ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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E – 539

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Nucl. Phys., 1961, v22, n2, p202.
Intern. Conf. on High Energy Physics. Rochester, 1960, p278-281

Дубна 1960 год

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Submitted to Nuclear Physics

During the last year a series of papers on the derivation of integral equations for scattering processes have appeared. These papers[1–6] have the common feature, that analytic properties of partial wave amplitudes are studied and employed from beginning. As a consequence, there appears a limitation on the domain of convergence in the unphysical region due to the spectral functions of Mandelstam's representation.

Recently, we derived[7,8,9] the system of equations for πN -scattering using rather different approach. It is based on the dispersion relations at fixed scattering angle which enable us to make detailed investigation into the analytical properties in the unphysical region. It leads us to a new recipe for the derivation of the integral equations for partial wave scattering amplitudes.

The essential point is the following. As is well known two kinds of approximation are usually made in the derivation of the integral equation from the dispersion relation and unitary condition.

1) The two particle intermediate state approximation of the unitary condition is made. The contribution to the imaginary parts of the scattering amplitudes from intermediate states involving three or more particles are usually neglected.

2) Only waves of the low angular momentum states are taken into account. The contribution from the waves of the highest angular momentum states are usually neglected.

The first approximation can be justified in the low energy phenomena. The second approximation is consistent with the first in the physical region. However, one has to be very careful in throwing away the higher waves in the unphysical region. As we shall show below in a number of important cases, the second approximation leads to large errors in the unphysical region even at low energies. For the sake of illustration, we discuss in detail the simplest case of the $\pi\pi$ -scattering. A short summary of the results on the πN -scattering is then given.

The dispersion relations for the $\pi\pi$ -scattering at fixed value of $\cos\theta = c$ has the following form:

$$A(\nu, c) = \frac{1}{\pi}\int_0^\infty d\nu' \frac{Im A(\nu', c)}{\nu' - \nu} + \frac{1}{\pi}\int_{-\infty}^{a} d\nu' \frac{Im A(\nu', c)}{\nu' - \nu} \qquad (1)$$

$\nu = q^2$ is the square of momentum in c.m.s. of the reaction I. The location of the point a depends on c and is shown in Fig. 1. In the unphysical region $-\infty < \nu < a$ $Im A$ is usually expressed in terms of the partial wave amplitudes of reaction II and III by means of analytic continuation with the help of the Legendre expansion. The expansion fails at the point β on the boundary of the region where the spectral function does not vanish. Even before reaching the point β, there is a large region, where the distance from the physical region is comparable with that from the point β. It is unlikely,

that the scattering amplitude in this region can be approximated with reasonable accuracy by keeping only a few terms of Legendre expansion. Using the spectral representation, we can put the dependence of the imaginary part of the scattering amplitude on $\cos\bar{\theta} = \bar{c}$ in an explicit form. This dependence is determined by the denominators $\tau \pm \bar{c}$ appearing in the representation, where $\tau = 1 + s'/2\bar{\nu}$ According to the theorem of Heine

$$\frac{1}{\tau \pm \bar{c}} = \sum_{l \geq 0} (2l+1) Q_l(\tau) P_l(\mp \bar{c}). \qquad (2)$$

The usual approximation consists in representing the denominators by the first two terms of the above expansion. The accuracy of this approximation can be estimated as follows. Let us put $\bar{\nu} = 3$, which corresponds to the threshold of the first inelastic process. Since the most important contribution comes from the neighbourhood of β, the value of s' is chosen to be that corresponding to the point β. $\bar{c}$ is determined by c and $\bar{\nu}$. The results of the calculation is given in the following table for two different scattering angles $c = 1$ and $c = 0$.

c	$\bar{c}$	$\frac{1}{\tau - \bar{c}}$	$Q_0(\tau) + 3\bar{c}Q_1(\tau)$	Error
1	1	0.281	0.273	3%
0	11/3	1.125	0.405	!

Thus, such approximation can lead to large errors for $c = 0$.

The partial wave dispersion relation can be obtained by integrating equation (1) over c from -1 to $+1$ after being multiplied by $P_l(c)$. It is then evident that there is a large domain in the unphysical region where such approximation can not be used even if the region beyond $\nu' = -L = -10$ is cut off.

To avoid introducing a new parameter the cutoff momentum L into the theory, Chew and Mandelstam let $L \to \infty$ after dropping the d and higher waves. Actually, if the equation (1) after one subtraction is used, the coefficient of the d- wave behaves like $\ln L$ at $L \to \infty$. It is doubtful. whether the contribution from the high energy unphysical region is sufficiently suppressed by one subtraction, so that the solution of the equation does not depend sensitively on the value of L. Actually, Fraser and Fulco[2] used similar method to treat the problem of the nucleon structure and the $N\bar{N}$ -annihilation. A part of their results depends sensitively on the value of the cutoff momentum.

The same situation exists for the case of πN- scattering. Due to the kinematics peculiarity ($M \neq \mu$) the cosine of the reaction's angle in $\pi + \pi \to N + \bar{N}$ takes complex values in the unphysi-

cal region. There also exists region where Legendre expansion converges, but the first two terms of it give a bad approximation.

To avoid these difficulties common to various processes we propose another general approach to the problem of deriving integral equations for low energy processes from the dispersion relations and unitary conditions. The idea is as follows. It is known experimentally, that only a few angular momentum states are important at low energies. To determine these states theoretically, it is sufficient to have expressions for the scattering amplitudes (and its derivatives) only at a few angles. There is no need of using dispersion relations for all scattering angles. Now it is evident from Fig. 1. that the range of the validity of analytic continuation by means of Legendre expansion varies greatly with different angles. We propose to choose those angles for deriving integral equations, for which the Legendre expansion fails only at the distant part of the unphysical region. The forward and backward directions are especially favorable in this case, since the path of the dispersion integral never intersect the boundary of the region where the spectral functions do not vanish, the problem of the breakdown of the Legendre expansion never arises.

This program is carried out in /10/ for the case of $\pi\pi$ -scattering. The Taylor expansion with respect to $\cos\theta$ in the neighbourhood of $\cos\theta = 1$ is employed there. The obtained equations for the partial waves are different from that of Chew and Mandelstam with regards to the structure of the integral in the unphysical region.

The numerical solution of these equations is now in progress. It is possible, that the solution of the equations is considerably different from that of Chew and Mandelstam, if the p-wave is important.

The above mentioned method is applied to the problem of the πN -scattering in the works /8,9/. It is to be pointed out, that CGLN /11/ actually employed this method in their work on the πN -scattering. They employed an expansion in the neighbourhood of $\cos\theta = 1$. However, if the unitary condition is limited to the two particles intermediate state approximation the influence of the process $\pi+\pi \to N+\bar{N}$ on πN- scattering is absent for $\cos\theta = 1$. The influence of this process on the πN- scattering is the greatest at $\cos\theta = -1$. Therefore the dispersion relations for the backward scattering is employed together with that for forward scattering in the paper /8/ for the πN- scattering. The equations obtained have to be solved simultaneously with that for the $\pi+\pi \to N+\bar{N}$ process. In the paper /9/ an expansion in the neighbourhood of $\cos\theta = -1$ is used. Under certain approximation the system of equations is obtained for the partial waves πN -scattering amplitude, in the kernel of which the $\pi\pi$ -scattering phase shifts appear. If the phase shifts of the $\pi\pi$ -scattering are put equal to zero in the equation obtained in /8,9/ then the principle terms (in an expansion in powers of 1/M) coinside with ones of CGLN[11]. It is then possible to estimate the sign and the order of magnitude of s-wave scattering length of $\pi\pi$ -scattering from the p-phase shifts of πN -scattering at the threshold. The value obtained is $0.9/\mu$ which agrees with the results of /12/. The same problem is discussed in the paper /4/. However, the effect of s-wave $\pi\pi$ -scattering is not taken into account at all there. For this

reason, the results obtained in [4] can not be considered reliable.

Recently, the method proposed in this paper has been used for the derivation of the integral equation for the partial waves of the π K - scattering [13] and the KN - scattering [14].

Note added in proof (May 31)

In a recent paper (UCRL – 9126) Chew and Mandelstam discovered that in the case of $\pi\pi$ -scattering 'when the p-wave is large at low energies, the integral equations previously formulated in [1] require a cutoff' and the solution becomes unstable.

Therefore our criticism to the equations of paper [1] receives further confirmation.

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Received by Publishing Department
on May 17, 1960.

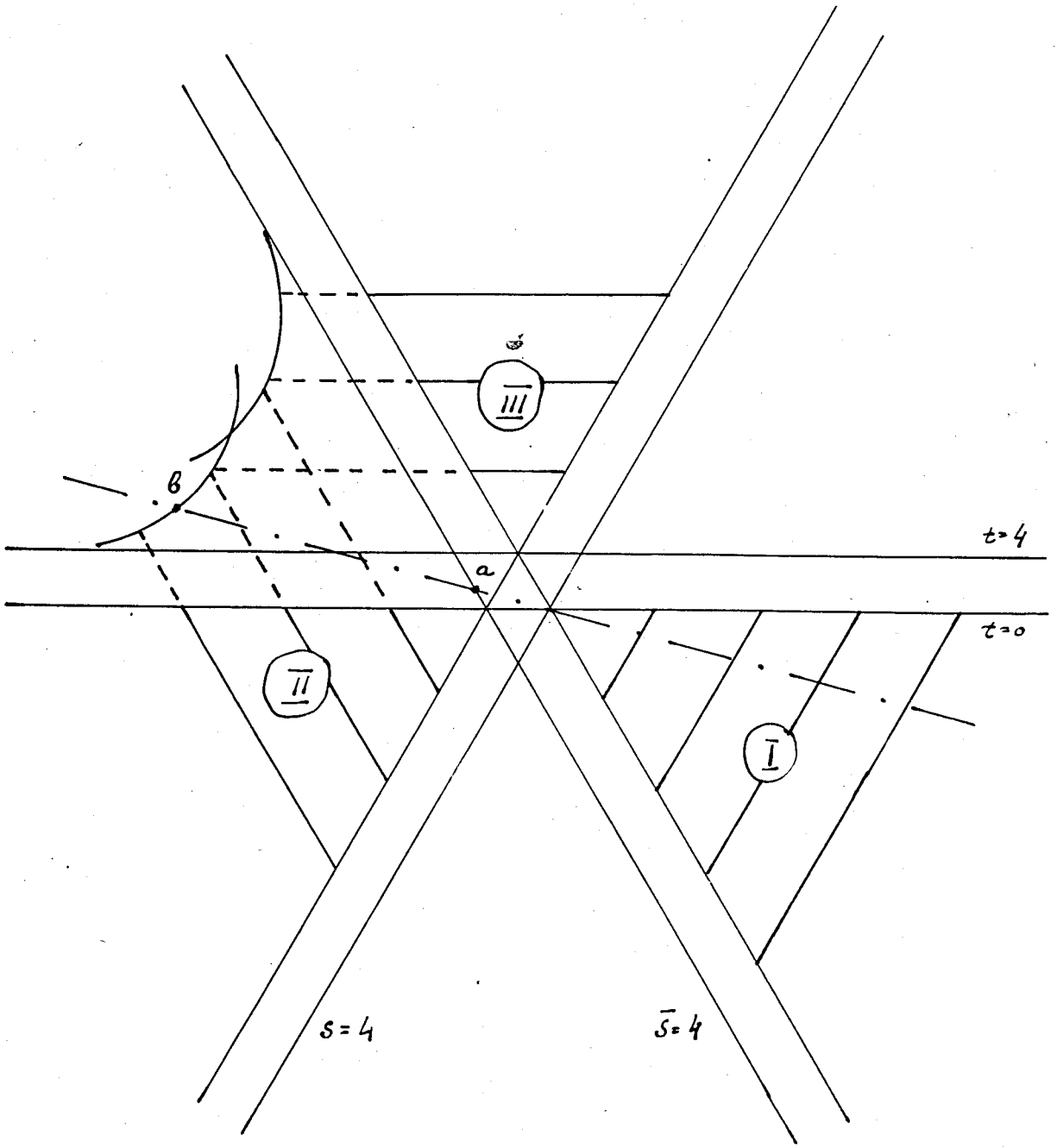

Fig. 1.