ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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THE METHOD OF CALCULATING STATISTICAL WEIGHTS AND DISTRIBUTIONS IN THE THEORIES OF MULTIPLE PRODUCTION

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Abstract

The effective method of calculating distributions and correlations in the arbitrary models of the multiple production has been found. The formulas are especially simple for the covariant model. It is shown how the study of this model can facilitate calculations in more complicated models.

Introduction

The success of the Fermi statistical theory in the prediction of the output of antiprotons at 25 BeV heightened the interest to this theory. On the other hand there appeared forcible proofs of the fact that the final states in the multiple production have not equal probability[1]. This betokens that in the general expression for the probability of the transition $w_{i\to f} = (2\pi/h)\,|H_{if}|^2\,\rho$ one cannot but take into account the dependence of the square of the transition matrix element $|H_{if}|^2 = \mathcal{F}(\vec{p}_1, \ldots \vec{p}_n) \equiv \mathcal{F}(p)$ on the characteristics of states (it depends at least on the directions of particle momenta).

New theories of multiple production will have therefore to deal with the integrals of the form

$$S = \int \mathcal{F}(\vec{p}_1, \ldots, \vec{p}_n)\,\delta\left(\sum \vec{p}_k - \vec{P}\right)\,\delta\left(\sum e_k - E\right)\,d\vec{p}_1 \ldots d\vec{p}_n$$

in calculating statistical weights and spectra. An example of this is the covariant formulation of the Fermi theory[2] in which it is implied that $\mathcal{F} = (2^n e_1 \ldots e_n)^{-1}$.

The usual methods of calculating statistical weights[3] comprizing the rather effective Monte-Carlo[4] methods consider the case when $F =$ Const but can not be applied when F is arbitrary.

In the present paper we treat the application of the Monte-Carlo method for obtaining statistical weights and quite generally, for obtaining kinematic distributions for an arbitrary interaction model with the square of the matrix element F. (The distributions of stars with many prongs according to kinematic characteristics: momentum distributions, angular and correlation ones et al are called by us kinematic distributions). To obtain these distributions it is necessary to find an effective way of derivation from the model the distributions which follow from this same model whatever complicated the model and/or kinematic characteristics are. It is shown[6] that the construction of a table of random stars which must correspond to the model solves this problem. However, there are two facts which decrease the efficiency of picking random stars[6], namely, it is difficult to fall into the physical region in which the momenta change, i.e. it is necessary to reject impossible sets of momenta, and, secondly, it is complicated to

reproduce the distribution of momenta which follows from the given model, i.e. it is necessary to reject unlikely sets of momenta.

The first aforementioned problem is solved in §1 by two methods. This can be done in a general form since the physicol region in which the momenta change is determined only by kinematic inequalities

The second difficulty is due to the kind of the model. However, we shall show in §1,2 how this difficulty can decrease in the factorizable models; in particular, convenient formulas will be obtained for the Fermi model and, especially, for the covariant model. Kinematic distributions which follow from more complicated models can be got with improved accuracy from the covariant model (§3). This is due to the fact that the efficiency of the Monte-Carlo method must increase with improving approximate models.

The kinematics of particles with variable mass and certain simple expressions for covariant weights are considered in Appendices A and B.

The simplicity of the formulas suggested in this paper will allow us very likely to calculate weights and kinematic spectra of several tens of particles in the covariant model and up to eight-ten particles - in other more complicated models.

§ 1. New Method of Obtaining Kinematic Distributions

1. Let the $(\vec{p}_1, \ldots, \vec{p}_n)$ – distributions density of the system of n particles with masses m_κ and energies e_κ be given by the square of the matrix element $F(\vec{p})$. The shape of the region D of the allowed values of the momenta is determined by the conservation laws. The complicated form of the region D makes the sampling of $\vec{p}_\kappa$ from D and the uniformity of this sampling difficult. Both the difficulties will vanish if we find the transformation of D into the $(3n-4)$ – dimensional cube. Such a transformation has been found by Ju.N. Blagoveschensky and the author[5]. The following physical considerations prompt an another solution of this same problem.

The choice of the momentum $\vec{p}_1$ in the rest system of particles with the energy E can be made simply. Owing to the equality of all the directions the direction $\vec{p}_1$ can be chosen randomly at an arbitrary point of the unit sphere surface. For the quantity e_1 one can choose arbitrary value between m_1 and $e_{1\,max} = [E^2 + m_1^2 - (\sum_2^n m_\kappa)^2]/2E$. However, once the momentum $\vec{p}_1$ has been chosen the other particles are already no longer at rest but they possess the momentum $-\vec{p}_1$ (and the energy $E_2 = E - e_1$). The sphere ω_2 - the region of the possible values of $\vec{p}_2$ - is deformed therefore into an ellipsoid d_2 [6].

We shall subject, however, the momenta of the particles 2, 3, ... n to the Lorentz transformation T_2 which would deform d_2 into ω_2, i.e. we shall pick $\vec{p}_2$ in the rest system of partic-

les $2,3,\ldots,n$. Then, the uniform sampling of the direction and the magnitude $\vec{p}_2$ will be easy again. The inverse transition T_2^{-1} yields $\vec{p}_2$ in the original frame of reference. The matrix T_2 is known since it depends only on E and $\vec{p}_1$. In the following we have the same situation. $\vec{p}_\kappa$ is picked in the rest system of particles $\kappa, \kappa+1, \ldots, n$, in which the region of allowed values $\vec{p}_\kappa$ is sphere ω_κ . Its radius is determined by the energy E_κ and the momentum $-\vec{P}_\kappa$ of the particles $\kappa, \kappa+1, \ldots, n$:

$$E_\kappa = E_{\kappa-1} - e_{\kappa-1} \; ; \quad \vec{P}_\kappa = \vec{P}_{\kappa-1} + \vec{p}_{\kappa-1} \; ; \quad M_\kappa^2 = E_\kappa^2 - P_\kappa^2 \; ;$$
$$E_1 = E \; ; \quad P_1 = 0 \; ; \quad \mu_\kappa = m_{\kappa+1} + m_{\kappa+2} + \ldots + m_n \; ; \qquad (1.1)$$
$$e_{\kappa\,max} = \left(M_\kappa^2 + m_\kappa^2 - \mu_\kappa^2\right)/2M_\kappa \; ; \quad p_{\kappa\,max}^2 = e_{\kappa\,max}^2 - m_\kappa^2 \; .$$

The transition into the Lab.sys. in performed by Lorentz transformation T_κ^{-1} . If we denote the quantities in the rest system of particles $\kappa, \kappa+1, \ldots, n$ by $'-'$, the cosine of the angle between $\vec{P}_\kappa$ and $\vec{p}_\kappa$ by η_κ , the azimuth of $\vec{p}_\kappa$ with respect to $\vec{P}_\kappa$ by φ_κ (notations are taken from paper /6/) then T_κ^{-1} will be

$$p_\kappa \eta_\kappa = \gamma_\kappa \left(\bar{p}_\kappa \bar{\eta}_\kappa + v_\kappa \bar{e}_\kappa\right) \; ; \quad \gamma_\kappa = E_\kappa / M_\kappa \; ;$$
$$p_\kappa \sqrt{1-\eta_\kappa^2} = \bar{p}_\kappa \sqrt{1-\bar{\eta}_\kappa^2} \; ; \quad \gamma_\kappa v_\kappa = -P_\kappa / M_\kappa \; ; \qquad (1.2)$$
$$e_\kappa = \gamma_\kappa \left(\bar{e}_\kappa + v_\kappa \bar{p}_\kappa \bar{\eta}_\kappa\right) \; ; \quad \gamma_1 v_1 = P_1 / M_1 \; .$$

In /6/ it has been shown that the shape of d_κ does not depend on $\varphi_1, \ldots, \varphi_{\kappa-1}$. This facilitated the calculations. The sphere ω_κ possesses the following convenient property: (*) its radius does not depend on $\bar{\eta}_{(1)}, \ldots, \bar{\eta}_{\kappa-1}$ as well. Indeed the value of the radius depends on the choice of the foregoing momenta $\vec{p}_1, \ldots, \vec{p}_{\kappa-1}$ only through the value of the effective mass M_κ . By using (1.1) we write

$$M_\kappa^2 = \left(E_{\kappa-1} - e_{\kappa-1}\right)^2 - \left(\vec{P}_{\kappa-1} + \vec{p}_{\kappa-1}\right)^2 = M_{\kappa-1}^2 + m_{\kappa-1}^2 - 2\left(E_{\kappa-1} e_{\kappa-1} + P_{\kappa-1} p_{\kappa-1} \eta_{\kappa-1}\right).$$

By substituting here $e_{\kappa-1}$ and $p_{\kappa-1}\eta_{\kappa-1}$ from (1.2) we obtain

$$M_\kappa^2 = M_{\kappa-1}^2 + m_{\kappa-1}^2 - 2M_{\kappa-1}\bar{e}_{\kappa-1} \; . \qquad (1.3)$$

Thus, M_κ depends only on the quantity $\bar{e}_{\kappa-1}$ but not on the direction of the $(\kappa-1)$ particle. We shall prove (*) by returning in Eq. (1.3) to $\kappa = 1$.

The property (*) seems to be strange and even contradictory to the common sense; it may seem that it proves the independence of the region of allowed values of the k -th particle momenta on the configuration of the momenta (fixed in value) $1,2,\dots$ ($k-1$) particles , which is not real. In fact two sets of momenta with the same $\bar{e}_1, \dots, \bar{e}_{k-1}$ and, say, different $\bar{\rho}_k^{(1)}$ and $\bar{\rho}_k^{(2)}$ when transforming into the laboratory frame of reference differ one from another not only by their configurations, but also the energy values (e_k in this case). So, in this frame of reference the values of the momenta are not fixed and any paradox does not arise.

The set of sphere ω_k ($k=1,2,\dots$) can be easily mapped into the cube. It should remember only that the density of the uniform sampling $\bar{\vec{p}}_k$ within ω_k will be different for various k due to the different size of ω_k Then, it should take into account the change in the density F due to the Lorentz transformations. But it is not difficult to do it by changing carefully the variables.

It will be convenient for us to take the phase space integral in the form

$$S(E,\vec{P}) = \int_{d_1} d^3\vec{p}_1 \dots \int_{d_k} d^3\vec{p}_k \dots \int_{(\infty)} d^3\vec{p}_{n-1}\, d^3\vec{p}_n\, F(\vec{p}_1,\dots,\vec{p}_n)\,\delta(\vec{p}_{n-1}+\vec{p}_n+\vec{P}_{n-1})\,\delta(e_{n-1}+e_n-E_{n-1}) \quad (1.4)$$

By substituting the variables $\bar{\vec{p}}_k = T_k p_k$ $(k<n-1)$ (1.2) we pass into the rest system of particles $k,\dots,n$. We find the Jakobian of this transition taking into account the fact that the transition matrix is triangle and the differential form $\frac{d^3\vec{p}_k}{e_k} = \frac{d^3\bar{\vec{p}}_k}{\bar{e}_k}$ is invariant. We find

$$S(E,\vec{P}) = \int_{\omega_k} \dots \int_{\omega_k} d^3\bar{\vec{p}}_k \dots \int_{\omega_{n-2}} d^3\bar{\vec{p}}_{n-2} \int_{(\infty)} d^3\vec{p}_{n-1}\, d^3\vec{p}_n\, F(\rho)\,\delta(\vec{p}_{n-1}+\vec{p}_n+\vec{P}_{n-1}) \times$$
$$\times\, \delta(e_{n-1}+e_n-E_{n-1}) \prod_1^{n-2} (e_k/\bar{e}_k)$$

We pass in the internal integral (the weight of the system of two particles) into the system where $P_{n-1}=0$. To do it we use the fact that

$$\delta^3(\vec{p}_{n-1}+\vec{p}_n+\vec{P}_{n-1})\,\delta(e_{n-1}-e_n-E_{n-1}) = \delta^4(p_{n-1}+p_n+P_n) = \delta^3(\bar{\vec{p}}_{n-1}+\bar{\vec{p}}_n)\,\delta(e_{n-1}+e_n-M_{n-1})$$

Now

$$S(E,\vec{P}) = \int_{\omega_k} \dots \int d^3\bar{\vec{p}}_k \dots \int_{(\infty)} d^3\bar{\vec{p}}_{n-1}\, d^3\bar{\vec{p}}_n\, F(\bar{\vec{p}})\,\delta(\bar{\vec{p}}_{n-1}+\bar{\vec{p}}_n)\,\delta(e_{n-1}+e_n-M_{n-1}) \prod_1^n (e_k/\bar{e}_k) \quad (1.5)$$

The integration over $\bar{\vec{p}}_n$ and $\bar{\vec{p}}_{n-1}$ means $\bar{\vec{p}}_n = -\bar{\vec{p}}_{n-1}$, the multiplication of the integrand by $\bar{e}_{n-1}\bar{e}_n/\bar{p}_{n-1}M_{n-1}$ and the substitution

$$\bar{p}_{n-1} = +\bar{p}_n = \left[\left(\frac{M_{n-1}^2 + m_{n-1}^2 - m_n^2}{2M_{n-1}} \right)^2 - m_{n-1}^2 \right]^{1/2} . \qquad (1.51)$$

It remains only to integrate over $\bar{\eta}_{n-1}$, $\bar{\varphi}_{n-1}$ in the limits $(-1, 1)$, $(0, 2\pi)$:

$$S_n(E, \vec{P}) = \int_{\omega_K} \dots \int d^3\vec{\bar{p}}_K \dots \int_{\omega_{n-2}} d^3\vec{\bar{p}}_{n-2} \int_{-1}^{1} d\bar{\eta}_{n-1} \int_{0}^{2\pi} d\bar{\varphi}_{n-1} \cdot F(\bar{p}) \frac{e_{n-1} e_n \bar{p}_{n-1}}{M_{n-1}} \prod_{1}^{n-2} \frac{e_K}{\bar{e}_K} \cdot (1.6)$$

This formula will be further initial for calculating phase space integrals and for obtaining random stars. We have to complicate it by reducing all the spheres to the unit radius and by mapping each of them into the unit three- dimensional cube. For this purpose we use the identity[5]

$$\int_{a_1'}^{a_1''} \dots \int_{a_m'}^{a_m''} \Phi(p_1, \dots, p_m) dp_1 \dots dp_m = \int_0^1 \dots \int_0^1 \tilde{\Phi}(\tilde{p}_1, \dots, \tilde{p}_m) W(\tilde{p}_1, \dots, \tilde{p}_m) d\tilde{p}_1 \dots d\tilde{p}_m , \quad (1.7)$$

where

$$W(\tilde{p}_1, \dots, \tilde{p}_m) = \prod_1^m (a_K'' - a_K') , \qquad \Phi(p_1, \dots, p_m) \equiv \tilde{\Phi}(\tilde{p}_1, \dots, \tilde{p}_m) \qquad (1.8)$$

and

$$p_K = a_K' + (a_K'' - a_K') \tilde{p}_K . \qquad (1.81)$$

In our case the 'linear' transformation (1.81)* is of the form

$$\begin{aligned} \bar{p}_K &\to \bar{p}_K p_{K\,max} && (k = 1.2, \dots, n-2) \\ \bar{\eta}_K &\to 2\bar{\eta}_K - 1 && (k = 1,2, \dots, n-1) \\ \bar{\varphi}_K &\to 2\pi \bar{\varphi}_K && (k = 1,2, \dots, n-1) \end{aligned} \qquad (1.82)$$

(1.6) expressed in terms of new variables (they are in the right hand side) takes the final form which is convenient for picking stars

$$S_n(E, \vec{P}) = \int_0^1 \dots \int_0^1 \prod_1^{n-2} d\bar{p}_K \, d\bar{\eta}_K \, d\bar{\varphi}_K \cdot d\bar{\eta}_{n-1} \, d\bar{\varphi}_{n-1} \cdot \Phi \qquad (1.9)$$

where

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* The expansion $\bar{p}_K$ is non-linear, but for given $\vec{\bar{p}}_1$, ..., $\vec{\bar{p}}_{K-1}$ it is linear.

$$\Phi = \mathcal{F} \cdot \frac{e_{n-1} e_n \bar{p}_{n-1}}{M_{n-1}} \prod_{1}^{n-2} \frac{\bar{p}_k^2 e_k}{\bar{e}_k} p_{k\,max} \cdot (4\pi)^{n-1} \qquad (1.10)$$

This means that if we shall choose uniformly from (0.1) random digits for each variable of the integration and reject (for the rejection method see /6/, /7/) a part of sets of $(3n-4)$ numbers so that the other sets may have (unnormalized) distribution density Φ , then we shall obtain stars from the remaining sets by the inverse recalculation in p_k, η_k, φ_k . All the statistical properties of these stars correspond exactly to those of the model possessing the square of the matrix element $\mathcal{F}$. The recalculation is performed as follows: if $\vec{p}_1, \dots, \vec{p}_{k-1}$ are already obtained, then using (1.8,2), (1.1), (1.3) we have to obtain $\vec{\bar{p}}_k$ in spherical coordinates and then using (2.5) – (2.7) from /6/ we have to obtain $\vec{p}_k$ in the rectangular coordinates and also the energy, effective mass and components of the momentum of the remaining particles $k+1, \dots, n$ in order to proceed to the determination of $\vec{p}_{k+1}$ /6/. We have the following sequence of the equations:

$$\vec{p}_k = \{X_k, Y_k, Z_k\} \; ; \quad R_k = (X_k^2 + Y_k^2)^{1/2}$$

$$x_k = \bar{p}_k \left[\frac{X_k}{p_k} \frac{E_k}{M_k} \bar{\eta}_k + \left(\frac{Z_k}{p_k} \frac{X_k}{R_k} \cos\bar{\varphi}_k - \frac{Y_k}{R_k} \sin\bar{\varphi}_k \right) \sqrt{1-\bar{\eta}_k^2} \right] + \frac{X_k}{M_k} \bar{e}_k \; ; \qquad (1.11)$$

$$y_k = \bar{p}_k \left[\frac{Y_k}{p_k} \frac{E_k}{M_k} \bar{\eta}_k + \left(\frac{Z_k}{p_k} \frac{Y_k}{R_k} \cos\bar{\varphi}_k + \frac{X_k}{R_k} \sin\bar{\varphi}_k \right) \sqrt{1-\bar{\eta}_k^2} \right] + \frac{Y_k}{M_k} \bar{e}_k \; ;$$

$$z_k = \bar{p}_k \left[\frac{Z_k}{p_k} \frac{E_k}{M_k} \bar{\eta}_k - \frac{R_k}{p_k} \cos\bar{\varphi}_k \sqrt{1-\bar{\eta}_k^2} \right] + \frac{Z_k}{M_k} \bar{e}_k \; ;$$

$$X_{k+1} = X_k + x_k \; ; \quad Y_{k+1} = Y_k + y_k \; ; \quad Z_{k+1} = Z_k + z_k \; ; \quad E_{k+1} = M_k E_k - e_k \; ;$$

$$p_{k+1}^2 = p_k^2 + p_k^2 + 2 p_k \gamma_k (\bar{p}_k \bar{\eta}_k + v \bar{e}_k) \; ; \quad M_{k+1}^2 = M_k^2 + m_k^2 - 2 M_k \bar{e}_k \; .$$

It should remember that each momentum $\vec{\bar{p}}_k$ is measured in its own frame of reference, they can not be therefore summed up, for example. The only exception are $\vec{\bar{p}}_{n-1}$ and $\vec{\bar{p}}_n$ obtained in their common rest system. Therefore, for $k=n$ (1.2) takes the form

$$e_n = \gamma_{n-1} (\bar{e}_n - v_{n-1} \bar{p}_{n-1} \bar{\eta}_{n-1}) \; ; \quad \bar{e}_n = M_{n-1} - \bar{e}_{n-1} \; . \qquad (1.2')$$

§ 2. Let us consider now factorizable models i.e. such models in which the square of the matrix element $F(\vec{p})$ can be somehow divided at least into two independent multipliers. From the physical point of view the factorizability corresponds to the possibility of breaking up the system of secondary particles at least into two non-interacting subsystems. Such are the Fermi model and the covariant model. The large class of factorized covariant models has been considered in /17/.

For the factorized model the efficiency of the sampling can be improved still more by a special method which is called by us the group method.

We break up the system of N particles into n groups $1, \dots, i, \dots, n$; let the

number of particles in the i-th group be n_i ($\sum_{i=1}^{n} n_i = N$); let the mass, energy and momentum of the particle j from the i-th group be now denoted by m_{ij}, e_{ij}, p_{ij}, and e_i, p_i, m_i denote the total energy, momentum and effective mass of particles from the i-th group:

$$\sum_{j=1}^{n_i} e_{ij} = e_i \quad ; \quad \sum_{j=1}^{n_i} \vec{p}_{ij} = \vec{p}_i \quad ; \quad m_i^2 = e_i^2 - p_i^2 .$$

We denote the minimum value of m_i by $\tilde{m}_i \equiv \sum_{j=1}^{n_i} m_{ij}$. We transform the phase space integral as follows

$$S_N(E,\vec{P}) = \int \prod_{ij} d^3\vec{p}_{ij}\, \delta\Big(\sum_{ij}\vec{p}_{ij} - \vec{P}\Big)\delta\Big(\sum_{ij} e_{ij} - E\Big)\, F(\vec{p}_{ij}) =$$

$$= \int \prod_{ij} d^3\vec{p}_{ij}\, \delta\Big(\sum_{i=1}^{n}\vec{p}_{ij} - \vec{P}\Big)\delta\Big(\sum_{i=1}^{n} e_i - E\Big) F(\vec{p}_{ij}) \prod_{i=1}^{n}\delta\Big(\sum_{j=1}^{n_i}\vec{p}_{ij} - \vec{p}_i\Big) \times$$

$$\times \delta\Big(\sum_{j=1}^{n_i} e_{ij} - e_i\Big)\, d^3\vec{p}_i\, de_i = \int \prod_{i=1}^{n} d^3\vec{p}_i\, de_i\, \delta\Big(\sum_{i=1}^{n}\vec{p}_i - \vec{P}\Big)\delta\Big(\sum_{i=1}^{n} e_i - E\Big) \times$$

$$\times \int \prod_{j=1}^{n_i} d^3\vec{p}_{ij}\, \delta\Big(\sum_{j=1}^{n_i}\vec{p}_{ij} - \vec{p}_i\Big)\delta\Big(\sum_{j=1}^{n_i} e_{ij} - e_i\Big) F(\vec{p}_{ij}).$$

Let the breaking up of N be now chosen so that $F(\vec{p}_{ij})$ is factorized:

$$F(\vec{p}_{ij}) = \prod_{i=1}^{n} F_i(\vec{p}_{ij}) . \qquad (1.12)$$

Let the analytic expression be also known for

$$S_i(e_i,\vec{p}_i) = \int \prod_{j=1}^{n_i} d^3\vec{p}_{ij}\, \delta\Big(\sum_{j=1}^{n_i}\vec{p}_{ij} - \vec{p}_i\Big)\delta\Big(\sum_{j=1}^{n_i} e_{ij} - e_i\Big) F_i(\vec{p}_{ij}) \qquad (1.13)$$

Then

$$S_N(E,\vec{P}) = \int \prod_{i=1}^{n} d^3\vec{p}_i\, de_i\, S_i(e_i,\vec{p}_i)\delta\Big(\sum_{i=1}^{n}\vec{p}_i - \vec{P}\Big)\delta\Big(\sum_{i=1}^{n} e_i - E\Big) \qquad (1.14)$$

This formula expresses the weight of the system in terms of the weights of the subsystems, generalizing naturally the formulas from /8/,/4/ . It is just the basis of the group method for calculating weights and sampling stars. If e_i, p_i are given then it is not difficult to pick the momenta of the particles of the i-th group (at least in the same way as in the § 1). It remains only to consider the methods of picking the quantities $e_i, \vec{p}_i$. We use again as above the transition into the

rest system of the groups $1, \ldots, n$. In this system all the directions have equal probability. The transformations T_i does not change the four-dimensional differential and the product of δ-functions, therefore

$$S_N(E,\vec{P}) = \int_{\delta_1} d^3\vec{p}_1 de_1 \tilde{S}_1(e_1,\vec{p}_1) \ldots \int_{\delta_{n-1}} d^3\vec{p}_{n-1} de_{n-1} \tilde{S}_{n-1}(e_{n-1},\vec{p}_{n-1}) \tilde{S}_n(M_{n-1}, -\vec{p}_{n-1}). \quad (1.15)$$

Here we denote $\tilde{S}_i(e_i,\vec{p}_i) \equiv S_i(e_i,\vec{p}_i)$; δ_i denotes the region of integration over $\vec{p}_i, e_i$; the total energy, momentum and effective mass of the system of the groups $1, \ldots, n$ are denoted by $E_i, \vec{P}_i, M_i$. The equations for the surfaces contining δ_i are given in Appendix (A). Sometimes it is appropriate to take as integration variables quadruples $\vec{p}_i, m_i$ or $e_i, m_i, \eta_i, \varphi_i$.

Eq. (1.15) is convenient for calculating the weight. For sampling it is to be complicated by mapping $\delta_1, \ldots, \delta_{n-1}$ into the unit cube. If we denote the limits in which p_i, e_i change by $(0, p_{i\,max})$ and (e_i', e_i''), then the initial formula for picking $e_i, \vec{p}_i$ will be

$$S_N(E,P) = \int_0^1 \ldots \int_0^1 \prod_1^{n-1} d\bar{p}_\kappa d\bar{\eta}_\kappa d\bar{\varphi}_\kappa d\bar{e}_\kappa \cdot \tilde{\Phi} \quad (1.16)$$

where

$$\tilde{\Phi} = \prod^{n-1} \bar{p}_\kappa^2 S_\kappa(\bar{e}_\kappa,\vec{\bar{p}}_\kappa)(e_\kappa''-e_\kappa')p_{\kappa\,max} \cdot (4\pi)^{n-1} \tilde{S}_n(M_{n-1}\bar{e}_{n-1}, -\vec{\bar{p}}_{n-1}). \quad (1.17)$$

The rejection method will now be applied twice: when picking $\bar{e}_i, \vec{\bar{p}}_i$ $(i=1,\ldots,n)$ and $\vec{p}_{ij}$ $(j=1,\ldots,n_i>2)$. Having chosen (as in the §1) from the hyper-cube a necessary number of quadruples of the quantities $(\bar{e}_i, \vec{\bar{p}}_i)$ we calculate $\tilde{\Phi}$ and compare it with a random number α from the interval $(0, \tilde{\Phi}_{max})$ where $\tilde{\Phi}_{max}$ is a certain upper estimate of the quantity $\tilde{\Phi}$. Those quadruples for which $\tilde{\Phi} > \alpha$ will possess the necessary distribution.

The group method (1.16) is more effective than the direct sampling (1.9). The unlikely combinations e_i, $\vec{p}_i$ will be found here rarely since the distribution density depends on smaller number of variables.

We consider, e.g., the Fermi model $F=1$. Let the number of particles N be even. Break up the system into groups; each group contains two particles. Then[6]

$$S_i(e_i,p_i) = \frac{4}{3}\pi \frac{e_i^2}{m_i^2} \frac{p_{i1}^*}{m_i} \left(3 e_{i1}^* e_{i2}^* - \frac{p_i^2}{e_i^2} p_{i1}^{*2}\right) \quad (1.18)$$

(the asterisks denote the quantities in the rest system of the pairs:

$$e^{*}_{i1} = (m_i^2 + m_{i1}^2 - m_{i2}^2)/2m_i \; ; \; p^{*2}_{i1} = e^{*2}_{i1} - m^2_{i1};).$$

Having picked (as in § 1) the set $(\frac{N}{2} - 1)$ of quadruples of the quantities $e_i, \vec{p}_i$ we calculate $\tilde{\Phi} = \prod_{i=1}^{N/2} S_i(e_i, \vec{p}_i)$. Let the maximum $\tilde{\Phi}_{max}$ of this product over all $e_i, \vec{p}_i$ be known even if roughly. If we reject all those sets $e_i, \vec{p}_i$ for which $\tilde{\Phi}$ turns out to be smaller than the number uniformly chosen from $(0, \tilde{\Phi}_{max})$ then the remaining sets will have the distribution of energies and pair momenta required by the Fermi model. It is not difficult now to obtain energies and momenta of particles in each pair knowing only their direction of emission, i.e. adding $2(N/2)$ quantities η^*_i, φ^*_i more, The general number of samplings is as usual $3N-4$ – it is impossible to describe the system of N various particles by smaller number of variables. However, $\tilde{\Phi}$ depends only on $2N-4$ variables (in fact on $\frac{3}{4}(2N-4)$ since there is no dependence on φ_K), but not on $3N-4$ as was the case with Φ in § 1. The function of smaller number of variables has less sharp maximum Φ_{max} ; this means that we need to reject smaller number of sets $e_i, \vec{p}_i$ in order to reproduce the necessary density.

The greater are the groups the better is the efficiency of the group method. It is true that for the groups of three and more particles the picking $\vec{p}_{ij}$ using $e_i, \vec{p}_i$ already picked requires to reject those sets $\vec{p}_{ij}$ for which $\Phi_i(p_{ij})$ (see (1.10)) is less than the random number from (0, $\Phi_{i\,max}(e_i, \vec{p}_i)$)) (for pairs of particles we do not need the rejection). This can slow the increase of the efficiency but can not stop it with the increase of the group magnitude.

At present there are no convenient analytic expressions (suitable for any energies, momenta and masses) for the weight of three, four etc. particle systems. It is the obstacle in the way of increasing the efficiency. In the following we shall see that in the covariant model some formulas for statistical weights turn out to be very simple. The covariant model, unlike the other ones, can therefore be tested in detail up to the creation of tens of particles.

In the paper by Granovsky and the author[16] a different principle for obtaining distributions and correlations has been applied. It is a machine analysis of the so called 'generalized table of random stars' The electronic computer took into account both the hardly and very probable sets $\vec{p}_i$ by putting in memory all $\Phi(p)$. Sorting Φ in accordance with the kinematic characteristic of interest the computer gave immediately necessary hystograms. In comparison with the direct method the group method would give here too better efficiency improving the accuracy of hystograms. Infortunately, the model of the paper[16] is not factorizable.

§2. Some models

We apply the equations of §1 to the models of Fermi and Srivastava-Sudarshan[2].

To calculate the integral (1.4) using the Monte-Carlo method we have to sum simply Φ (1.10) or $\tilde{\Phi}$ (1.17) obtained after the uniform picking of points in the hyper-cube. If the points p in the hyper-cube are distributed ununiformly but with the density $\Phi^*(p)$, then

$$S(E,P) = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum^{\mathcal{N}} \frac{\Phi(p)}{\Phi^*(p)} . \qquad (2.1)$$

The determination of the hystograms – distributions of p with respect to the quantity $q(p)$ – can also be reduced to the "sorting of p with respect to q", i.e. to the calculation of the sum over all those p, for which $q(p)$ is close to q_0

$$S(q_0) = \overline{[\Phi(p) / \Phi^*(p)]}_{(p\in q(p)\approx q_0)}. \qquad (2.2)$$

In this paragraph it is implied that $\Phi^*(p) = 1$; more general case is referred to §3.

1. The Fermi model $F = 1$ yields no essential simplifications as compared to the general case. In (1.9) one succeeds in integrating only over $\bar{\varphi}_1, \dots, \bar{\varphi}_{n-1}$. It is impossible to integrate in the remaining $(2n-3)$ – dimensional integral over $\bar{\eta}_1, \dots, \bar{\eta}_{n-2}$ since Φ depends in a complicate manner on $\bar{\eta}_1, \dots, \bar{\eta}_{n-2}$. The integration over $\bar{\eta}_{n-1}$ yields (1.18). It is convenient to pass to the variables $\bar{e}_i$ instead of $\bar{p}_i$. We have

$$S_n(E,0) = \underbrace{\int_0^1 \dots \int_0^1}_{(2n-4)} \prod_1^{n-2} d\bar{e}_k \, d\bar{\eta}_k \cdot \Phi^{(F)}, \qquad (2.3)$$

where

$$\Phi^{(F)} = (4\pi)^{n-2} \cdot \frac{4\pi}{3} \frac{E_{n-1}^2}{M_{n-1}^2} \frac{p_{n-1}^*}{M_{n-1}} \left(3 e_{n-1}^* e_n^* - \frac{P_{n-1}^2}{M_{n-1}^2} p_{n-1}^{*2}\right) \prod_1^{n-2} \bar{p}_k e_k (e_{k\,max} - m_k). \qquad (2.4)$$

In the Eq. (1.15), in which $\tilde{S}_i^{(2)}$ is taken from (1.18), we can also integrate only over $\bar{\varphi}_1, \dots, \bar{\varphi}_{n-1}$. It is convenient to choose m_i, $\bar{e}_i$, $\bar{\eta}_i$ as integration variables (see Appendix A). We have

$$S_{2n}(E,P)=\int_{\tilde m_1}^{M_1-\tilde M_1} m_1 dm_1 \int_{m_1}^{e_{1\max}} \bar p_1 d\bar e_1 \int_{-1}^{1} d\bar\eta_1 \cdot 2\pi S_1^{(2)}(e_1,\vec{\bar p}_1)\dots$$

$$\dots\int_{\tilde m_{n-1}}^{M_{n-1}-\tilde M_{n-1}} m_{n-1} dm_{n-1} \int_{m_{n-1}}^{e_{n-1\max}} \bar p_{n-1} d\bar e_{n-1} \int_{-1}^{1} d\bar\eta_{n-1}\, 2\pi S_{n-1}^{(2)}(e_{n-1},\vec{\bar p}_{n-1})\, S_n^{(2)}(e_n,\vec{\bar p}_n). \qquad (2.5)$$

Here $e_{\kappa\,\max}$ is taken from (A.2), $M_i^2=E^2-P^2$; $\tilde M_\kappa=\sum_{k+1}^{n}\tilde m_i$; for other notations and relations see above. According to this formula the weight of four particles in their rest system is expressed by the double integral.

In the analytical calculations according to (2.5) we have to obtain formulas for weights which are similar to the Block's formulas[9]. In calculating by the Monte-Carlo method the integration region is to be mapped into the cube (see (1.7) – (1.8,2). and (A)).

2. The covariant model $F=(2^n e_1\dots e_n)^{-1}$ leads to

$$S_n(E,P)=\int\dots\int_{\omega_{n-2}} d^3\bar p_{n-2}\int_{-1}^{1} d\bar\eta_{n-1}\int_{0}^{2\pi} d\bar\varphi_{n-1}\cdot 2^{-n}(\bar p_{n-1}/M_{n-1})\prod_{1}^{n-2} e_\kappa^{-1}. \qquad (2.6)$$

The integrand does not depend on $\bar\varphi_i$ as well as on $\bar\eta_i$. As we have pointed out already (property (M)) the limits of integration do not depends on $\bar\varphi_i$, $\bar\eta_i$ too. The weight is therefore expressed by $(n-2)$-dimensional integral

$$S_n(E,P)=S_n(M,0)=\frac{(2\pi)^{n-1}}{2}\int\dots\int\bar p_\kappa d\bar e_\kappa\int\dots\int\bar p_{n-2}\,d\bar e_{n-2}\cdot\frac{\bar p_{n-1}}{M_{n-1}} \qquad (2.7)$$

where the limits $\bar e_\kappa$ are calculated by means of the Eqs. (1.1), (1.3)

$$m_\kappa\le\bar e_\kappa\le(M_\kappa^2+m_\kappa^2-\mu_\kappa^2)/2M_\kappa \qquad (2.81)$$

$$M_\kappa^2=M_{\kappa-1}^2+m_{\kappa-1}^2-2M_{\kappa-1}\bar e_{\kappa-1} \qquad (2.8,2)$$

and $\bar{p}_{n-1}/M_{n-1}$ (1.5,1) can be written in the form

$$\frac{\bar{p}_{n-1}}{M_{n-1}} = \frac{1}{2}\left[1 - 2\frac{m_{n-1}^2+m_n^2}{M_{n-1}^2} + \left(\frac{m_{n-1}^2-m_n^2}{M_{n-1}^2}\right)^2\right]^{1/2} = \frac{1}{2M_{n-1}^2}\prod_{\varepsilon=\pm 1}\left[M_{n-1}^2 - (m_{n-1}+\varepsilon m_n)^2\right] . \quad (2.9)$$

The expression (2.7) is also obtained immediately from the recurrent formula of Srivastava-Sudarshan (2)/2/

$$S_n(M) = 2\pi \int \bar{p}_n \, d\bar{e}_n \, S_{n-1}\left(\sqrt{M^2+m_n^2-2M\bar{e}_n}\right) \quad (2.10)$$

and the expression for S_2 (2.16)* . For calculating with the use of the Monte-Carlo method it is convenient to take instead of (2.7)

$$S_n(M_1) = \frac{(4\pi)^{n-1}}{2}\int_0^1 \dots \int_0^1 d\alpha_1 \dots d\alpha_{n-2}\,\frac{\bar{p}_{n-1}}{M_{n-1}}\prod_1^{n-2}\bar{p}_\kappa\,\frac{(M_\kappa-m_\kappa)^2-\mu_\kappa^2}{2M_\kappa} \quad (2.11)$$

where

$$\bar{e}_\kappa = m_\kappa + \alpha_\kappa\frac{(M_\kappa-m_\kappa)^2-\mu_\kappa^2}{2M_\kappa}, \quad M_{\kappa+1}^2 = (M_\kappa-m_\kappa)^2(1-\alpha_\kappa)+\mu_\kappa^2\alpha_\kappa, \quad (2.12)$$

or in other variables

$$S_n(M_1) = \frac{\pi^{n-1}}{2M_1^2}\int_0^1 \dots \int_0^1 \frac{d\beta_1 \dots d\beta_{n-2}}{M_2 \dots M_{n-1}}\prod_1^{n-2}(M_\kappa-m_\kappa-\mu_\kappa)\,R(M_{\kappa-1},m_{\kappa-1},M_\kappa)\times R(M_{n-1},m_{n-1},m_n) \quad (2.13)$$

where

$$M_{\kappa+1} = \mu_\kappa + \beta_\kappa(M_\kappa - m_\kappa - \mu_\kappa)$$

$$R^2 = [(M_{\kappa-1}+m_{\kappa-1})^2 - M_\kappa][(M_{\kappa-1}-m_{\kappa-1})^2 - M_\kappa^2] . \quad (2.14)$$

In both formulas the integrand vanishes on the surface of the hyper-cube. Therefore, the number theoretical method of Korobov/12/ turns out to be extremely effective for calculating the integral. It yields in these conditions the accuracy which can not be reached by the Monte-Carlo method. Besides, the Eq. (2.11) is simpler than the corresponding formula by Cerulus and Hagedorn/4/ for the Fermi's weight, which has been used for calculating the production of fifteen particles. (2.11) will allow one to go on.

* It is interesting to note the outward resemblance between (2.7) and the erroneous formula of Jakovlev /10//11/.

In calculating the weight of the system of n particles with the total energy M_1 by means of the Eqs. (2.11)–(2.13) we can obtain also for any subsystem the dependence of its weight on its energy (in the energy interval allowed for the given M_1) or any non-angular distributions and correlations[4]. So, the weight of the system of particles $k, \dots, n$ with the energy M_k is proportional to the sum of products of all those multipliers in (2.13) which do not depend on $M_1, \dots, M_{k-1}$ and for which M_k is close to the given value. We shall not discuss this question in more detail.

For the analytic calculations it is convenient to use Eq. (2.10) or its twice (which is analogous to (2.13))

$$S_n(M_n) = \frac{\pi}{M_n^2} \int_{M_n}^{M_n - m_n} S_{n-1}(M_{n-1}) \sqrt{\left[(M_n - m_n)^2 - M_{n-1}^2\right]\left[(M_n + m_n)^2 - M_{n-1}^2\right]}\, \frac{dM_{n-1}^2}{2} \qquad (2.15)$$

Of great number of formulas following from (2.10), (2.13) we choose only those which can be used in the group method:

a) The weight of the system of two particles is

$$S(M, m_1, m_2) = \pi p_1^*/M = \frac{\pi}{2M^2}\sqrt{(M^2 - 4m^2)(M^2 - 4\mu^2)}; \quad \left(\begin{matrix} 2m \\ 2\mu \end{matrix} = m_1 \pm m_2\right) \qquad (2.16)$$

b) The weight of the system of three particles having the same mass $S(M, m, m, m)$ with the error not greater than 1.6 percent is represented by the expression $(\alpha = m/M)$

$$S(M, m, m, m) = \frac{1}{8}\pi^2 M^2 (1 + 0{,}997\alpha^2)(1 - \alpha)^{3/2}(1 - 9\alpha^2)^2; \quad \left(0{,}997 = 9\,\frac{\pi - 2\sqrt{2}}{2\sqrt{2}}\right). \qquad (2.17)$$

In the derivation of this formula we find the increasing function α which would transform for $\alpha \to 0$ and $\alpha \to 1/3$ into the ultra-relativistic and non-relativistic formulas respectively (see Appendix B).

Now we pass to the calculations using the group method. In (1.15) the covariance of the weight enable us to take away the "$\sim$" and integrate over the directions. The formula expressed in terms of the variables (m, $\bar{e}$) takes the form

$$S_N(M_1) = (4\pi)^{n-1} \int \dots \int_{\tilde{m}_n}^{M_n - \tilde{\mu}_n} S_n(m_n)\, m_n\, dm_n \int_{m_n}^{e_{n\,max}} \bar{p}_n\, d\bar{e}_n \int_{\tilde{m}_{n+1}}^{M_{n+1} - \tilde{\mu}_{n+1}} \dots$$

$$\dots \int_{\tilde{m}_{n-1}}^{M_{n-1} - \tilde{\mu}_{n-1}} S_{n-1}(m_{n-1})\, m_{n-1}\, dm_{n-1} \int_{m_{n-1}}^{e_{n-1\,max}} \bar{p}_{n-1}\, d\bar{e}_{n-1}\, S_n\left(\sqrt{M_{n-1}^2 - 2M_{n-1}\bar{e}_{n-1} + m_{n-1}^2}\right). \qquad (2.18)$$

Thus, the weight of the system of N particles is expressed through the $2(n-1)$ dimentional integral, n being the number of groups of N particles. In breaking up by pairs (S_n from (2.16)) the dimensionality of integral is $N-2$ (the same as in (2.11)). However, if at least a part of particles possesses equal masses it is convenient to group them by three and use (2.17), this will decrease the dimensionality of the integral.

So, in calculating the weight of the nine identical particles we can do with the four-dimensional integral.

A particular case of (2.18) is the duplication formula. Let the index of S denotes not the label of the group, but the number of particles in it. We break up N particles into two groups with the known dependences of their weights on the energy. Then (2.18) yields

$$S_N(M_1) = 4\pi \int_{\tilde{m}_1}^{M_1-\tilde{m}_2} m_1 S_\nu(m_1)\,dm_1 \int_{m_1}^{e_{1\max}} \bar{p}_1 S_{N-\nu}\left(\sqrt{M_1^2 - 2M_1\bar{e}_1 + m_1^2}\right) d\bar{e}_1 . \quad (2.19)$$

The duplication formula proper is obtained when $N=2\nu$.

If both groups are identical then to calculate S_N we need to know the only function $S_{N/2}(m_1)$. If in turn $N/2$ particles can be broken up again into two identical groups then (2.19) yields $S_{N/2}$ if we know at least the only function $S_{N/4}(m)$ etc. In the most favourable case when all the particles are identical and their number N is $3\cdot 2^{n-1}$ or 2^n, we have only to calculate energy dependances of $(n-1)$ weights S_6, S_{12}, S_{24} or S_4, S_8, S_{16} only (which is equivalent to the calculation of the $n-1 \equiv \log_2(N/2)$ three-dimensional integrals). Thus, in using (2.19) the labour-consuming character of the calculation increases very slowly i.e. logarithmically with the increasing number of particles. The calculation of probabilities of the production of several tens of particles can be achieved with the modern computers*.

If only a part of particles has equal masses then the skillful use of Eq. (2.19) will reduce calculations. So, it is convenient to break up the system NN 8π into two groups $N4\pi$ and each of them into $N\pi$ and 3π. For calculating the weight of $N3\pi$ we can use Eqs. (2.15) and

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* There arises, however, a specific difficulty: the calculated functions increase extremely rapidly (for large N) with the energy.

(2.17). It is only in that case when it is impossible to find among N particles two identical groups of more than two particles that Eq. (2.19) will not yield simplifications as compared to (2.11).

The duplication formula might be useful for the Fermi model too. In this case, however, the calculation of $S_N(E,P)$ through $S_{N/2}(E,P)$ is equivalent to the calculation of the five-dimensional integral (instead of the three-dimensional one in (2.19)).

Note that (2.19) can be written more simply if one mark the dependance of S on masses of particles together with the energy.

By comparing (2.19) with (2.10) we can write

$$S_N(M_1; m_{11}, \dots, m_{1\nu}, m_{21}, \dots, m_{2,N-\nu}) =$$
$$= 2 \int_{m_{11}+\dots+m_{1\nu}}^{M_1 - m_{21} - \dots - m_{2,N-\nu}} S_\nu(m_1; m_{11}, \dots, m_{1\nu})\, S_{N-\nu+1}(M_1; m_1, m_{21}, \dots, m_{2,N-\nu})\, m_1 dm_1 . \quad (2.20)$$

We consider in brief the question on the construction of tables of random stars for the systems with large number of particles. If all the particles are identical we have only to describe one of them (if we are interested in the momentum and angular distribution), two of them if we need pair correlations etc. As to other particles, we can integrate over them. If in the system there are two kinds of particles then we can describe in the table one representative of each kind of particles (if we need momentum and angular distributions and correlations between different particles) and two representatives for more complicated spectra.

So, the table for the system $N(n+1)\pi$ can be obtained from the expression

$$S_{N(n+1)\pi}(M) = 4\pi^2 \int \bar{p}_N d\bar{e}_N \int \bar{p}_\pi d\bar{e}_\pi S_{n\pi}(M_{n\pi})$$

(where

$$M^2_{n\pi} = M^2_{(n+1)\pi} + m^2_\pi - 2M_{(n+1)\pi}\bar{e}_\pi$$
$$M^2_{(n+1)\pi} = M^2 + m^2_N - 2M\bar{e}_N \ ),$$

if the function $S_{n\pi}$ is found preliminarily and then the usual method of picking random stars is used.

§ 3. The Use of Important Sampling in Simulating

The convergence of the procedure for calculating integral (2.1) with the use of the Monte-Carlo method can be accelerated as the information on the integrand develops . So, the closer is $\Phi^*(\rho)$ to the $\Phi(\rho)$ the more precise is $S(E,P)$ for the given computing time. The use of the density $\Phi^*(\rho)$ close to $\Phi(\rho)$ is called an important sampling./7/

This characteristic feature of the probabilistic methods will lead to interesting results in the problem of simulating the process of multiple production. We denote by $\Phi(\rho)$ the distribution density of ρ in the model which explains in all details the results of multiple production experiments ('exact model'), and $\Phi_1(\rho)$, $\Phi_2(\rho)$ ... are distributions in various approximate models. Let each subsequent approximate model be closer to the precise model than the foregoing one. Let the table of random stars be obtained for the model Φ_1 . This means that each state (set of momenta ρ) is found in the table with the frequency proportional to its weight $\Phi_1(\rho)$. Let now the models $\Phi_\kappa(\rho)$ closer to $\Phi(\rho)$ be investigated. The calculation in these new models can then be made with improved accuracy if an important sampling with the density $\Phi^*(\rho)=\Phi_1(\rho)$ is used, i.e. if the computer is able to extract in consecutive order from the memory sets ρ and their weights $\Phi_1(\rho)$, calculate $\Phi_\kappa(\rho)$ and sum up $\Phi_\kappa(\rho)/\Phi_1(\rho)$ using (2.1) or (2.2). The same table of random stars $(\rho, \Phi_1(\rho))$ for the model $\Phi_1(\rho)$ will simplify calculations for many models $\Phi_\kappa(\rho)$ improving $\Phi_1(\rho)$.

There arises an interesting situation: as the theory develops (i.e. their predictions approach the experimental data), the calculations using the Monte-Carlo method will not become complicated if we take as the 'fundamental model' the model rather close to the 'exact' one. This situation resembles the perturbation theory. The latter can be considered as a way of finding the eigenfunctions of complicated perturbed system by making use of the information about these functions obtained in studying a simple non-perturbed system.

By calculating the dispersion of the statistical weight the machine itself can estimate whether the investigated model is close to the basic one.

As such a 'fundamental' model we can now take the covariant model because it is simple and their predictions are rather close to the experimental results/2,13

For the ultra-relativistic particles we do not need to commit in memory the corresponding table. It can be reproduced using the following procedure. It is known/7,6/ that to obtain the set ρ distributed in (a,b) with the density $\Phi^*(\rho)$ we have to take for ρ the set of solutions of the equation

$$\int_a^\rho \Phi^*(\rho)\,d\rho = \alpha \int_a^b \Phi^*(\rho)\,d\rho \qquad (3.1)$$

where α is the set of numbers uniformly distributed over (0,1). For the systems of ultra-relativistic particles one can calculate the integrals in (3.1) (Appendix B1). We consider a system of such particles. Let the momenta of particles 1,2, ..., $n-\nu-1$ be picked somehow and the momentum of $(n-\nu)$-th particle is to be picked. The distribution over the $(n-\nu)$-th particle momenta in the rest system of particles $n-\nu, \ldots, n$ is

$$\frac{d^3 \bar{p}_{n-\nu}}{\bar{e}_{n-\nu}} S_\nu (M_{n-\nu+1}) \qquad (3.2)$$

where

$$M^2_{n-\nu+1} = M^2_{n-\nu} + m^2_{n-\nu} - 2 M_{n-\nu} \bar{e}_{n-\nu} . \qquad (3.3)$$

Since the particles are ultra-relativistic, then (3.2) is proportional to

$$d\bar{e}_{n-\nu} \cdot \bar{e}_{n-\nu} \; M^{2(\nu-2)}_{n-\nu+1} \qquad (3.4)$$

where

$$M^2_{n-\nu+1} = M_{n-\nu} \left(M_{n-\nu} - 2\bar{e}_{n-\nu} \right). \qquad (3.5)$$

Instead of the $\bar{e}_{n-\nu}$- distribution we find the $M_{n-\nu+1}$ distribution using (3.5). It is proportional to

$$dM^2_{n-\nu+1} \cdot \left(M^2_{n-\nu} - M^2_{n-\nu+1} \right) M^{2(\nu-2)}_{n-\nu+1} \qquad \left(0 \le M_{n\,\nu+1} \le M_{n-\nu} \right) \qquad (3.6)$$

(3.6) should be substituted into (3.1). For the ratio

$$M^2_{n-\nu+1} / M^2_{n-\nu} = t_\nu \qquad (3.7)$$

we have the equation

$$(\nu-1)\, t_\nu^\nu - \nu\, t_\nu^{\nu-1} + \alpha = 0. \qquad (3.8)$$

Now

$$M_{n-\nu+1} = M_{n-\nu} \sqrt{t_\nu}$$

$$\bar{e}_{n-\nu} = \frac{1}{2} M_{n-\nu} (1 - t_\nu) \qquad (3.9) .$$

or finally (M_1 - is the energy of the system as a whole)

$$M_{n-\nu+1} = M_1 \left(t_\nu t_{\nu+1} \dots t_{n-1}\right)^{1/2}$$

$$\bar{e}_{n-\nu} = \frac{1}{2} M_1 \left(t_{\nu+1} t_{\nu+2} \dots t_{n-1}\right)^{1/2} (1-t_\nu) \qquad (\nu \geq 2) \qquad (3.10)$$

$$\bar{e}_{n-1} = \bar{e}_n = \frac{1}{2} M_1 \left(t_2 \dots t_{n-1}\right)^{1/2}$$

Thus, if we choose randoinly ($n-2$) random numbers α uniformly over (0.1), solve ($n-2$) equations (3.8) for ν from $n-1$ to 2 (Fig. 1) and then calculate $\bar{e}_{n-\nu}$ using (3.10), then $\bar{e}_{n-\nu}$ will be distributed with the density (3.4) presented by the covariant model.

We note that this density depends only on the number of particles but not on their energy (which is natural for particles without mass).

Let us take the concret example $n = 6$. Then, we have to calculate

$$\bar{e}_1 = \frac{1}{2}(1-t_5) \quad , \text{ where } \quad 4t_5^5 - 5t_5^4 + \alpha_5 = 0 ;$$

$$\bar{e}_2 = \frac{1}{2}(1-t_4)\sqrt{t_5} \quad , \text{ where } \quad 3t_4^4 - 4t_4^3 + \alpha_4 = 0 ;$$

$$\bar{e}_3 = \frac{1}{2}(1-t_3)\sqrt{t_4 t_5} \quad , \text{ where } \quad 2t_3^3 - 3t_3^2 + \alpha_3 = 0 ; \qquad (3.10')$$

$$\bar{e}_4 = \frac{1}{2}(1-t_2)\sqrt{t_3 t_4 t_5} \quad , \text{ where } \quad t_2^2 - 2t_2 + \alpha_2 = 0 ;$$

$$\bar{e}_5 = \bar{e}_6 = \frac{1}{2}\sqrt{t_2 t_3 t_4 t_5} \, .$$

By picking ($\bar{\eta}_\kappa, \bar{\varphi}_\kappa$) ($k \leq 5$) from ($-1,+1$), ($0,2\pi$) and $\bar{\eta}_6 = -\bar{\eta}_5$, $\bar{\varphi}_6 = \pi + \bar{\varphi}_5$ we get the complete description of one case of the production of six ultra-relativistic particles which corresponds to the covariant model. For any other, more precise model with the square of the matrix element F the weight of the system of n ultra-relativistic particles can be easily determined using the Eq. (2.1).

$$S_n(E, P) = \lim_{N \to \infty} \frac{1}{N} \sum_1^N F \cdot 2^n e_1 \dots e_n \qquad (3.11)$$

(where $\bar{e}_\kappa$ are obtained by making use of (3.10)).

(3.11) is true, of course, for an arbitrary system of particles too, but in this case the computer is not able to obtain immediately sets ($\bar{\varepsilon}_\kappa, \bar{\eta}_\kappa, \bar{\varphi}_\kappa$) and it will have to work out these sets beforehand (in the covariant model) and put them in memory.

§ 4. The Calculation of the Multiple Production on Nuclei by the Monte-Carlo Method

In preparing experiments on the accelerators the question arises about the momentum and angular spectrum of secondary particles (for example, in calculating antiproton or meson channels or in calculating the multiple production on the emulsion nuclei).

We outline a scheme for obtaining such a spectrum. We restrict ourselves to the case when the incident proton in the nucleus interacts only with one nucleon of the nucleons. The generalization of the scheme stated below to the case of the successive interaction of proton with several nucleons of the nucleus (cascade calculations) is quite obvious.

1. The sampling of the direction and the energy of the nucleon of the nucleus. The calculation of the energy of the system pN in the c.m.s.

2. The picking for the nucleon the state p or n . The picking of the reaction. (We restrict ourselves to the most important reactions, all these reactions have equal probability).

3. The choice of the charge state of the system of secondary particles. (It is better instead of this to attribuate to secondary particles the weights W which express the probability of their appearance in the given reaction).

4. The sampling (if it is necessary) of the mass of the bound state πN or $\pi\pi$ following the Breit-Wigner formula.

5. The sampling of the momenta for the model following the formulas of the paragraph 1. The calculation of the weight Φ of the picked state. The sampling of the decay of the isobar or unstable particle, if it is necessary. The transition into the Lab.sys. The sorting of particles (we are interested in) by momenta and angles etc. taking into account their weights W . According to the results of sorting - the sending Φ with the weight W in the corresponding cells of memory.

6. The return to 1. After several hundreds of samplings have been made we can already estimate roughly the probabilites of some reactions, after this in (2) we can use an important sampling for choosing the reactions ; due to the application of the procedure 1-6 the distributions necessary will be

accumulated in the memory of the machine.

We see that this scheme includes easily all such effects (the motion of the particle-target; the change of the mass, secondary decays) which are omitted in usual analytic calculations due to the fact that their influence is difficult to take into account or these effects are estimated approximately. Here they are taken into account, in principle, accurately; the accuracy is limited only to the computer calculation time. That small accuracy which is necessary for spectra in preparing experiments can be thereby achieved at several hours of the work of the computer.

In the covariant model we can simplify considerably the choice of the reaction (point 2) and the sampling of the momenta (point 5) if we have the dependence $S_n(M_n)$ beforehand. The sampling of the momenta should be performed in this case as indicated at the end of the paragraph 2.

Conclusions

In describing the system of n particles it is convenient to specify the momentum of the particle k ($k = 1,2, \ldots, n-1$) in the rest system of particles $k, k+1, \ldots, n$. In this case nothing restricts the direction of the momentum, and the magnitude of the momenta is restricted only to the energy conservation requirement. This facilitates the calculation of statistical weights and the determination of momenta allowed by kinematics in simulating the process of multiple production.

If in the model of multiple production studied by us the fraction of particles is assumed to be non-interacting with other particles the preliminarily calculation of statistical weights of independent groups facilitates the simulating of the system as a whole.

The table of random stars constructed for the given model F_1 can be used for studying other models F_K too. In constructing hystograms it should only to attribuate to each star from the table the weight Φ_K / Φ_1 instead of 1. The maximum efficiency of constructing the table is achieved in the covariant model (the ultra-relativistic case).

The Monte-Carlo method is suitable for the wide class of functions F, for any masses and energies of particles, it is able to give complicated hystograms and is rather convenient for taking into account the bound states (πN, $\pi\pi$) and the decay of unstable particles. It is therefore believed to be more suitable for studying the multiple production than the usual analytic methods.

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Appendix A

The kinematics of the system of particles with variables mass.

We determine the boundaries in which the kinematic characteristics of the i-th group - e_i, m_i, p_i - change under the assumption that the characteristics of other groups are not specified. Let for the sake of definiteness $i = 1$ and the system as a whole rests ($P = 0$).

We start with the limits for the effective mass m_1 :

$$\tilde{m}_1 \leqslant m_1 \leqslant M_1 - \tilde{\mu}_1 \; ; \; \tilde{\mu}_1 = \sum_{i>1} \tilde{m}_i \; ; \; \tilde{m}_i = \sum_{j=1}^{n_i} m_{ij} \; . \qquad (A1)$$

Necessarity of these conditions is obvious. m_1 reaches its lower boundary when the particles of the group 1 move with equal velocities and the upper one when all other particles move in a similar manner. Both these states are possible physically, hence follows sufficiency of (A1).

Let m_1 be now fixed. Then the limits for e_1 are

$$m_1 \leqslant e_1 \leqslant e_{1\,max} = \frac{M_1^2 + m_1^2 - \tilde{\mu}_1^2}{2 M_1} \; . \qquad (A2)$$

The lower limit is reached when the group 1 does not move, the upper one when the group 1 and other particles move in opposite directions and have equal momenta, all particles of the groups $i \neq 1$ having identical velocities. The fact that (A2) is necessary follows from the well known theorem of Sternheimer.

The region in which (e_1 , m_1) changes is represented in Fig. 2 by the curvilinear triangle ABC. On the planes (m_1 , p_1) and (p_1, e_1) this same region δ_1 will be obtained by simple recalculation. The coordinates A,B,C are

$$\begin{aligned} & m_A = e_A = \tilde{m}_1 \; ; \quad p_A = p_B = 0 \; ; \\ & m_B = e_B = M_1 - \tilde{\mu}_1 \; ; \\ & m_C = \tilde{m}_1 \; ; \; e_C = \frac{M_1^2 + \tilde{m}_1^2 - \tilde{\mu}_1^2}{2 M_1} \; ; \; p_C^2 = e_C^2 - m_C^2 \; . \end{aligned} \qquad (A3)$$

If the effective masses of the groups 2,3, ..., n are fixed then in these equations the corresponding terms in $\tilde{\mu}_1 = \tilde{m}_2 + \tilde{m}_3 + \ldots + \tilde{m}_n$ should be replaced by $m_2, m_3, \ldots, m_n$.

Δ ABC transforms in this case into $\Delta AB_2C_2, \ldots,$ $\tilde{A}\tilde{B}\tilde{C}$ (we get the latter when the masses of all other groups are fixed).

By using this Fig. it is easy to write another pairs of limits:

$$(e_1, m_1)\quad \begin{aligned} &\tilde{m}_1 \le e_1 \le M_1 - \tilde{M}_1 \\ &\tilde{m}_1 \le m_1 \le e_1 \ ; \quad (e_1 \le e_{1c}); \\ &m_{1\,min} \le m_1 \le e_1 \ ; \quad (e_1 \ge e_{1c}),\ m^2_{1\,min} = \tilde{M}_1^2 + 2M_1 e_1 - M_1^2 \ ; \end{aligned} \qquad (A4)$$

$$(m_1, p_1)\quad \tilde{m}_1 \le m_1 \le M_1 - \tilde{M}_1 \ ; \ 0 \le p_1 \le p_{1\,max} = \left(e^2_{1\,max} - m_1^2\right)^{1/2} \ ; \qquad (A5)$$

$$(p_1, m_1)\quad 0 \le p_1 \le p_{1c} \ ; \ \tilde{m}_1^2 \le m_1^2 \le m^2_{1\,max} = M_1^2 + \tilde{M}_1^2 - 2M_1\left(p_1^2 + \tilde{M}_1^2\right)^{1/2} \ ; \qquad (A6)$$

$$(p_1, e_1)\quad \begin{aligned} &0 \le p_1 \le p_c \ ; \ e_1' \le e_1 \le e_1'' \ ; \\ &e_1' = \left(p_1^2 + \tilde{m}_1^2\right)^{1/2} \ ; \ e_1'' = M_1 - \left(p_1^2 + \tilde{M}_1^2\right)^{1/2} . \end{aligned} \qquad (A7)$$

$$(e_1, p_1)\quad \tilde{m}_1 \le e_1 \le M_1 - \tilde{M}_1 \ ; \quad \begin{aligned} &0 \le p_1 \le \left(e_1^2 - \tilde{m}_1^2\right)^{1/2} \quad (e_1 \le e_{1c}) \\ &0 \le p_1 \le \sqrt{(M_1 - e_1)^2 - \tilde{M}_1^2} \quad (e_1 \ge e_{1c}) . \end{aligned} \qquad (A8)$$

We consider a special case when either the group 1 consists of one particle or all groups $i \neq 1$ are reduced to one particle. In the first case $m_1 = m_{11}$, B and C coincide, δ_1 is reduced to the arc AC. In the second one $\tilde{m}_1 \le m_1 \le M_1 - m_{21}$; by fixing m_1 we fix thereby the masses of both groups; this means that the energy is fixed as well: $e_1 = (M_1^2 + m_1^2 - m_{21}^2)/2M_1$, and $\triangle ABC$ is reduced to the arc BC.

Appendix B

Covariant statistical weights

The weight of the system of n ultrarelativistic particles/10/ is obtained by substituting $S_{n-1}(M_{n-1}) = a_{n-1} M_{n-1}^{2n-6}$ in (2.15). Hence

$$a_n = 2\pi \, \frac{a_{n-1}}{(2n-4)(2n-2)}$$

from where

$$S_n(M_n) = \frac{(\pi/2)^{n-1}}{\Gamma(n)\,\Gamma(n-1)}\,(M_n^2)^{n-2}\,. \qquad (3.1)$$

In particular, $S_2 = \frac{\pi}{2}$, $S_3 = \frac{1}{8}\pi^2 M_3^2$, $S_4 = \frac{\pi^3}{96} M_4^4, \ldots$ The weight of the system of n non-relativistic particles is obtained from the Fermi weight if one takes into account the fact that in this approximation $F \approx 2^{-n}(m_1 \ldots m_n)^{-1}$:

$$S_n(M_n) = 2^{-n}(2\pi)^{\frac{3}{2}(n-1)}\left[\frac{m_1 \ldots m_n}{(m_1+\ldots+m_n)^3}\right]^{1/2}\frac{\left(M_n - \sum_1^n m_i\right)^{\frac{3}{2}n-\frac{5}{2}}}{\Gamma\left(\frac{3}{2}(n-1)\right)}\,. \qquad (3.2)$$

In particular, $S_2 = \sqrt{2}\,\pi\sqrt{\frac{m_1 m_2}{(m_1+m_2)^3}}\,T$; $S_3 = \frac{\pi^3}{2}\sqrt{\frac{m_1 m_2 m_3}{(m_1+m_2+m_3)^3}}\,T^2$;

$$S_4 = \sqrt{2}\,\pi^4\sqrt{\frac{m_1 m_2 m_3 m_4}{(m_1+m_2+m_3+m_4)^3}}\,T^{3/2}, \ldots$$

The weight of the system of n ultra-relativistic particles and one particle with an arbitrary mass m is obtained from (2.10) and (3.1) and equals

$$S_{n+1}(M, m, 0, \ldots, 0) = \frac{\pi^n m^2 (Mm)^{n-2}}{\Gamma(n)\,\Gamma(n-1)}\,f_n(\alpha), \text{ where } \alpha = \ln\frac{M}{m}\,, \qquad (3.3)$$

$$f_n(\alpha) = \int_0^\alpha \mathrm{sh}^2 u\,(\mathrm{ch}\,\alpha - \mathrm{ch}\,u)^{n-2}\,du =$$

$$= \sum_{\ell=1}^{n}\frac{(-)^\ell}{2^{\ell-1}}\,\mathrm{ch}^{n-\ell-2}\alpha\left(C_{n-2}^{\ell-2}\,\mathrm{ch}^2\alpha - C_{n-2}^{\ell}\right)\sum_{k=0}^{[\frac{\ell}{2}]} C_\ell^k\,\frac{\mathrm{sh}\,(\ell-2k)\alpha}{\ell-2k} - \alpha\,\mathrm{ch}^{n-2}\alpha \qquad (3.4)$$

and $\quad \mathrm{sh}\,0\alpha\,/\,0 = \alpha/2\,.$

The weight of the system of three particles with arbitrary masses is obtained by substituting (2.16) into (2.15)

$$S_3(E) = \frac{\pi^2}{2E^2}\int_{m_1+m_2}^{E-m_3}\left\{\left[M^2-(m_1-m_2)^2\right]\left[M^2-(m_1+m_2)^2\right]\left[(E-m_3)^2-M^2\right]\left[(E+m_3)^2-M^2\right]\right\}^{1/2}\frac{dM}{M}\,. \qquad (3.5)$$

We denote

$$M^2=y\,;\ a_1=(m_1-m_2)^2\,;\ a_2=(m_1+m_2)^2\,;\ a_3=(E-m_3)^2\,;\ a_4=(E+m_3)^2\,; \qquad (36)$$

$$a_1 \le a_2 \le y \le a_3 \le a_4\ ,$$

then

$$S_3(E)=\left(\frac{\pi}{2E}\right)^2 \int_{a_2}^{a_3} \frac{dy}{y}\sqrt{(y-a_1)(y-a_2)(a_3-y)(a_4-y)} \qquad (37)$$

which is reduced to elliptic integrals by the replacement (/14/, p. 47)

$$\sin^2\varphi = \frac{a_3-a_1}{a_3-a_2}\cdot\frac{y-a_2}{y-a_1}\ . \qquad (38)$$

We have

$$S_3(E)=\frac{\pi^2}{2E^2}\,\frac{(a_3-a_1)^{5/2}(a_4-a_2)^{1/2}(a_2-a_1)^2}{a_1(a_3-a_2)^2}\int_0^{\pi/2}\frac{\sin^2\varphi\cos^2\varphi\sqrt{1-k^2\sin^2\varphi}}{(m+\sin^2\varphi)(n+\sin^2\varphi)}\,d\varphi\,, \qquad (39)$$

where

$$k^2=\frac{a_3-a_2}{a_4-a_2}\cdot\frac{a_4-a_1}{a_3-a_1}<1\ ;\ m=-\frac{a_3-a_1}{a_3-a_2}\ ;\ n=\frac{a_2}{a_1}m\ ;\ \begin{matrix}|n|>|m|>1\ ,\\ |nk^2|>|mk^2|>1\ .\end{matrix} \qquad (310)$$

The integral in (39) can be broken up into the sum of four integrals

$$J=\sum_1^4 D_i\,J_i \qquad (311)$$

where

$$J_1=\int_0^{\pi/2}\frac{d\varphi}{(n+\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}\ ,\quad J_{1+j}=\int_0^{\pi/2}\frac{d\varphi}{(m+\sin^2\varphi)^j\sqrt{1-k^2\sin^2\varphi}}\ ;\ (j=1,2,3). \qquad (312)$$

To determine D_i we have first to calculate

$$A=m(1+m)(1+k^2m)\,;\ B=1+2m+2k^2m+3k^2m^2\,;\ C=1+k^2+3k^2m \qquad (313)$$

and then to obtain D_i from the formulas

$$D_4=\frac{A}{m-n}\ ,\ D_3=\frac{D_4-B}{m-n}\ ,\ D_2=\frac{D_3+C}{m-n}\ ,\ D_1=k^2-D_2\ .\qquad (B\,14)$$

J_1 and J_2 (B 12) are complete elliptic integrals of the 3-d kind and expressed (/15/ p. 71) in terms of complete and incomplete integrals of the first and second kind:

$$J_2=\frac{1}{m}\left\{K(k)+\frac{\sqrt{(a_4-a_2)(a_3-a_1)}}{a_2-a_1}\left[K(k)E(k,\theta_m)-E(k)F(k,\theta_m)\right]\right\}$$

$$J_1=\frac{1}{n}\left\{K(k)+\sqrt{\frac{a_1a_2}{a_3a_4}}\frac{\sqrt{(a_4-a_2)(a_3-a_1)}}{a_2-a_1}\left[K(k)E(k,\theta_n)-E(k)F(k,\theta_n)\right]\right\}\qquad (B\,15)$$

$$\left(\sin^2\theta_m=\frac{a_4-a_2}{a_4-a_1}\ ,\quad \sin^2\theta_n=\frac{a_1}{a_2}\sin^2\theta_m\right).$$

J_3 and J_4 are expressed in terms of J_2 and $K(k), E(k)$ (/14/, p. 59)

$$J_3=\frac{B}{2A}J_2-\frac{1+k^2m}{2A}K(k)+\frac{1}{2A}E(k)\qquad (B\,16)$$

$$J_4=\frac{3B}{4A}J_3-\frac{C}{2A}J_2+\frac{1}{4A}k^2K(k)\ .$$

So, the weight is to be calculated by using (B 9) where instead of the integral one calculates (B11 , from Eqs. (B 10) – (B 16). Due to the small accuracy of the tables of elliptic integrals we should prefer the numerical integration with the use of the simple Eq. (B 7) to all these formulas. If $m_3=0$ (B 7) is reduced to elementary functions

$$S_3(E,m_1,m_2,0)=\frac{\pi^2}{2E^2}\left\{\frac{a_1+a_2+2E^2}{4}\sqrt{Q}+\frac{\sqrt{a_1a_2}E^2}{2}\operatorname{arsh}\frac{2\sqrt{a_1a_2Q}}{(a_2-a_1)E^2}-\right.$$

$$\left.-\left[\frac{a_1+a_2}{4}E^2-\frac{(a_2-a_1)^2}{16}\right]\operatorname{arsh}\frac{2\sqrt{Q}}{a_2-a_1}\right\};\quad Q=(E^2-a_1)(E^2-a_2).\qquad (B\,17)$$

In particular $(\alpha=m/E)$

$$S_3(E,m,m,0)=\frac{1}{8}\pi^2E^2\left[(1+2\alpha^2)\sqrt{1-4\alpha^2}-8\alpha^2(1-\alpha^2)\operatorname{arch}\frac{1}{2\alpha}\right]\qquad (B\,18)$$

$$S_3(E,m,0,0)=\frac{1}{8}\pi^2E^2\left(1-\alpha^4+4\alpha^2\ln\alpha\right).\qquad (B\,19)$$

The approximate formulas for the weight of three particles with equal mass

We rewrite the non-relativistic expression for S_3 (B 2) so as to obtain S_π (B 1) as the ultra- relativistic limit. With the accuracy up to T^2 we have (T - kinetic energy)

$$m_1 + m_2 + m_3 = M_3 - T, \quad m_1 = \frac{M_3 + m_1 - m_2 - m_3}{2} \quad etc... \qquad (B\ 20)$$

$$T = \frac{M_3^2 - (m_1+m_2+m_3)^2}{2(m_1+m_2+m_3)} = \frac{M_3}{2}\left[1-\left(\frac{m_1+m_2+m_3}{M_3}\right)^2\right] ; \qquad (B\ 21)$$

$$S_3 = \frac{\pi}{2\sqrt{2}}\,\frac{\pi^2}{8}\,M_3^2\left[1-\left(\frac{m_1+m_2+m_3}{M_3}\right)^2\right] \times$$

$$\times\left[\left(1-\frac{-m_1+m_2+m_3}{M_3}\right)\left(1-\frac{m_1-m_2+m_3}{M_3}\right)\left(1-\frac{m_1+m_2-m_3}{M_3}\right)\right]^{3/2}. \qquad (B\ 22)$$

For $m_1, m_2, m_3 \to 0$ Eq. (B.22) differs from S_3 (B1) by a factor $\pi/2\sqrt{2} \approx 1$. We introduce therefore instead of $\pi/2\sqrt{2}$ the factor

$$1 + \frac{\pi - 2\sqrt{2}}{2\sqrt{2}}\left(\frac{m_1+m_2+m_3}{M_3}\right)^2. \qquad (B\ 23)$$

We obtain for $m_1 = m_2 = m_3 = m$ (2.17) . The difference between $S_3(M_3)$ (2.17) and the exact expression (B 7) does not exceed anywhere 1,6%.

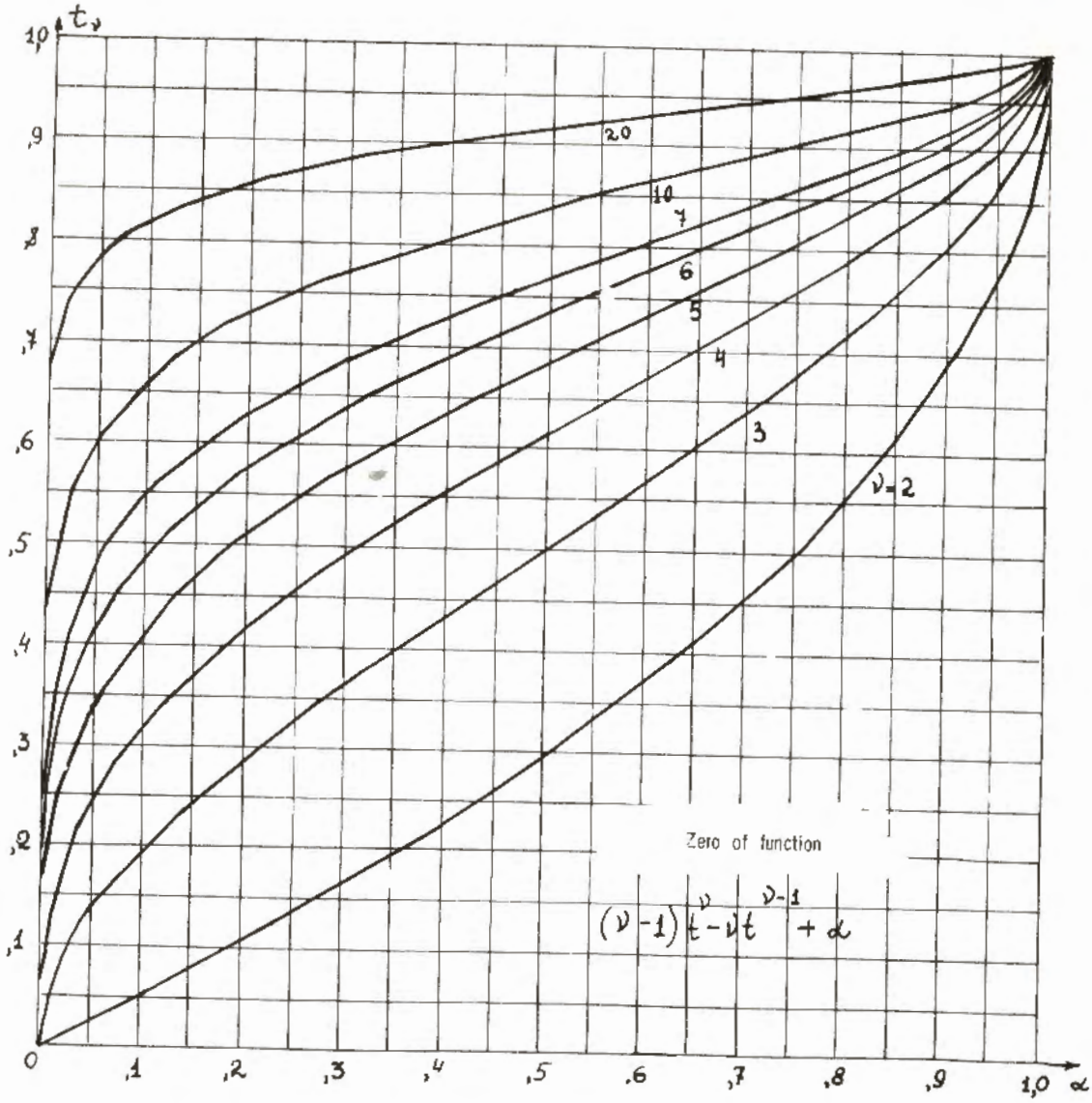
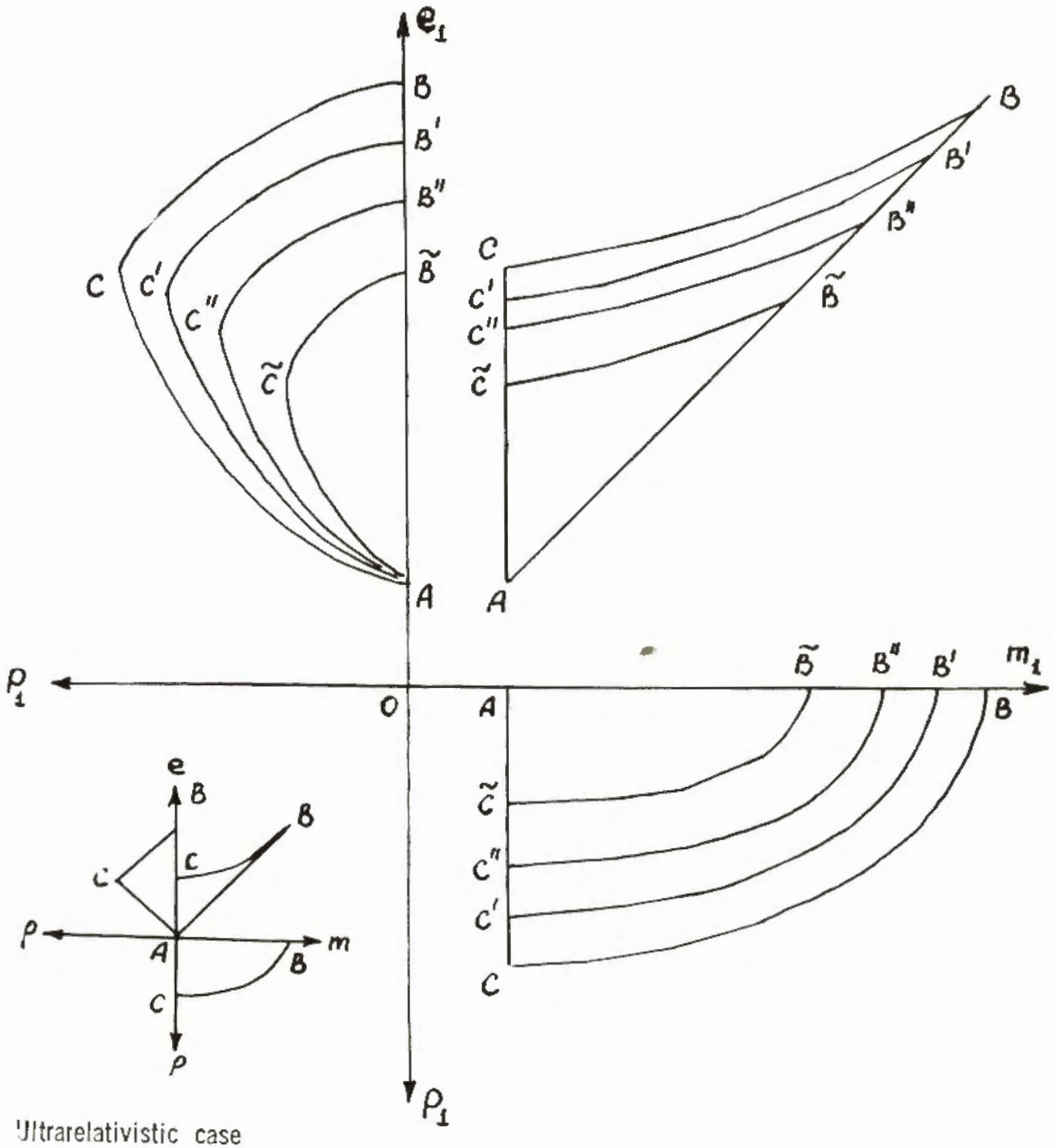

t_ν
1,0
,9
,8
,7
,6
,5
,4
,3
,2
,1
0
,1
,2
,3
,4
,5
,6
,7
,8
,9
1,0
α
20
10
7
6
5
4
3
ν=2
Zero of function
$(\nu-1)t^{\nu}-\nu t^{\nu-1}+\alpha$

Fig. 2.



Ultrarelativistic case

Fig. 2.

Addendum

In some cases we took the quantities M_k (2.13),(3.6) as the integration variables. A successive use of them outlines some new ways of calculations. We introduce instead of M_k the kinetic energy $\bar{\tau}_k$ of the system of particles $k,\dots,n$ in their rest system

$$\bar{\tau}_k = M_k - \sum M_{k-1} . \tag{1}$$

This determines uniquely the energy $\bar{e}_{k-1}$ of the particle $k-1$ from (1.4). By comparing with (1.1),it is easy to obtain the form of the region D in the variables $\bar{\tau}_k$ (instead of $\bar{e}_{k-1}$)

$$\bar{\tau}_1 \geq \dots \geq \bar{\tau}_k \geq \bar{\tau}_{k+1} \geq \dots \geq \bar{\tau}_{n-1} \geq 0 . \tag{2}$$

D has now plane boundaries. (1.6) passes into

$$S_n(E_1,P_1) = \frac{1}{M_1} \int d\bar{\tau}_2 \iint d\bar{\Omega}_1 \dots \int d\bar{\tau}_{n-1} \iint d\bar{\Omega}_{n-2} \iint d\bar{\Omega}_{n-1} \cdot \mathcal{F} \prod_1^{n-1} \bar{p}_k \prod_1^{n} e_k , \tag{3}$$

and instead of stretching D into a cube it is possible, having chosen (n-2) random numbers in (0,1), to take the largest one as $\bar{\tau}_2/\bar{\tau}_1$, the next by the magnitude as $\bar{\tau}_3/\bar{\tau}_1$ etc.

This ensures hitting the cube. In the calculations by the Monte-Carlo method it is necessary to introduce the factor $\frac{1}{(n-2)!}$ into the sum of random numbers Φ/Φ^* (2.1).

Since the region D is independent of $m_1, \dots, m_n, E$, but depends only upon n, then the important sampling for $m_1,\dots,m_n=0$, may turn out to be valid in the wide range of m_k, E. Considering (3.10) as formula for $\bar{\tau}_{n-\nu+1}$ and calculating $\vec{p}_1, \dots, \vec{p}_n$ by (1.4), (1.3), (1.11′) one can make calculations of different reactions for different energies by (3.11), Thus, the reserve in the last phrase of § 3 falls out.

Further, in the group method it is also possible to restrict the region $\mathcal{S}_1$ by the planes

$$0 \leq \bar{t}_1 \leq \bar{\tau}_1 , \quad 0 \leq \bar{t}_2 \leq \bar{\tau}_1 - \bar{t}_1 \tag{4}$$

if group 1 is set by the quantities $\bar{t}_1 = m_1 - \tilde{m}_1$, $\bar{t}_2 = M_2 - \sum \tilde{M}_1$, $\bar{\eta}_1$, $\bar{\varphi}_1$ instead of the quantities $m_1, \bar{e}_1, \bar{\eta}_1, \bar{\varphi}_1$. The duplication formula becomes now symmetrical

$$M_1 S_{n+\nu}(M_1) = 4\pi \int_0^{\bar{\tau}_1} d\bar{t}_1 \int_0^{\bar{\tau}_1 - \bar{t}_1} d\bar{t}_2 \cdot m_1 S_n(m_1) \cdot \bar{p}_1 \cdot m_2 S_\nu(m_2) . \tag{5}$$

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