponding expansion is assymptotically negligible for $V \to \infty$ In this connection the conclusion was made that the influence of H_1 is to be neglected after the limiting process $V \to \infty$.

It is evident that such reasoning can not claim to be mathematically correct. But it should also be emphasized that in the problems of statistical mechanics even more crude approaches are used. Very widespread are, for example, approximation methods based upon selective summation of, in a certain sense, 'main terms' of perturbation theory series. The other terms, although they even do not tend to zero for $V \to \infty$, are omitted.

The real doubts concerning the validity of our mentioned results arose when varions attempts to apply the usual Feynman diagram technique have failed. When I'm speaking here about the usual Feynman technique I mean the diagram technique where the anomalous Pairings:

$$\overbrace{\overset{+}{a}_{f}\, \overset{+}{a}_{-f}} \quad , \quad \overbrace{a_{-f}\, a_{f}} \qquad (3)$$

are not taken into account. The necessity of introducing such anomalous pairings is clearly seen when we perporm our canonical u - V transformation. And if we permit to introduce these pairings, as it was done for example in the works of Belayev, Gorkov and Zubarev, all is right and the correct results are immediately obtained.

But if we apply the Feynman's technique in the orthodox way - as in the quantum field theory - we must put such pairings (3) equal to zero.

In view of this situation Zubarev, Tserkovnikov and myself have recently investigated the whole infinited system or 'chain' of linked equations for the Green's functions corresponding to the hamiltonian H_0.

We have succeeded to show that the Green's functions for the hamiltonian H_0 satisfy any equation of this chain for the exact hamiltonian H with the error of the order of $1/V$

This confirms the result of our earlier mentioned paper and it becomes clear that the additional term H_1 is 'not effective'.

However one may consider the situation from a purely mathematical point of view. - As soon as have fixed the hamiltonian, say in the form (1), we have a quite definite mathematical problem which should be solved rigorously without any 'physical assumptions'.

Then it is not enough that the approximate expressions satisfy the exact equations up to the terms of order of smallness of $1/V$ and we must estimate the difference between the exact and the approximate expressions.

In order to get a full insight in the behaviour of the model dynamical system I have adhered to such purely mathematical standpoint and solved the problem in the preprint I have mentioned in the beginning of this report.

The chief aim of this work was not only to have a fully convincing proof of allready known results but to acquire a deeper understanding of the situations with the anomalous Green's functions (3).

I must stress that the reasonings and proofs became very involved because of the renunciation of the so called 'simple physical considerations' and the need of using instead the complicated mathematical majoration technique.

But I shall explain here the main ideas which are very simple. So we shall consider the dynamical system with the hamiltonian H in the case of the zero temperature: $\theta = 0$.

Because of methodical reasons it would be more convenient for us to consider a somewhat more general hamiltonian, containing terms which are the sources for the creation and annihilation of pairs:

$$
\begin{aligned}
H = \sum_{(f)} T(f)\, \overset{+}{a}_f a_f - \nu \sum_{(f)} \frac{\lambda(f)}{2} \left(a_{-f} a_f + \overset{+}{a}_f a_f \right) - \\
- \frac{1}{2V} \sum_{(f,f')} \lambda(f)\, \lambda(f')\, \overset{+}{a}_f \overset{+}{a}_{-f} a_{-f'} a_{f'}
\end{aligned} \qquad (4)
$$

where ν is a parameter which we shall assume to be greater or equal to zero: $\nu \geq 0$.

Note that the case $\nu < 0$ needs no special consideration because it may be reduced to the case $\nu > 0$ by a simple gauge transformation of the Fermi-operators:

$$
a_f \rightarrow i\, a_f \, ; \quad \overset{+}{a}_f \rightarrow - i\, \overset{+}{a}_f
$$

Let us emphasize also that the case $\nu > 0$ P will be considered only in so far as it is of interest for understanding the situation in the real case, where $\nu > 0$.

For our present investigation we shall not need those concrete properties (2) of the functions $\lambda(f), T(f)$ which were mentioned above. It will be quite sufficient to satisfy the following general conditions:

1) The functions $\lambda(f)$, $T(f)$ are real, partly continuous and have the simmetry properties:

$$
\lambda(-f) = -\lambda(f) \, ; \quad T(-f) = T(f)
$$

2)

$$|\lambda(f)| \leqq \text{const} \qquad \text{for} \quad |f| \to \infty$$
$$T(f) \to \infty$$

3)

$$\frac{1}{V} \sum |\lambda(f)| \leqq \text{Const}$$

4)
$$\lim_{V \to \infty} \frac{1}{2V} \sum_{(f)} \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)x + T^2(f)}} > 1 \qquad \text{for}$$

sufficiently small positive x .

Let us represent now H in the form:

$$H = H_0 + H_1$$

where

$$H_0 = \sum_{(f)} T(f) \overset{+}{a}_f a_f - \frac{1}{2} \sum_{(f)} \lambda(f) \left\{ (\nu + \sigma^*) a_{-f} a_{+f} + (\nu + \sigma) \overset{+}{a}_f \overset{+}{a}_{-f} \right\} + \frac{|\sigma|^2}{2} V \tag{5}$$

$$H_1 = -\frac{1}{2V} \left(\sum_{(f)} \lambda(f) \overset{+}{a}_f \overset{+}{a}_{-f} - V\sigma^* \right) \left(\sum_{(f)} \lambda(f) a_{-f} a_f - V\sigma \right) \tag{6}$$

where σ is an arbitrary complex number.

Starting from this representation one can get very easily the expression for an upper limit for the lowest eigenvalue E_H of the hamiltonian H

Let $E_0(\sigma)$ be the lowest eigenvalue of H_0, considered as a function of σ. As $H_1 \leqq 0$ we immediately see that:

$$E_0(\sigma) \geqq E_H$$

for any value of G. So the best inequality will be

$$\min_{(G)} E_0(G) \geqq E_H \tag{7}$$

In order to evaluate the left hand part of (7) it is sufficient to note that H_0 is a quadratic form in the Fermi-operators.

Thus by the corresponding canonical transformation we can obtain the following identity:

$$H_0 = \sum_{(f)} \sqrt{\lambda^2(f)|\nu+G|^2+T^2(f)}\,(\overset{+}{a}_f u_f + a_{-f} v_f^*)(a_f u_f + \overset{+}{a}_{-f} v_f) + \tag{8}$$
$$+ \frac{1}{2} V \left\{ |G|^2 - \frac{1}{V} \sum_{(f)} \left[\sqrt{\lambda^2(f)|\nu+G|^2+T^2(f)} - T(f) \right] \right\}$$

where

$$u_f = \frac{1}{\sqrt{2}} \left(1 + \frac{T(f)}{\sqrt{\lambda^2(f)|\nu+G|^2+T^2(f)}} \right)^{½}$$

$$v_f = -\frac{\epsilon(f)}{\sqrt{2}} \left(1 - \frac{T(f)}{\sqrt{\lambda^2(f)|\nu+G|^2+T^2(f)}} \right)^{½} \frac{\nu+G}{|\nu+G|}$$

$$\lambda(f) = \epsilon(f)\,|\lambda(f)|$$

Here obviously: $u(-f) = u(f)$; $v(-f) = -v(f)$

u is real, v - complex, and $u^2 + |v|^2 = 1$

It follows that the operator amplitudes:

$$\alpha_f = a_f u_f + \overset{+}{a}_{-f} v_f$$

will be of the fermion type.

Hence the lowest eigenvalue $E_0(G)$ of H_0 is attained for the occupation numbers:

$$\overset{+}{\alpha}_f \alpha_f = 0$$

We thus obtain:

$$E_0(G) = \frac{1}{2}\left\{G^*G - \frac{1}{V}\sum_{(f)}\left[\sqrt{\lambda^2(f)|\nu + G|^2 + T^2(f)} - T(f)\right]\right\}$$

Consider now the minimum problem - to find G for which $E_0(G)$ gets its lowest value.

Two different cases must than be considered: $\nu = 0$ and $\nu > 0$. In the first case $\nu = 0$ we may write:

$$E_0(G) = \frac{1}{2} V F(G^*G)$$

$$F(x) = x - \frac{1}{V}\sum_{(f)}\left\{\sqrt{\lambda^2(f)x + T^2(f)} - T(f)\right\}$$

We see that $E_0(G)$ does not depend upon the phase of G, being the function of its module only.

For this reason the minimum condition enables us to determine only the module of G:

$$\sqrt{x} = |G|$$

but not its phase.

The examination of $F(x)$ shows that (in virtue of (4°)) it has only one minimum point $x_0 > 0$ in the relevant interval $x \geqq 0$

We thus have:

$$\frac{V}{2} F(x_0) \geqq E_H$$

Let us turn now to the case $\nu > 0$. Here not only the module of G but also its phase are fully determined by the minimum condition for $E_0(G)$.

It can be shown that σ must be real:

$$\sigma = \sqrt{x_0} - \nu$$

Where $x = x_0 > \nu^2 > 0$ is determined by the minimum condition for the function:

$$F(x) = (\sqrt{x} - \nu)^2 - \frac{1}{V} \sum \left\{ \sqrt{\lambda^2(f)\, x + T^2(f)} - T(f) \right\} \qquad (9)$$

in the interval $x \geqq 0$.

So in both cases we have:

$$\frac{V}{2} F(x_0) \geqq E_H \qquad (10)$$

In order to prove that the suplementary term H_1, in the hamiltonian H is not effective and that we have the assymptotic equality:

$$\lim_{(V \to \infty)} \left\{ \frac{E_H}{V} - \frac{1}{2} F(x_0) \right\} = 0$$

we shall obtain now a lower limit for E_H . It would be very desirable to get rid of the term:

$$H_1 = -\frac{V}{2} (\overset{+}{L} - \sigma)(L - \sigma); \quad L = \frac{1}{V} \sum_{(f)} \lambda(f)\, a_{-f}\, a_f$$

and to obtain thus the identity of the type (8) for the complete hamiltonian H . This could have been achieved by considering σ not as a C - number byt as the operator:

$$\sigma = L$$

But unfortunately it is impossible to carry out the canonical transformation from the fermions a to the fermions α with σ being an operator.

Nevertheless we allways can try to generalize the identity (8) for the total hamiltonian H by putting $\sigma = L$ with some prescriptions as to 'the correct order' in products of operator and by adding supplementary commutator terms. Following this way we have in fact obtained such identity which we h shall write down.

Let us first consider the operators:

$$P_f = \frac{1}{\sqrt{2}} \left(\sqrt{K \lambda^2(f) + T^2(f)} + T(f) \right)^{1/2}$$

$$q_f = -\frac{\epsilon(f)}{\sqrt{2}} \left(\sqrt{K \lambda^2(f) + T^2(f)} - T(f) \right)^{1/2} K^{-1/2} (L + \nu)$$

$$K = (L + \nu)(L^+ + \nu) + \beta^2$$

where β is an arbitrary real number (at the last end we shall put $\beta \to 0$).

We then have the identity:

$$\mathcal{H} = \sum_{(f)} \left(\overset{+}{a}_f P_f + a_{-f} \overset{+}{q}_f \right) \left(P_f a_f + q_f a_{-f} \right) +$$
$$+ \frac{1}{2} V \left\{ L L^+ - \frac{1}{V} \sum_{(f)} \left[\sqrt{\left(K + \frac{2\bar{S}}{V}\right) \lambda^2(f) + T^2(f)} - T(f) \right] \right\} + V \mathcal{R} \tag{11}$$

where

$$\bar{S} \geqq \frac{1}{V} \sum_{(f)} |\lambda(f)|^2$$

The expression of $\mathcal{R}$ contains many terms with commutators. Note that the commutators:

$$\left| \left[a_f \cdot \overset{+}{L} \right] \right| = \frac{2}{V} \left| \lambda(f) \overset{+}{a}_{-f} \right| \leqq \frac{2|\lambda(f)|}{V}$$

$$\left| \left[L, \overset{+}{L} \right] \right| \leqq \frac{2}{V} \frac{1}{V} \sum_{(f)} \lambda^2(f) \leqq \frac{2\bar{S}}{V}$$

etc. are of the order of smallness of $\dfrac{1}{V}$.

By using these simple estimates and by applying our majoration technique we have proved that:

$$< \phi^* \mathcal{H} \phi > \geqq \frac{\mathcal{D}}{V} \quad ; \quad \mathcal{D} = Const$$

for any normalized wave function ϕ.

We thus get the inequiality:

$$\langle \phi^* H \phi \rangle \geqq -\mathcal{D} + \frac{V}{2} \langle \phi^* \Big\{ L\overset{+}{L} - \frac{1}{V} \sum_f \Big[\sqrt{\{(L+\nu)(\overset{+}{L}+\nu)+\beta^2+\frac{2\bar{S}}{V}\} \lambda^2(f) + T^2(f)} -$$

$$- T(f) \Big] \Big\} \phi \rangle + \langle \phi^* \sum_{(f)} (\overset{+}{a}_f p_f + a_{-f} \overset{+}{q}_f)(p_f a_f + q_f \overset{+}{a}_{-f}) \phi \rangle \geqq -\mathcal{D} + \frac{1}{2} V \times$$

$$\times \langle \phi^* \Big\{ L\overset{+}{L} - \frac{1}{V} \sum_{(f)} \Big[\sqrt{\{(L+\nu)(\overset{+}{L}+\nu)+\beta^2+\frac{2\bar{S}}{V}\} \lambda^2(f) + T^2(f)} - T(f) \Big] \Big\} \phi \rangle$$

which is the starting point for our investigation.

In particular this inequality yields the following lower limit for E_H:

$$E_H \geqq \frac{V}{2} F(x_0) - Z \tag{13}$$

Here Z is a constant when $V \longrightarrow \infty$

Therefore in virtue of (10) we may write:

$$\left| \frac{E_H}{V} - \frac{F(x_0)}{2} \right| \leqq \frac{Z}{V} \underset{V \to \infty}{\longrightarrow} 0 \tag{14}$$

By inspecting the inequalities (10), (12) one can also obtain an information concerning the assymptotic behaviour of the operators L, $\overset{+}{L}$

So in the case $\nu = 0$ we have:

$$< \phi^* / L^+ L - C^2 /^2 \phi > \leqq \frac{J}{V}$$

$$< \phi^* / L \overset{+}{L} - C^2 /^2 \phi > \leqq \frac{J}{V}$$

(15)

J Const.; $C^2 = x_0$

where ϕ is the wave function for H corresponding to E_H, or more generally ϕ is a wave function for which:

$$< \phi^* H \phi > - E_H \leqq J_1 = \text{Const.}$$

Thus, for states with the average energy assumptotically close to the lowest energy E_H the operator $\overset{+}{L} L$ with an assumptotic accuracy is equal to the C - nucber $C^2 = x_0$. Such states, however, do not possess similar properties for single operators $\overset{+}{L}$ and L

Let us consider, in fact, the state ϕ_H with the lowest energy E_H. Generally speaking a case of degeneration may occur, so that we shall have not one but a certain linear manifold $\{ \Phi_H \}$ of possible states with the same lowest energy E_H

Since in our case ($\nu = 0$) the operator

$$N = \sum_{(f)} \overset{+}{a}_f a_f$$

representing the total number of particles commutes exactly with H one allways can find in this manifold $\{ \Phi_H \}$ such state Φ_H' for which N takes a certain definite value, say N_0

Then obviously:

$$< \phi_H^{*\prime} L \phi_H > = 0 ; \quad < \phi_H^{*\prime} L \phi_H' > = 0$$

Therefore L can not take even approximatively a definite value in the state ϕ_H' since otherwise the operator $L \overset{+}{L}$ would be approximatively equal to zero and not to $C^2 = x_0 > 0$ in this state.

Consider now the linear manifold $\{\phi\}$ of states with energy assumptotically close to E_H. Since L, $\overset{+}{L}$ approximatively commute with . H it is natural to expect that it is possible to choose such ϕ, in the manifold $\{\phi\}$, for which both L and $\overset{+}{L}$ assume definite values with the assymptotic accuracy. This expectation turns out to be quite true. It is just this circumstance which explains the success of the approximate method where we replace the hamiltonian H with the exact conservation low for N by the hamiltonian H_0 for which N is no longer the exact integral of motion.

Now it can also be shown that the approximation method could be formulated in such a way that the conservation low for N would not be violated, even formally.

To this end we may introduce the operators

$$\alpha_f = u_f a_f + v_f \frac{L}{c} \overset{+}{a}_{-f} \tag{16}$$

satisfying the commutation relations for the Fermi-amplitudes with the assymptotic accuracy.

Such operators are in some respect analogous to the operators:

$$b_f = \overset{+}{a}_0 N_0^{-1/2} a_f$$

introduced by me more than ten years ago in the theory of superfluidity.

Let us now turn our attention to the case $\nu > 0$. In this case we have proved more strict inequalities:

$$\begin{gathered} \langle \phi^* (\overset{+}{L} + \nu - c)(L + \nu - c) \phi \rangle \leqq \frac{I_0}{V} \\ \langle \phi^* (L + \nu - c)(\overset{+}{L} + \nu - c) \phi \rangle \leqq \frac{I_\nu}{V} \end{gathered} \tag{17}$$

Here I_ν, is a constant when $V \to \infty$ and ν is fixed > 0.

Therefore, when we include in the hamiltonian the terms with the sources of pairs:

$$-\nu \sum_{(f)} \frac{\lambda(f)}{2} (a_{-f} a_f + \overset{+}{a}_f \overset{+}{a}_{-f}) \tag{18}$$

the operators L , $\overset{+}{L}$ themselves become assymptotically c -numbers equal to $C-\nu$ for states with the energy assymptotically close to E_H .

We see that here an analogy arizes with the theory of ferromagnetism in an isotopic medium.

When the external magnetic field is absent the direction of the magnetisation axis is not fixed. If however a magnetic field, whatever weak, acting along a definite direction is switched on then the vector of magnetisation at once orients inself, in just that direction.

In our problem we have essentially the same situation. If the pair source terms (18) in the hamiltonian were replaced by the terms of the form:

$$-\nu \sum_{(f)} \frac{\lambda(f)}{2} \left(a_{-f} a_f e^{i\varphi} + \overset{+}{a}_f \overset{+}{a}_{-f} e^{-i\varphi} \right) \qquad (19)$$

with a phase φ , than the relations

$$L \sim C - \nu$$

were changed into

$$L \sim (C-\nu) e^{-i\varphi}$$

This result can most easily be established if we notice that the hamiltonian with the source terms (19) may be reduced to our usual form (4) by the gauge transformation :

$$a_f \longrightarrow e^{-i\varphi/2} a_f ; \qquad \overset{+}{a}_f \longrightarrow e^{i\varphi/2} \overset{+}{a}_f \qquad (20)$$

In our preprint we also have studied the averages of the products of second quantized operators in Heisenberg representation:

$$A_{f_1}(t_1) \ldots A_{f_s}(t_s)$$

$A_f(t)$ being equal to $a_f(t)$ or to $\overset{+}{a}_f(t)$.

Here the equations of motion:

$$i \frac{d a_f}{dt} = T(f) a_f - \lambda(f) \overset{+}{a}_{-f} (\nu + L)$$
$$i \frac{d \overset{+}{a}_f}{dt} = - T(f) \overset{+}{a}_f + (\nu + \overset{+}{L}) a_{-f} \lambda(f) \qquad (21)$$

began to play the most important role.

We see that the difference between the equations of motion for the hamiltonians H , H_o resides in the fact that in the case of H, L is the operator and in the case of H_o, L is replaced by the C - number $C-\nu$.

Let us first consider the situation where $\nu > 0$.

Then the inequalities (17) may be applied.

By systematically using them together with the majoration technique we were able to prove that:

$$\lim_{V\to\infty} \left\{ \langle A_{f_1}(t_1) \dots A_{t_s}(t_s) \rangle_H - \langle A_{f_1}(t_1) \dots A_{f_s}(t_s) \rangle_{H_o} \right\} = 0 \quad (22)$$

Because the limit

$$\lim_{V\to\infty} \langle A_{f_1}(t_1) \dots A_{f_s}(t_s) \rangle_{H_o} \qquad (23)$$

obviously exists (it may even be easily calculated in the closed form) we see at once that limiting value of the average:

$$\langle A_{f_1}(t_1) \dots A_{f_s}(t_s) \rangle = \lim_{V\to\infty} \langle A_{f_1}(t_1) \dots A_{f_s}(t_s) \rangle_H$$

exists and is equal to (23).

The case $\nu = 0$ is somewhat more involved because in this case only the weaker inequalities (15) do hold.

To treat this situation it is profitable to deal with the operators (16) and to write down the equations of motion for them. In such a way we have proved that the relations (22) are still valid when $\nu = 0$, provided the numbers of creation and distruction operators among $A_{f_1} \dots A_{f_s}$ are the same.

Suppose, on the contrary, that the difference between the numbers of creation and distruction amplitudes in the considered product is equal to $n \neq 0$.

It is easy to see that in this case the limit

$$\lim_{\binom{\nu>0}{\nu\to 0}} \langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle_\nu =$$

$$= \lim_{\binom{\nu>0}{\nu\to 0}} \left\{ \lim_{V\to\infty} \langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle_{H^\nu} \right\} = \tag{24}$$

$$= \lim_{\binom{\nu>0}{\nu\to 0}} \left\{ \lim_{V\to\infty} \langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle_{H_0^\nu} \right\}$$

also exists. This limit is interpreted by us as the 'quasi-average':

$$\langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle$$

So we adopt the following definition of the quasi-average

$$\langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle = \lim_{\binom{\nu>0}{\nu\to 0}} \langle A_{f_1}(t_1)\dots A_{f_s}(t_s)\rangle_\nu \tag{25}$$

which shows that in the case when $n = 0$ the quasi-average is just the usual one.

It is enteresting to note that if the source terms (18) of the hamiltonian were replaced by source terms (19) then the value of the quasi-average would be multiplied by $e^{i n\varphi/2}$.

We have here a kind of 'hidden' degeneracy because no degeneracy would manifest itself in those averages which only have the physical meaning that is for which $n = 0$.

The same conclusions are also valid if we consider the averages of field operators in τ-representation.

$$\langle \dots \overset{+}{\psi}(t_j, \vec{\tau}_j, s_j)\dots \psi(t_\kappa, \vec{\tau}_\kappa, s_\kappa)\dots\rangle \tag{26}$$

Here the average:

$$\langle \dots \rangle$$

is defined as the usual average:

$$\lim_{V\to\infty} \langle \dots \rangle_{H} = \lim_{V\to\infty} \langle \dots \rangle_{H_0}$$

for the case when $\nu > 0$ or for the case when $\nu = 0$, $h = 0$. If $\nu = 0$, $h \neq 0$ (26) is to be defined as the quasi-average.

The investigation of the averages of such type (26) may present some interest because here one has a very rare non trivial model when it is possible to prove 'the principle of vanishing correlation' by direct calculation.

Consider for example the average:

$$\langle \overset{+}{\psi}(t_1,\vec{r}, s_1)\, \overset{+}{\psi}(t_2,\vec{r}_2, s_2)\, \psi(t_2',\vec{r}_2', s_2')\, \psi(t_1',\vec{r}_1', s_1')\rangle \tag{27}$$

Let us fix $t_1,\ t_2,\ t_2',\ t_1'$ and let

$$|\vec{r}_\alpha - \vec{r}_\beta'| \to \infty;\quad \alpha = 1, 2;\ \beta = 1, 2$$

Then the assumptotic form of the expression (27) will be equal to the product

$$\langle \overset{+}{\psi}(t_1,\vec{r}_1, s)\, \overset{+}{\psi}(t_2,\vec{r}_2, s_2)\rangle\langle \psi(t_2',\vec{r}_2', s_2')\, \psi(t_1',\vec{r}_1', s_1')\rangle$$

of usual averages ($\nu > 0$) or of auasi-averaaes ($\nu = 0$).

Such properties lead to the possibility of introducing the notion of the quasi-average not by means of supplying the given hamiltonian with infinitesimal source terms but by considering the assumptotic forms of usual averages.

We have given here an account of our investigation of the case $\theta = 0$. It must be stressed that the same results can be obtained, and even in a more simple way, for the case $\theta > 0$, the case $\theta = 0$ being the most involved from the mathematical point of view.

So far only the simplest model system caracterized by the reduced hamiltonian was treated. But all results were established by rigorous mathematics.

Let us turn now to the general situation in the theory of superconductivity and superfluidity renouncing the claims of the full mathematical rigour.

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We shall examine first the notion of the degeneracy of a state of the statistical equilibrium.

The notion of the degeneracy is familiar in quantum mechanics and its meaning concerns the eigenfunctions of an operator, say of the hamiltonian.

At first sight it may appear that any state of the statistical equilibrium is always non degenerate.

In fact, the average of a given dynamical variable A

$$\langle A \rangle = \frac{Sp\, A\, e^{-H/\theta}}{Sp\, e^{-H/\theta}} \tag{28}$$

is always unambigously defined.

But the real situation is not so simple as that. To get an idea of the inherent difficulties let us consider the case of an ideally isotopic ferromagnetic medium with the temperature below the Curie point. Denote by $\vec{\mathfrak{M}}$ the magnetisation vector. If no external magnetic field is present then the average value of calculated by the usual formula (28) is obviously equal to zero.

We thus see that usual averages are not the best tool for describing such states of statistical equiblium. Let us switch on an infinitesimal external magnetic field $\nu\vec{e}\ (\nu > 0)$; then:

$$\lim_{\substack{\nu \to 0 \\ \nu > 0}} \langle \vec{\mathfrak{M}} \rangle$$

will be $\vec{e} A$ where $A \neq 0$ (because the temperature is below the Cuire point).

Here we may introduce the notion of the quasi-average. The quasi-average $\langle A \rangle_{\vec{e}}$ of a dynamical variable may be defined as

$$\lim_{\substack{\nu > 0 \\ \nu \to 0}} \langle A \rangle_{\nu\vec{e}}$$

where

$$\langle A \rangle_{\nu\vec{e}} = \lim_{V \to \infty} \frac{Sp\, A e^{-\frac{H_{\nu\vec{e}}}{\theta}}}{Sp\, e^{-\frac{H_{\nu\vec{e}}}{\theta}}}$$

and where $H_{\nu\vec{e}}$ is the hamiltonian containing the external magnetic field terms.

We clearly see that quasi-averages do depend upon $\vec{e}$ and we have the degeneracy connected with the rotation group of the orth $\vec{e}$.

It is also evident that the physical properties in the considered case are best caracterized not by usual averages but by quasi-averages; the usual average bieng merely the mean value

$$\int \langle A \rangle_{\vec{e}} \, d\vec{e}$$

of the quasi-average, taken over all possible direction of $\vec{e}$.

We may remove the degeneracy by fixing $\vec{e}$ say in the direction of the Z - axis that is by introducing an infinitesimal magnetic field acting along this axis.

Such a procedure is always used in the theory of ferromagnetism. The special notion of the quasi-average has not been explicitely introduced in this theory owing probably to the triviality of the situation.

Let us consider now the case of a cristal state. Then the average density

$$\langle \overset{+}{\psi}(\vec{r})\, \psi(\vec{r}) \rangle \qquad (29)$$

is expected to be a periodic function of coordinates. But if we apply the formula (28) we just get for (29) a constant value due to the existence of the translation group. In momentum representation the usual averages (28) must verify the relation:

$$\langle \overset{+}{a}_{k}\, a_{k'} \rangle = 0, \qquad \vec{k} = \vec{k}'$$

conditioned by the conservation of momenta.

Wee see that in this situation too the notion of the quasi-average must be introduced.

Let us add to the translationally invariant hamiltonian H the infinitesimal external field terms:

$$\nu \int U(\vec{r})\, \overset{+}{\psi}(\vec{r})\, \psi(\vec{r})\, d\vec{r}, \quad \nu > 0 \qquad (30)$$

where $U(\vec{r})$ is a periodic function of $\vec{r}$ with appropriate periods. The main role of such terms is to get rid of the translation group and to fix the position of the cristal structure.

We define the quasi-average $\langle A \rangle$ as

$$\lim_{\nu\to 0} \langle A \rangle_\nu \ ; \quad \langle A \rangle_\nu = \lim_{V\to\infty} \frac{Sp\,(A e^{-\frac{H_\nu}{\theta}})}{Sp\ e^{-\frac{H_\nu}{\theta}}},$$

where H_ν is the hamiltonian supplied with the terms (30).

Then the quasi-average

$$\langle \overset{+}{\psi}(\vec{r})\,\psi(\vec{r}) \rangle$$

is no more a constant; it is a periodic function of representing in fact the average distribution of density in the cristal.

The quasi-averages in momentum representation

$$\langle \overset{+}{a}_k \ a_{k'} \rangle \qquad (31)$$

need not be zero, even if $\vec{k} \neq \vec{k}'$. The selection rule based on the momentum conservation law does not work here.

It is easy to note that the quasi-averages (31) are defined up to the factor $e^{i(\vec{k}-\vec{k}')\vec{\xi}}$ with an indeterminate $\vec{\xi}$.

In fact, if we replace $U(\vec{r})$ by an equally acceptable function $U(\vec{r}+\vec{\xi})$ then (31) are to be multiplied by $e^{i(\vec{k}-\vec{k}')\vec{\xi}}$

The degeneracy is removed by fixing the form of the external field $U(\vec{r})$

Let us turn now oue attention to the most general case of the statistical equilibrium of a macroscopic dynamical system. We first consider the usual averages:

$$\langle \ldots \overset{+}{\psi}(t_\alpha, \vec{r}_\alpha, s_\alpha) \ldots \psi(t_\beta, \vec{r}_\beta, s_\beta) \ldots \rangle = \lim_{V\to\infty} \frac{Sp\left\{ (\ldots \overset{+}{\psi}(t_\alpha, \vec{r}_\alpha, s_\alpha) \ldots \psi(t_\beta, \vec{r}_\beta, s_\beta) \ldots) e^{-\frac{H}{\theta}} \right\}}{Sp\, e^{-\frac{H}{\theta}}} \qquad (32)$$

or the corresponding quantities in momentum representation:

$$\langle \ldots \overset{+}{a}_{p_\alpha, s_\alpha} (t_\alpha) \ldots a_{p_\beta, s_\beta} (t_\beta) \ldots \rangle \tag{33}$$

We may remark that the additive conservation laws lead to the selection rules for these averages. For example the law of conservation of the total number of particles demands the annihilation of all those averages (32), (33) for which the number of the creation operators is not equal to the number of the destruction operators.

The law of conservation of the total momentum demands the annihilation of all those averages (33) for which

$$\sum \vec{p}_\alpha - \sum \vec{p}_\beta \neq 0$$

etc.

We shall formulate now the general definition of the degeneracy and that of the non degeneracy of a state of the statistical equilibrium.

Introduce in the given hamiltonian the infinitesimal external field or source terms which violate the mentioned conservation laws and observe the effect of these terms on the values of the usual averages. We shall say that the considered state of the statistical equilibrium is not degenerate if these averages exhibit only infinitesimal variations.

We shall speak about the degeneracy of a state of the statistical equilibrium if some of the averages (32), (33) obtain finite increments and the selection rules become violated an a finite amount...

As we are considering only the stable systems those combinations of averages which correspond to physical quantities independent from phase-angles, directions etc. may exhibit only infinitesimal variations even in the cases of the degeneracy.

For degenerate states of statistical equilibrium the notion of the quasi-average may be introduced just in the way explained above - the quasi-average $\langle A \rangle$ being the limiting value of the usual average $\langle A \rangle_\nu$ corresponding to the hamiltonian H_ν with infinitesional extra terms removing the degeneracy.

We stress once more that in defining the quasi-averages the conventional limiting process of the statistical mechanics $V \to \infty$ must first be carried out, before we tend ν to zero.

Let us consider for example the situation in the theory of superconductivity (below the transition point) where the degeneracy is physically conditioned by the appearance of a condensate of bout pairs in the s-state.

In such cases the degeneracy may be removed by including into the hamiltonian the source terms of the form:

$$-\nu \sum_{(f)} w(f)\left(\overset{+}{a}_f \overset{+}{a}_{-f} + a_{-f}\, a_f\right) \tag{34}$$

where

$$\nu > 0; \quad f = (p, s); \quad -f = (-p, -s)$$
$$w(-f) = -w(f)$$

and where $w(f)$ is a real function.

These terms remove the gauge - group

$$a_f \longrightarrow a_f\, e^{i\varphi}$$

connected with the conservation law of the total number of fermions.

We have previously discussed the influence of such terms in the case of the model system caracterized by the reduced hamiltonian.

In the general case the situation remains the same*. In particular the quasi-averages

$$\langle a_{-f}\; a_f \rangle$$

have real non zero values in spite of the selection rules.

Let us also remark that in the theory of the superfluidity of Bose-systems the source terms removing the degeneracy may be taken in the form:

$$-\nu \sqrt{V}\; \left(a_0 + \overset{+}{a}_0\right)$$

Because of these terms we can consider both a_0 and $\overset{+}{a}_0$ as macroscopic c -numbers equal to $\sqrt{N_0}$ with an assymptotic accuracy.

The general notions of the degeneracy of the states of the statistical equilibrium and the quasi-averages were introduced here in view of their application to the perturbation theory treatment of problems of the statistical mechanics.

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* There is a difference with the previously considered case of the model system. In that case all mentioned results were proved with full mathematical rigour and in the general case we must be satisfied by more euristic arguments of the kind adopted in treating most problems of the statistical mechanics.

As is well known in the recent years some powerful techniques using the partial summation have been developped in the framework of the perturbation theory.

We are refering here to the Feynman diagrams technique and its various modifications and generalisations.

Among somewhat older approaches we also may mention the Fock's method very convenient for the computation of 'the first approximation '.

Nevertheless we wish to point out now that some aspects of these techniques need to be clarified. It is instructive to notice, for example, that varions attempts to use the Feynman diagrams approach in the orthodox way have failed in the theory of cristals as well as in the theory of superconductivity. By the orthodox way of using the Feynman diagrams we mean the procedure of introducting the diagrams containing only those lines which are 'permitted' by all selection rules.

Refering to our previous discussion we see that the main cause of the difficulty resides in the fact that the degeneracy was not removed before using the perturbation theory.

We may now announce as a kind of the general principle the following prescription:

In order to apply any form of the perturbation theory treatment for the study of a degenerate state of the statistical equilibrium we must first remove the degeneracy or, what amounts to the same,, we must work not with the usual averages, obeying all selection rules, but with the quasi- averagds which do not satisfy some of them.

Therefore the diagrams must contain also 'anomalous lines' which are to be introduced always in the (as last partially) summed form.

Such lines correspond to 'dangerous diagrams' in the sense that they give a finite contribution in spite of the fact that they are formally conditioned by the infinitesimal extra terms in the hamiltonian.

We may remark that these infinitesimal external field or source terms can be ommited in actual computations. From the purely technical point of view their only role consists in granting us a kind of 'permit' for using the anomalous Green's functions based upon the corresponding quasi-averages.

For example in order to obtain the correct results in the theory of the cristal state the diagrams must contain not only the lines $\overline{\overset{+}{a}_p\, a_p}$ 'conserving' the momentum but also the anomalous lines $\overline{\overset{+}{a}_p\, a_{p'}}$

In the theory of superconductivity the diagrams must contain not only the usual lines $\overline{\overset{+}{a}_p\, a_{p'}}$ but also the anomalous lines $\overline{a_{-f}\, a_f}$, $\overline{\overset{+}{a}_f\, \overset{+}{a}_{-f}}$, etc.

The main cause of the success of our u - v transformation in obtaining the correct results for the theory of superconductivity was due to the fact that it enabled us to work with the quasi-averages.

It is still an open question whether this situation has some meaning for the quantum field theory.

In view of the recent remarks made by Nambu we may think about the possibility that in the quantum field theory too the Feynman diagrams may contain more lines than it is permitted by the selection rules.

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