ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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PION–NUCLEON SCATTERING AT LOW ENERGIES. II

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Abstract

Using the results of the preceding paper[1] the integral equation for α_{33} and expressions for the other phase-shifts are obtained. It contain the phase-shifts of $\pi\pi$ - scattering δ_0 and δ_1. The important role of backward πN- scattering dispersion relations is established. The phase-shift δ_0 is shown to influence greatly the πN -scattering. The scattering length of δ_0 is evaluated from consideration of the small p-phase-shifts of πN -scattering near the threshold.

1. Transition to Partial Waves

In the foregoing paper/1/* the analytical properties of the scalar coefficients of the pion - nucleon scattering amplitude in complex q^2-plane for fixed $\cos\theta = c$ (in the c.m.s.) have been analysed. The contribution from the cut of crossing-reaction III was approximately expressed by the functions

$$F_i(q^2,c) = e^{u_i(q^2,c)} \quad ; i = 0, 1 \tag{1.1}$$

Here u_0 and u_1 are determined by $\pi\pi$-scattering phase-shifts (see formula (1.5.5)**. The function F_1 for $c = -1$ is the electromagnetic formfactor of the pion/2/. Therefore, for the sake of brevity we shall call F_0, F_1 s- and p- formfactors (although actually this term is a formal one for F_0).

These formfactors F_i entered the kernels of the final system of integral relations (5.4), (1.5.6) and (1.5.10) which determine the amplitudes of πN -scattering.

The integral terms of these relations are explicitly expressed by the partial scattering amplitudes. In order to get integral equations for partial wave amplitudes we must go over to partial waves on the left-hand side of our relations. The usual way is integrating with $P_\ell(c)$ over the region $-1 \leq c \leq +1$. But this is not the only possible way, e.g. CGLN/3/ used to that end the expansion in c about $c = 1$.

Let us investigate more closely the analyticity properties of Φ in the unphysical region of reaction I in order to choose the best way of 'projecting' on the partial waves. When deriving the integral relation the expansion of Φ in cosine of reaction III in unphysical region has been used. The convergence region of such expansion is determined by singularities of Mandelstam

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** I.e. the formula/5.5/ from I.

representation. The nearest ones, after the poles being extracted, which restrict the expansion of $Re\,\Phi$ are the asymptotes $s=(M+\mu)^2$, $\bar{s}=(M+\mu)^2$. The region of convergence for $Jm\,\Phi$ is wider and determined by the spectral functions boundaries (the curves C_{13}, C_{23} in Mandelstam's notation/4/).

Using Heine's theorem/5/ we can find the ellipse of analyticity in the complex plane $\cos\Theta_3$. Then we go over to the variables q^2, c by means of (1.5.1). The curves C_R and C_I (Fig. 1) can be obtained which restrict the Legendre expansion of $Re\,\Phi$ and $Jm\,\Phi$ correspondingly.

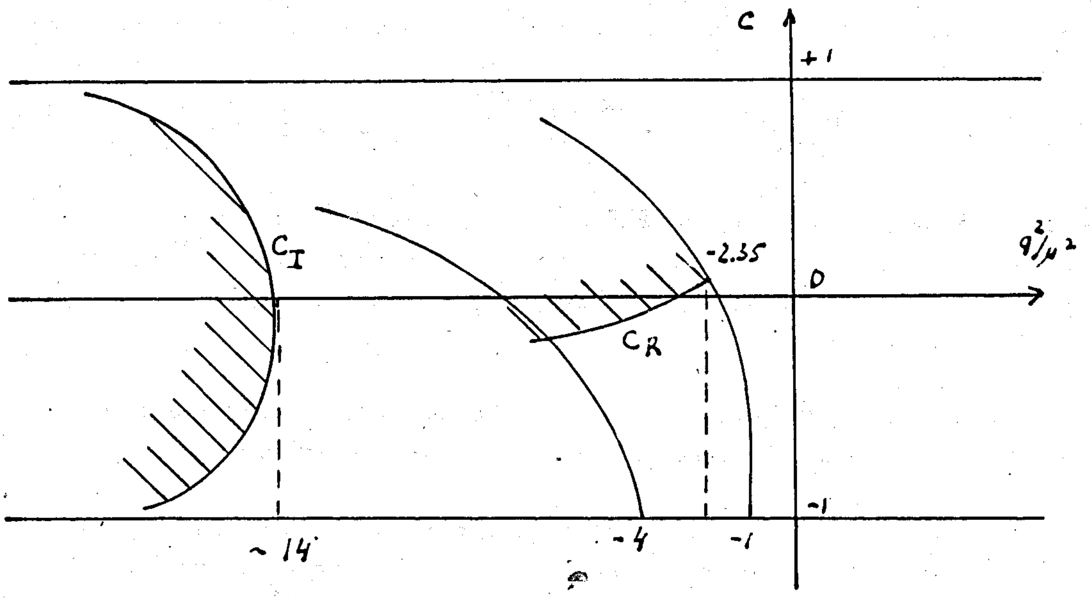

Fig. 1.

One can see, by inspection of Fig. 1, that the usual way of going over to partial wave amplitudes enables us to take into consideration the nearest singularity from reaction III up to $q^2=-2,35$ only. (From here on we put $\mu=1$).

Practically we restrict ourselves to two s- and p-waves of the reaction III. This approximation gives a great error in a certain region below the curve C_R where $\cos\Theta_3$ has a large complex value. Therefore we make a transition to partial waves by expanding our functions into a Taylor series at the point $c=-1$ (see in this connection also /6/).

The advantages of this procedure are the following:

1) the nearest part of unphysical contribution from reaction III is taken into account with a considerable accuracy;

2) the amplitude of reaction III enters the integrand for the physical value of $\cos\theta_3 = -1$, so we do not use in fact the analytical continuation into the unphysical region of $\cos\theta_3$ and always stay in the region of convergency. We can expect our formulae to be correct in some region below $q^2 = -4\mu^2$ where the inelastic contribution is still small;

3) at $c = -1$ the unphysical cuts from crossing reaction II are absent and the cut from reaction III is the only unphysical one;

4) at $c = -1$ our formulae are very simple, because relation (1.4.7) is greatly simplified.

2. The Equations for Partial Waves

Bearing in mind the application at low energies we restrict ourselves to s- and p-waves only. We have in this approximation

$$
\begin{aligned}
f_s^{(\pm)}(q^2) &= f_1^{(\pm)}(q^2,-1) + f_1^{(\pm)\prime}(q^2,-1) \\
f_{p1/2}^{(\pm)}(q^2) &= f_2^{(\pm)}(q^2,-1) + \frac{1}{3} f_1^{(\pm)\prime}(q^2,-1) \\
f_{p3/2}^{(\pm)} &= \frac{1}{3} f_1^{(\pm)\prime}(q^2,-1)
\end{aligned}
\tag{2.2}
$$

By (1.2.3) $f_{1,2}^{(\pm)}(q^2,-1)$ are expressed through $A^{(+)}$, $\alpha = A^{(-)}/(s-\bar{s})$, $\beta = B^{(+)}/(s-\bar{s})$, $B^{(-)}$ as follows:

$$
\begin{aligned}
f_1^{(+)}(q^2,-1) &= \frac{p_0+M}{8\pi W}\left\{A^{(+)}(q^2) + 4p_0\omega(W-M)\beta(q^2)\right\} \\
f_1^{(-)}(q^2,-1) &= \frac{p_0+M}{8\pi W}\left\{4p_0\omega\,\alpha(q^2) + (W-M)B^{(-)}(q^2)\right\} \\
f_2^{(+)}(q^2,-1) &= \frac{p_0-M}{8\pi W}\left\{-A^{(+)}(q^2) + 4p_0\omega(W+M)\beta(q^2)\right\} \\
f_2^{(-)}(q^2,-1) &= \frac{p_0-M}{8\pi W}\left\{-4p_0\omega\,\alpha(q^2) + (W+M)B^{(-)}(q^2)\right\}
\end{aligned}
\tag{2.3}
$$

$$f_1^{(+)'}(q^2;-1) = \frac{p_0+M}{8\pi W}\left\{A^{(+)'}(q^2)+(W-M)\left(2q^2\beta(q^2)+4p_0\omega\beta'(q^2)\right)\right\}$$

$$f_1^{(-)'}(q^2;-1) = \frac{p_0+M}{8\pi W}\left\{2q^2\alpha(q^2)+4p_0\omega\alpha'(q^2)+(W-M)B^{(-)'}(q^2)\right\}$$

Here $\omega = q_0$, $W = p_0+q_0$, $\phi(q^2)$ and $\phi'(q^2)$ are $\phi(q^2;c)$ and $\frac{d}{dc}\phi(q^2;c)$ in $c=-1$ after subtraction at the point $q^2=0$.

Let us consider the subtraction. Supposing, as usual, a linear rate of increase of the amplitude at infinity and taking into account crossing-symmetry properties, it is easy to show that for the functions $A^{(+)}$, α and $B^{(-)}$ one subtraction is sufficient, and that there is no need of such subtraction for β. It is convenient to take the experimental values of scattering lengths a_1, a_3 and $a_{33} = \lim\limits_{q\to 0}\frac{\alpha_{33}}{q^3}$ as subtraction parameters. We also subtract the functions $\phi'(q^2)$ at the point $q^2=0$ without introducing additional parameters, to ensure the correct vanishing at $q^2\to 0$. Let us introduce the following notations

$$\Phi = \Phi_0+\Phi_p+\Phi_{\pi N}+\Phi_{\pi\pi} \quad ; \quad \Phi' = \Phi_p'+\Phi_{\pi N}'+\Phi_{\pi\pi}' \qquad (2.4)$$

Here Φ_p is the pole term contribution, $\Phi_{\pi N}$ is that part of the integral terms which contains no formfactor and $\Phi_{\pi\pi}$ is the one which contains them. $\Phi_{\pi\pi}$ is zero if $F_0=F_1=1$.

All the terms in the right-hand side of (2.4) are vanishing at $q^2=0$ except subtractional constants Φ_0. These, except β_0, can be expressed through experimental values

$$a_- = \frac{1}{3}(a_1-a_3)\ ,\quad a_+ = \frac{1}{3}(a_1+2a_3)\ ,\quad a_{33}\ .$$

Let us make an additional approximation. We shall restrict ourselves in the integrand to α_{33} only, because of the relative values of the phase shifts.

The integrals in the right-hand side of (2.4) will depend then on $\mathrm{Im}\, f_{33} = \psi(q^2)$ and formfactors only. As a result we get the integral equation for f_{33} and the other phase shifts are determined by ψ. Such an approximation corresponds to the first step when solving the complete set of integral equations for partial wave amplitudes, by iteration procedure, the experimental value of α_{33} being the zero approximation.

The subtractional constants can be expressed as follows:

$$\frac{A^{(+)}(0)}{4\pi} = (1+\frac{1}{M})a_+ + M\frac{\beta(0)}{\pi} \; ; \; \frac{B^{(-)}(0)}{4\pi} = (1+\frac{1}{M})a_- - M\frac{\alpha(0)}{\pi}$$

$$\frac{\beta(0)}{\pi} = -\frac{4f^2}{1-\frac{1}{4M^2}} + \frac{1}{\pi^2}\int_0^\infty \frac{Jm\,\beta(x)}{x}dx$$

$$\frac{\alpha(0)}{\pi} = \frac{\beta(0)}{\pi} - 6(1+\frac{1}{M})a_{33} + \tag{2.6}$$

$$+\lim_{q^2\to 0}\frac{1}{q^2}\{A^{(+)\prime}(q^2) - B^{(-)\prime}(q^2) + 4M\beta'(q^2) - 4M\alpha'(q^2)\}$$

and the pole terms are

$$\beta_p = \frac{4\pi f^2 q^2}{(1-\frac{1}{4M^2})(1+q^2-\frac{1}{4M^2})} \; ; \; \beta_p' = \frac{8\pi f^2 q^2}{(1+q^2-\frac{1}{4M^2})(2p_0\omega-1-2q^2)}$$

$$B_p^{(-)} = \frac{8\pi f^2 q^2(1-\frac{1}{2M})}{(1-\frac{1}{4M^2})(1+q^2-\frac{1}{4M^2})} -$$

$$-16\pi Mf^2\left[\frac{q^2}{p_0\omega}\left(\frac{p_0}{\omega+1}-\frac{1}{M+p_0}\right)\ln\frac{W}{p_0-\omega} + \ln\left(\frac{M-1}{p_0-\omega}\cdot\frac{W}{M+1}\right)\right]$$

$$B_p^{(-)\prime} = -32\pi M^2 f^2\frac{q^2}{2p_0\omega-1-2q^2} + \tag{2.7}$$

$$+4\pi M^2 f^2\frac{q^2}{p_0^2\omega^2}\left[2-\frac{M^2+1+2q^2}{p_0\omega}\ln\frac{W}{p_0-\omega}\right]$$

Let us go over to integral terms. After putting $c = -1$ in (1.5.4), (1.5.6) and (1.5.10) we have taking into account (1.6.3)

$$\phi_{\pi N} = \frac{q^2}{\pi}\int_0^\infty \frac{Jm\,\phi(x)}{x(x-q^2)}dx \tag{2.8}$$

$$\Phi_{\pi\pi} = \frac{q^2}{\pi}\int_0^\infty \frac{Im\,\Phi(x)}{x-q^2}\, G_i(x,q^2)\,dx \tag{2.9}$$

where

$$G_i(x,q^2) = \frac{E_i(x) - E_i(q^2)}{F_i(x)} \quad ; \quad F_i(x) = 1 - x E_i(x) .$$

By differentiating of the aforementioned expressions with respect to c at $c = -1$ one can find

$$\left.\begin{aligned} \Phi'_{\pi N} &= \frac{q^2}{\pi}\int_0^\infty \frac{Im\,\Phi'(x)}{x(x-q^2)}\left\{1 - 2(x-q^2)\varphi(x,q^2)\right\}dx + \\ &\qquad + \frac{q^2}{\pi}\int_0^\infty Im\,\Phi(x)\,\frac{d\varphi(x,q^2)}{dx}\,dx \\ \varphi(x,q^2) &= \frac{\frac{x}{q^2}\left[K(q^2) - M\right] - \left[K(x) - M\right]}{x-q^2}\cdot\frac{1}{W^2(x)} \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} \Phi'_{\pi\pi} &= -\frac{q^2}{\pi}\int_0^\infty \frac{Im\,\Phi'(x)}{x-q^2}\, G_i(x,q^2)\,\frac{P_0^2(x)+\omega^2(x)}{W^2(x)}\,dx + \\ &- \frac{q^2}{2\pi}\int_0^\infty \frac{Im\,\Phi(x)}{x-q^2}\,\Delta_i(x,q^2)\,dx - \frac{q^2}{\pi}\int_0^\infty Im\,\Phi(x)\,\frac{d}{dx}\left\{\frac{x K(x)\, G_i(x,q^2)}{W^2(x)(x-q^2)}\right\}dx \\ \Delta_i(x,q^2) &= \frac{F_i'(x)\left(x E_i(q^2) - 1\right) + F_i'(q^2)\, F_i(x)}{F_i^2(x)} \end{aligned}\right\} \tag{2.11}$$

In formulae (2.9)–(2.11) $i=0$ for $A^{(+)}$, $i=1$ for $\alpha, B^{(-)}$ but $\beta_{\pi\pi}=0$. The sign of Δ, G is chosen to be positive when the $\pi\pi$ -scattering length is positive. The imaginary parts of Φ and Φ' entering (2.8)–(2.11) are expressed through ψ in the following way

$$Jm\, A^{(+)}(x) = -\frac{16\pi}{3}\frac{\psi(x)}{x}\left[p_o(x)\omega(x) + x - 2M\omega(x)\right]$$

$$Jm\, A^{(+)\prime}(x) = 8\pi\frac{\psi}{x}\left[(p_o - M)\omega + x\right]$$

$$Jm\, B^{(-)} = \frac{8\pi}{3}\frac{\psi}{x}(2p_o - M)$$

$$Jm\, B^{(-)\prime} = -4\pi\frac{\psi}{x}(p_o - M).$$

$$Jm\, \alpha = \frac{3\pi}{2}\frac{\psi}{x}\,\frac{p_o\omega + x - 2\omega M}{\omega p_o}$$

$$Jm\, \alpha' = -\frac{\pi}{3}\frac{\psi}{x}\,\frac{3\omega p_o\left[(p_o-M)\omega + x\right] + x(p_o\omega + x - 2\omega M)}{\omega^2 p_o^2}$$

$$Jm\, \beta = -\frac{4\pi}{3}\frac{\psi}{x}\,\frac{2p_o - M}{p_o\omega} \qquad (2.12)$$

$$Jm\, \beta' = 2\pi\frac{\psi}{x}\left[\frac{(2p_o - M)x}{3p_o^2\omega^2} + \frac{p_o - M}{p_o\omega}\right]$$

So, we have for f_{33} the integral equation which contains pion formfactors and the other phase shifts are expressed by means of ψ, the subtractional constants and formfactors.

3. The Expansion in $1/M$ and Evaluation of the Small p-Waves

Let us look into the properties of the equations obtained. After the expansion in $1/M$ the main terms of the πN-part coinside with those of the CGLN equations after in the latter ones an analogous subtraction has been made. But our equations contain in addition $\pi\pi$ -terms. Note that the formal transition to the static limit $(M \to \infty)$ is impossible even in p-waves because the contribution to $\Phi_{\pi\pi}$ from s-wave is proportional to M.

Small p-waves for $q^2 \to 0$ $\alpha_{ik} \approx a_{ik} q^3$ are determined by the subtractional constants a_+, a_-, a_{33} and integrals which contain $\psi(x)$ and formfactors. After the calculations having been performed we get with an accuracy of $1/M$

$$
\begin{aligned}
a_{31} &= -0{,}141 + 2M\Psi_0 \\
a_{13} &= -0{,}164 + 2M\Psi_0 - \Psi_1 \\
a_{11} &= 0{,}075 - 2M\Psi_0 + 2\Psi_1 .
\end{aligned}
\tag{3.1}
$$

Here

$$
\Psi_0 = \frac{1}{3\pi}\int_0^\infty \frac{\omega(x)\psi(x)}{x^2}\,\Delta_0(x,0)\,dx
$$

$$
\Psi_1 = \frac{1}{3\pi}\int_0^\infty \frac{\psi(x)}{x\,\omega(x)}\left[2G_1(x,0) + \Delta_1(x,0)\right]dx . \tag{3.2}
$$

The values $f^2 = 0.08$, $a_+ = -0.016$, $a_- = -0.094$, $a_{33} = 0.232$ have been used.

The integrals with f_{33} were computed over the interval from 0 to $q^2 = 6$. The values of f_{33} in this interval have been taken from experiment/7,8/. Taking into account the integral $\int_6^\infty$

we find the accuracy of our calculations to be 5% what corresponds in (3.1) to a few thousandths.

By comparing the right hand side of (3.1) with the experimental data we can estimate the integrals Ψ_0, Ψ_1 entering it. A resonable correspondence can be obtained with the help of Ψ_0 only. In the table the experimental data and the values of the right-hand sides of (3.1) for $2M\Psi_0 = 0{,}12$ are given.

Table

	Experiment	Theory
a_{31}	–0.039 ± 0.022	–0.02
a_{13}	–0.044 ± 0.005	– 0.04 – Ψ_1
a_{11}	–0.038 ± 0.038	– 0.045 + 2 Ψ_1

By inspection of table one can see that the large experimental errors make it impossible to draw any conclusion about the integral Ψ_1 which depends on the p -wave of $\pi\pi$ -scattering. However, the sign and order of magnitude of Ψ_0 can be considered to be reliable. This result is in accordance with the conclusions of paper/9/.

If we take for the s-wave $\pi\pi$ -scattering the 'scattering length' approximation (see/9/)

$$tg\, \delta_0(q) = \alpha_s q$$

which gives us $\Psi_0 \simeq 0{,}01 \frac{\alpha_s}{1+\alpha_s}\left(1+\frac{2\alpha_s}{1+\alpha_s}\right)$ we obtain from $2M\Psi_0 = 0.12$ the value $\alpha_s = 0.9/\mu$. This result also agrees with the paper/9/ where $\alpha_s \sim 1/\mu$ has been obtained. We stress in particular that the numbers of the right-hand side of (3.1) are not reliable. The matter is that the difference of large terms occur and small terms of the order $1/M$ become important.

However, this terms can not be computed with necessary accuracy because the terms which arise in the expansion of denominators of the kind $1 + 2\frac{\omega(x)}{M}$ give badly convergent integrals. That is why a reliable computation of the small p -wave has to be done without an expansion in $1/M$. This computation and the solution of the integral equation for f_{33} are now in progress.

4. The Discussion of Results

1. From the double Mandelstam representation the system of integral equations for the partial wave amplitudes of πN -scattering have been obtained. The important role of πN-backward scattering has been ascertained. It is this dispersion relations which are already sufficient to get the system of integral equations. As there is little hope for the strict proof of the Mandelstam representation the proof of backward scattering dispersion relation becomes of great interest.

2. The s-phase shift of $\pi\pi$ -scattering is shown to enter the expression for πN-partial waves with a large coefficient M. Due to this fact we succeeded in determining the sign and order of magnitude of δ_0 from the consideration of the small p-waves of πN--scattering near the threshold only, dispite the roughness of our computation and large experimental errors. We hope that a more precise computation of s- and p - waves in the energy interval 100–200 Me V will enable us to get some more information about p-wave of $\pi\pi$ - scattering.

3. These results corresponds to the paper/9/ and contradict to the conclusion of paper/10/. Putting aside the question of different methods to obtain equations for partial wave amplitudes (see in this connection/6/) we note the following:

a) having in mind to get information about resonance behaviour of p-wave of $\pi\pi$ -scattering the authors left out of consideration the s-wave which gives the main contribution to πN scattering; b) there is no subtaction in this paper, the necessity of which was already emphasized by CGLN. As a result, for example, δ_1 gives a contribution to a_{13} in this work. Besides, the integrals in this paper which depend on $\mathrm{Im}\, f_{33}$ were evaluated very roughly. The value a_{33} as defined by (5.3) of/10/, being computed by our method is 0.176 instead of 0.213.

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References

1. A.V.Efremov, V.A. Meshcheryakov, D.V. Shirkov, 'Equations for Low-Energy Meson-Nucleon Scattering', JINR-Preprint D–503, 1960.
2. W. Frazer, J. Fulco, Phys.Rev. Lett. 2, 365 (1959).
3. G.F. Chew, M. Goldberger, F. Low, Y. Nambu, Phys.Rev. 106, 1335 (1957).
4. S. Mandelstam, Phys.Rev. 112, 1344 (1958).
5. E. Wittaker, G. Watson, 'A Course of Modern Analysis', § 154.
6. A.V. Efremov, V.A. Meshcheryakov, D.V. Shirkov and H.Y. Tzu, ' On the Derivation of Equations from the Mandelstam Representation', JINR - Preprint E-539.
7. Kiev Conference on High Energy Physics (1959) Pontecorvo's Report; CERN – Symposium, Volume II, p. 225, 230 (1956).
8. Geneva Conference on High Energy Physics (1958).
9. K. Ishida, A. Takahashi, Y. Ueda, 'Pion-Pion Interaction and Pion-Nucleon Scattering', Preprint.
10. J. Bowcock, W. Cottingham and D. Lurie, 'Effect of a Pion-Pion Scattering Resonance at Low Energy Pion-Nucleon Scattering', CERN - preprint.

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