ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

M.I. Shirokov

E - 507

ON THE RELATIVISTIC GENERAL THEORY OF REACTIONS

жэтф, 1960, т 39, в3, с633-638

Дубна 1960

M.I. Shirokov

E - 507

ON THE RELATIVISTIC GENERAL THEORY OF REACTIONS

Submitted to JETP

Abstract

Formulae of the relativistic general theory which express the cross section and the polarization in terms of phase-shifts can be of different forms for different definitions of the relativistic spin operator. However, in the rest system of a particle its spin operators coincide. This allows one represent the general theory in a form which is the same for all equivalent definitions of the spin.

Introduction

For the relativization of formulae which express the differential cross section and the polarization in terms of phaseshifts it is necessary to define relativistic operator of the particle spin and find transformation functions from the representation in the momenta and spin projections into the representation involving conserved total angular momentum of participant particles. In[1,2] for the description of particles with spin the representation of L. Foldy-Ju. Shirokov[3] is applied. The authors of[1] obtained this representation starting from the definition of the spin as the internal angular momentum of particle relative to its center of mass. However, there may be different relativistic definitions of the center of mass coordinate (see e.g.[4,5]). In§2 of this paper we find (and this is one of the results of the present work) an ununitary representation of the inhomogeneous Lorentz group in which the operator of the internal angular momentum differs from Pryce-Foldy- Ju. Shirokov's spin. Other approches to the relativistic spin definition are possible (see e.g.[5] and[6]).

To various spin definitions there may correspond various transformation functions and, respectively, expressions of various forms for the cross section and the polarization 1). In the present paper it is shown that there exists a form of the general theory whose transformation functions and aforementioned expressions are identical for all equivalent representations of the inhomogeneous Lorentz group. We assume that the masses of all the particles are nonvanishing and their spins are arbitrary.

Note that for describing the states of participant particles we should make use of wave functions which transform just according to the inhomogeneous Lorentz group representation. The theory of inhomogeneous Lorentz group representations yields the description of such systems all the states of which can be obtained from some fixed state by using displacements, rotations and Lorentz transformations (and by superposing these states), so, there exist no relativistically invariant differences between various states of the system[7]. The homogeneous Lorentz group contains no displacements and can not therefore des-

---

1) *Nevertheless one can show that in this case the angular correlations (e.g. azimuthal asymmetries in the experiments on double and triple scattering) will be the same (though they may be expanded in various complete sets of functions of angles).*

cribe all the states of the free particles.

1. Spin in the Particle Rest System and General Theory of Reactions

1. One may caracterize the state of a free particle with spin in the following manner: 1) one indicates the momentum $\vec{p}$ of the particle (e.g. in the c.m.s. of the reaction); 2) in the Lorentz system where the particle is at rest one indicates its spin state.

The spin operator $\vec{S}^{o}$ of the particle in its rest system is equal to its total momentum $\vec{M}$ (since the orbital momentum in this system is zero), consequently all the spin operators coincide in the particle rest system. The spin projection m is defined as an eigenvalue of the operator $\Sigma = (\vec{S}^{o} \cdot \vec{n}) = (\vec{M} \cdot \vec{n})$ where $\vec{n}$ is unit vector parallel to the direction of $\vec{p}$ (let us note that in the rest system we can point out the direction parallel to $\vec{p}$).

2. The general reasoning of the formal theory of reactions is stated in[8], section 1, and in[9], introduction. We have to express the elements

$$(m_c \; m_d \; \vec{p}' \mid S \mid m_a \; m_b \; \vec{p}) \tag{1.1}$$

of the S-matrix for reactions of the type $a + b \to c + d$ in terms of phase-shifts [2] (more exactly, in terms of the elements of this matrix in the representation which includes total angular momentum). It is implied that we had already took into account the total momentum conservation; $\vec{p}'$ and $\vec{p}$ are relative momenta of particles, m are the eigenvalues of the operators Σ of individual particles. In contrast to[9] the spin functions by means of which the S-matrix element (1.1) is formed are considered to be referred to the particle rest systems (besides, in[9] m_d denotes the eigenvalue of $(\vec{M}_d \cdot (\vec{-n}))$, and here m_d is assumed to be an eigenvalue of the operator $(M_d n')$. However, the total angular momentum (as a quantity which is common for initial and final state) must be referred to one Lorentz system only (for convenience, to the c.m.s. of the reaction) and of course, to one quantization exis Z. The expression of (1.1) in terms of elements $(\lambda_c \; \lambda_d \; \vec{p}' \mid S_o \mid \lambda_a \; \lambda_b \; \vec{p})$ referred to the c.m.s. of the reaction must be of the form

$$(m_c \, m_d \, \vec{p}' | S | m_a \, m_b \, \vec{p}) = \varphi^*(m_c \, m_d \, p') \, (m_c \, m_d \, \vec{p}' \mid S_o \mid m_a \, m_b \, \vec{p}) \, \varphi \, (m_a \, m_b \, p). \tag{1.2}$$

Since we deal with Lorentz transformations whose velocities are parallel to the momenta of particles (see Appendix).

2) *The invariance under four-dimensional rotations can be expressed just as it has been already done by Stapp[6], but the S-matrix unitarity may be expressed in a simple manner only by introducing phase-shift expansion.*

S_0- matrix elements in (1.2) can now be expressed in terms of elements of the S_0 in the representation of a square and Z-projection of the operator $\vec{J} = \vec{M}_1 + \vec{M}_2$ of the total angular momentum (which is a "spin" of the system of interacting particles in the c.m.s. of the reaction); of the eigenvalues of the operators Σ_1 and Σ_2 and E (total energy). The corresponding transformation function has been obtained in papers by Chou Kuang-chao/10/ and Jacob and Wick/9/. Its derivation (see/10/, §2) does not use any assumptions about the concrete representation which describes interacting particles (in particular, even representations with zero rest mass are allowed). By using the diagonality of S with respect to the square $J(J+1)$ and the projection M of the total angular momentum we have

$$(m_c m_d \vec{p}' | S | m_a m_b \vec{p}) = \sum_{J,M} \mathcal{D}^J_{m_c+m_d, M}(-\pi, \vartheta', \pi-\varphi')\, q^*(m_c m_d p') \times$$

$$\times (m_c m_d p' JM | S_0 | m_a m_b p JM)\, q(m_a m_b p)\, \mathcal{D}^J_{M, m_a+m_b}(\varphi, \vartheta, 0). \qquad (1.3)$$

Now if we introduce the matris $\tilde{S}$:

$$(m_c | \tilde{S}^{JE} | m_a) = q^*(m_c p')(m_c | S_0^{JE} | m_a)\, q(m_a p) \qquad (1.4)$$

we perform in fact the transformation inverse to (1.2) i.e. $\tilde{S} = S$. Note that using the phase-shift analysis we can find only the product of all the factors in the right hand side of (1.4) but not each separate factor.

The expression of the tensors of polarization of the reaction products in terms of elements of (1.1) and in terms of the beam and target tensors is defined by non-relativistic formulas see e.g./11/. The elements ($m_c\, m_d \,|\, S^J | m_a\, m_b$) can be introduced into these equations. In particular, in/9/ the angular distribution and the polarization vector are expressed in terms of these elements. Of course, the polarization vector, for example, is to be defined as a mean value of the spin vector of the particle in its rest system. If it known originally in other Lorentz system, e.g. in the Lab. syst. (polarized beam) then we need to find its expression in the rest system of a particle. In this case we may have need of a concrete form of representation. In other respects the form of the general theory stated above (which differs only slightly from that discussed in detail by Jacob and Wick/9/) is the same for various representations of particles possessing spin.

3. In the aforementioned general theory of reactions there arises one more problem the statement and the solution of which will be illustrated by an example of the proton double scattering.

Let the proton polarization be found by means of the azimuthal asymmetry of the angular distribution of the scattering II and used for the phase-shift analysis of the scattering I. The asymmetry II allows one to find the components P_{z2}, P_{y2}, P_{x2} of the polarization vector, respectively, along the directions of the proton momentum $\vec{P}_2$ in the c.m.s. of II (or in the lab. sys. since the target II is at rest), along the perpendicular y_2 to the plane of the scattering I etc, referred to the rest system K_2 of the proton , as to the Lorentz system. For the phase-shift analysis of I we need the components P_{z1}, P_{y1}, P_{x1} (ort Z_1 is parallel to the proton momentum $\vec{p}_1$ in the c.m.s. of 1, and ort $y_1 \| y_2$), referred to the rest system K_1 of the proton (which differs from K_2 , see below). By turning the components P_{z2}, P_{y2}, P_{x2} at the angle between $\vec{p}_2$ and $\vec{p}_1$ we get the components P'_{z1}, P'_{y1}, P'_{x1} referred respectively to the orts z_1, y_1, x_1 (for detail see[11], § 3) but expressed in the same Lorentz system K_2 . In order to proceed from K_2 into K_1 we have to perform the following Lorentz transformations: 1) from K_2 into the Lab.syst. K_ℓ means of the velocity $\vec{\beta} \| \vec{p}_1$; 2) from K_ℓ into c.m.s. of 1 by means of the velocity $\vec{\beta}_2$ parallel to the beam of the scattering 1; 3) from the c.m.s. of 1 into K_1 . The corresponding velocity $\vec{\beta}_1$ is calculated as a relativistic sum of the velocities $\vec{\beta}_2$ and $-\vec{\beta}$. The product of these three transformations is a three- dimensional rotation, see[12], § 22, [6] and foot-note 5 in[1]. So, the orths z_1, y_1, x_1 with respect to space axes of K_2 have the orientation different form that with respect to the K_1 axes. The finding of the axis and the angle Ω of the rotation under discussion is a purely kinematic problem. In particular Ω is the angle between the velocities $\vec{\omega}$ and $\vec{\omega}''$ (see Moller[12] § 22, Eqs. (59) and (59′)). To find $\sin\Omega$ we need only to perform a vector multiplication of the expressions for $\vec{\omega}$ and $\vec{\omega}''$.

The results are given in the paper[1]. They are expressed in terms of the rotation of the spin vector with respect to unchanged space axes (which is equivalent to the aforementioned rotation of the K_1 axes with respect to the K_2 axes).

4. So, the general theory of reactions can be represented in the form whose basis is Eq.(1.3) which is the same for different (but equivalent) representations of the inhomogeneous Lorentz group describing the particle with spin. The Eq. (1.3) in its form concides with the corresponding non - relativistic one. In the problems of phase-shift analysis or of finding the angular correlation unlike the non-relativistic case the polarization tensors are to be subjected to some rotation of the relativistic origin. The axis and the angle of this rotation are the same for different representations of particles with spin.

2. The Example of the Ununitary Representation of the Inhomogeneous Lorentz Group

The principles of the theory of representations of the inhomogeneous Lorentz group are assumed to be known (see, for example[13]).

1. Let us express the generators $M_{\mu\nu}$ of the infinitesimal rotations in the space-time in the form

$$M_{\mu\nu} = \rho_\mu p_\nu - \rho_\nu p_\mu + J_{\mu\nu} . \qquad (2.1)$$

Here p_μ are displacement generators (momentum operators), ρ_μ and $J_{\mu\nu}$ are certain new operators introduced instead of the six operators $M_{\mu\nu}$. They must obey simpler commutation relations than $M_{\mu\nu}$, which should facilitate its determination.

The operators $M_{\mu\nu}$ themselves form a tensor representation of the Lorentz group (so called infinitesimal representation; see[14] § 44 and § 76). In the representation of L. Foldy - Ju. Shirokov[3] $J_{\mu\nu}$ and $\rho_\mu p_\nu - \rho_\nu p_\mu$ taken separately are not tensors of the second rank. Indeed, using the direct Lorentz transformation we can test that $\vec{J}$ and $[\vec{J} \times \vec{p}]/\varkappa + p_0$ are not space and time components of the antisymmetric second rank tensor ($\varkappa$ is the rest mass).

In order that $J_{\mu\nu}$ will be a tensor, we represent $M_{\mu\nu}$ in the form $\tau_\mu p_\nu - \tau_\nu p_\mu + J_{\mu\nu}$ where τ_μ are such operators that

$$[\tau_\mu, \tau_\nu] = 0 \qquad , \qquad [\tau_\mu, p_\nu] = i\,\delta_{\mu\nu} \qquad (2.2)$$

(it is implied that $\hbar = c = 1$; $\mu, \nu = 1,2,3,4$; $p_4 = i p_0$). In accordance with the reasoning presented in §1[1] we should assume for the four-vector τ_μ such covariant commutations so that $[\vec{\tau}, \sum_\nu p_\nu p_\nu] = 0$ (e.g. $[\tau_\mu, p_\nu] = i(\delta_{\mu\nu} + p_\mu p_\nu/\varkappa^2)$ and $[\tau_\mu, \tau_\nu] = i(\tau_\mu p_\nu - \tau_\nu p_\mu)/\varkappa^2$. It can be shown that $\vec{\tau}$ can be expressed in this case in terms of $M_{\mu\nu}$ and p_μ and called consequently the centre of mass coordinate. But the result concerning the spin operator turns out to be the same as for (2.2).

Unlike the paper[1] the subsequent calculations can be made in a covariant form. From commutations

$$[M_{\mu\nu}, p_\lambda] \equiv [\tau_\mu p_\nu - \tau_\nu p_\mu + J_{\mu\nu}, p_\lambda] = i(p_\nu \delta_{\mu\lambda} - p_\mu \delta_{\nu\lambda})$$

and (2.2) follows that $[J_{\mu\nu}, p_\lambda] = 0$. The operator τ_μ as a four-vector must have the same commutations $[M_{\mu\nu}, \tau_\lambda] = i(\tau_\nu \delta_{\mu\lambda} - \tau_\mu \delta_{\nu\lambda})$ and therefore $[J_{\mu\nu}, \tau_\lambda] = 0$. Further we make sure that the commutations $[J_{\mu\nu}, J_{\lambda\sigma}]$ are of the same form as $[M_{\mu\nu}, M_{\lambda\sigma}]$. By introducing the notation $\{J_1, J_2, J_3\} = \{J_{23}, J_{31}, J_{12}\}$ and $J_{j4} = i K_j$ they can be rewritten in the form

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \quad ; \quad [J_i, K_j] = i\varepsilon_{ijk} K_k \quad ; \quad [K_i, K_j] = -i\varepsilon_{ijk} J_k . \qquad (2.3)$$

It is only these commutation relations which restrict the operators $J_{\mu\nu}$. The task is reduced to their determination. The commutations (2.3) are caracteristic of the homogeneous Lorentz group whose representations are known, see for example/15/, part II, § 2. $\vec{J}^2 - \vec{K}^2$ and $(\vec{J}\cdot\vec{K})$ are invariants of (2.3) and they are simultaneously invariants of our inhomogeneous Lorentz group. But besides them the inhomogeneous Lorentz group has one more invariant $\Gamma^2 = \sum_\mu \Gamma_\mu \Gamma_\mu$

$$\Gamma_\mu = \frac{1}{2i}\,\varepsilon_{\mu\nu\sigma\lambda}\, M_{\nu\sigma}\, p_\lambda \qquad \text{or}$$
$$\vec{\Gamma} = \vec{J}p_0 - [\vec{K}\times\vec{p}\,] \quad , \qquad \Gamma_4 = i(\vec{M}\cdot\vec{p}) = i(\vec{J}\cdot\vec{p}). \qquad (2.4)$$

We have to choose from all irreducible representations of the homogeneous group such representations so that the matrix Γ^2 in the basis of the functions $f_{\ell m}$ of the representation (ℓ_0, ℓ_1) see/15/, ($m = -\ell, -\ell+1, \dots +\ell$, $\ell = \ell_0, \ell_0+1 \dots$) be proportional to the unit one. To provide this it is necessary that

$$\Gamma^2 f_{\ell m} \equiv \{ \vec{K}^2 p^2 - (\vec{p}\cdot\vec{K})^2 - (\vec{p}\cdot\vec{J})^2 + \vec{J}^2 p_0^2 + p_0(\vec{p}\cdot[\vec{J}\times\vec{K}]) - p_0(\vec{p}\cdot[\vec{K}\times\vec{J}]) \} f_{\ell m} =$$
$$= \tilde{\gamma}^2 f_{\ell m} \qquad (2.5)$$

for any function $f_{\ell m}$ ($\tilde{\gamma}^2$ is the value of the invariant Γ^2 in the representation under consideration). However by applying $(\vec{p}\cdot\vec{K})^2$ and $p_0\{(\vec{p}\cdot[\vec{J}\times\vec{K}]) - (\vec{p}\cdot[\vec{K}\times\vec{J}]\}$ to $f_{\ell m}$ we can get the functions $f_{\ell\pm1,\,m}$ and $f_{\ell\pm2,\,m}$. Let us consider, for example, the operator $p_0 p_1\{J_2K_3 - J_3K_2 - K_2J_3 + K_3J_2\}$ which is independent of other operators. Demand that $K_3 f_{\ell m}$ would not contain $f_{\ell\pm1,\,m}$ /15/.

$$K_3 f_{\ell m} = C_\ell\sqrt{\ell^2 - m^2}\, f_{\ell-1,\,m} - m A_\ell f_{\ell m} - C_{\ell+1}\sqrt{(\ell+m+1)(\ell-m+1)}\, f_{\ell+1,\,m}. \qquad (2.6)$$

It is necessary for this that $C_\ell = C_{\ell+1} = 0$. Since $C_\ell = i\sqrt{(\ell^2-\ell_0^2)(\ell^2-\ell_1^2)}/\sqrt{4\ell^2-1}\cdot\ell$ this can be possible only for $\ell = \ell_0$, $\ell+1 = \ell_1$ I.e. only in the representations $(\ell_0, \ell_1) = (\ell_0, \ell_0+1)$ Γ^2 will be invariant. In this case ℓ takes the only value $\ell = \ell_0$ and then $A_\ell = i$, see/15/. From Eqs. (3′) in/15/ p. 11, § 2 it follows that $\vec{K} = -i\vec{J}$ ($\vec{K} = i\vec{J}$ satisfies also the commutation relations (2.3) of the homogeneous group and (2.5)). From (2.5) we find that $\Gamma^2 f_{\ell m} = \vec{J}^2 \varkappa^2 f_{\ell m} = \ell(\ell+1)\varkappa^2 f_{\ell m}$, i.e. that the square of the spin $\vec{J}^2$ is Lorentz invariant. From (2.4) it follows that $\varkappa^2\vec{J} = \vec{\Gamma}p_0 + i[\vec{p}\times\vec{\Gamma}] + i\Gamma_4\vec{p}$ i.e. $\vec{J}$ can be expressed in terms of $M_{\mu\nu}$ and p_μ only and therefore $\vec{J}$ is conserved.

2. The representation

$$\vec{M} = [\vec{r}\times\vec{p}\,] + \vec{J} \;; \quad \vec{N} = \vec{r}p_0 - r_0\vec{p} - i\vec{J} \;; \quad [J_i, J_j] = i\varepsilon_{ijk}J_k \qquad (2.7)$$

is uninitary one and before using it for the description of the states of a particle it is necessary to define the invariant norm (Ψ,Ψ). One of the ways of doing this is doubling of the number of the wave function components (doubling of the representation dimension).

In the following we will follow Ju.M. Shirokov's statement/13/, § 6.

If under finite transformations the doubled $\psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix}$ is transformed by the operator $\begin{pmatrix} \mathcal{U} & 0 \\ 0 & \mathcal{U}^{+-1} \end{pmatrix}$ where $+$ notes the Hermitian conjugation, then the corresponding infinitesimal operators are of the form $\begin{pmatrix} M & 0 \\ 0 & M^{+} \end{pmatrix}$, for example. Since τ_μ and p_μ may be considered to be Hermitian ones then in the doubled representation only

$$\vec{J}^{\,d} = \begin{pmatrix} \vec{J} & 0 \\ 0 & \vec{J}^{+} \end{pmatrix} \qquad \text{and} \qquad \vec{K}^{d} = \begin{pmatrix} -i\vec{J} & 0 \\ 0 & iJ^{+} \end{pmatrix} \tag{2.8}$$

change essentially.

As the metrix matrix h we can take any matrix of the form $\begin{pmatrix} 0 & \exp i\alpha \\ \exp(-i\alpha) & 0 \end{pmatrix}$ and then $h^{-1} K^{+} h = K$ and $h^{-1} J^{+} h = J$. The operators $h J^{d}$ and $h K^{d}$ are Hermitian and

$$\langle J \rangle \equiv (\psi^{+} h J^{d} \psi) \equiv \iiint \frac{d^3 p}{p_0} \sum_{\alpha} \psi_\alpha^{*} (h J^{d})_{\alpha\beta} \psi_\beta \tag{2.9}$$

is a real number.

Under the Lorentz transformation $\psi \to \mathcal{U}\psi = \psi'$ the mean value $\langle J \rangle$ will change:

$$\langle J \rangle' = (\psi'^{+} h J^{d} \psi') = (\psi^{+} \mathcal{U}^{+} h\, \mathcal{U}\mathcal{U}^{-1} J^{d} \mathcal{U} \psi) = (\psi^{+} h\, J' \psi) = \langle J' \rangle \tag{2.10}$$

(The definition $\mathcal{U}^{+} h\, \mathcal{U} = h$ of the metrix matrix is here used). In other words, by finding $\langle J \rangle'$ we may take the mean value over the same functions, but of the transformed operator $J'_{\mu\nu} = \mathcal{U}^{-1} J_{\mu\nu}\, \mathcal{U} = a_{\mu\lambda} a_{\nu\delta} J_{\lambda\delta}$. Thus, $\langle \vec{J} \rangle$ transforms as space components of the antisymmetrical tensor, see /12/, § 53

$$\langle \vec{J}' \rangle = \gamma \langle \vec{J} \rangle + \vec{\beta} (\vec{\beta} \cdot \langle \vec{J} \rangle)(1-\gamma)/\beta^2 + \gamma [\vec{\beta} \times \langle \vec{K} \rangle] \,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.11}$$

i.e. in a different way as compared to the polarization vector in the Foldy-Shirokov's representation, see e.g./1/, Appendix 2.

In particular, in various Lorentz systems $\langle \vec{J} \rangle^2$ has different values (which does not contradict the invariance of J^2, since $\langle \vec{J} \rangle^2 \neq \langle \vec{J}^2 \rangle$).

3. Our aim was to show one possible ununitary representation of the inhomogeneous Lorentz group. We shall not discuss many details of this representation. We make only several remarks.

1) Ju.M. Shirokov has pointed out that it is possible to find the metric matrix without doubling the number of components (for example, for spin ½ $h = -p_0/2 + (\vec{p}\cdot\vec{J})$). However then $\langle \vec{J} \rangle$ is not real in all the Lorentz systems, i.e. can not play the role of the operator of the physical quantity.

2) The Dirac spinors, which describe the states with positive and negative energy transform according to the doubled representation (2.7), (2.8) (see/16/). However, the excess components in the doubled representation can make another meaning.

3) If we take as β_μ in (2.1) the operator $g_\mu = \sum_\nu M_{\mu\nu} p_\nu$ (see/13/) then we obtain the representation (40) in/13/ in which the operators of the internal angular momentum J_x, J_y, J_z do not commute as the Pauli matrices (see commutations (3.6) in/4/ for the operator $\vec{\Sigma}$).

4) In the case of ununitary representations with metric matrix h, the S-matrix instead of the unitarity property must possess the following property $S = h^{-1}(S^+)^{-1}h$ to provide the norm ($\psi^+ h \psi$) to be conserved in time.

In conclusion I wish to express my gratitude to Ju.M. Shirokov and I.V. Polubarinov for discussing the questions broached in § 2.

Appendix

Spin Operator in the Rest System

Because $\vec{S}^0$ is equal to $\vec{M}$ in the rest system, then $[S_i^0, S_j^0] = i\varepsilon_{ijk} S_k^0$. These commutations define the representation of the three- dimensional rotation group which may be assumed to be unitary/15/ and consequently S_k^0 may be considered as Hermitean matrices. In the rest system $\vec{\Gamma} = \varkappa \vec{M}$, $\Gamma_4 = 0$, see (2.4) and $\vec{S}^0 = \vec{\Gamma}/\varkappa$. From here it follows that $(\vec{S}^0)^2$ equals to the Lorentz invariant $\Gamma^2/\varkappa^2$.

If we do not consider the concrete representation we do not know the transformation of spin functions when the Lorentz frame of reference changes. But for our general theory we need to know only how the spin functions transform under Lorentz transformations Λ with velocities $\vec{\beta}$ parallel to the particle momentum $\vec{p}$. The operator $\Sigma = (\vec{S}^0\vec{n}) = \Gamma_4/i\varkappa|\vec{p}|$ is invariant under such transformation. Indeed, let $\vec{\beta} = \alpha\,\vec{p}/p_0$, $0<\alpha<1$ then

$$\frac{\Gamma_4'}{|\vec{p}\,'|} = \frac{\gamma\{\Gamma_4 - i(\vec{\beta}\cdot\vec{\Gamma})\}}{|\vec{p} + \vec{\beta}[(\vec{\beta}\cdot\vec{p})(\gamma-1)/\beta^2 - p_0\gamma]|} = \frac{\Gamma_4}{|\vec{p}|}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(the equality $\sum_\mu \Gamma_\mu p_\mu = 0$ is used). Let us denote representation of transformation Λ by $\mathcal{U}_\Lambda$ so that $\psi = \mathcal{U}_\Lambda \psi'$.

We has showed that $\Sigma' = \mathcal{U}_\Lambda^{-1} \Sigma\, \mathcal{U}_\Lambda = \Sigma$, i.e. that $[\Sigma, \mathcal{U}_\Lambda] = 0$. This means that matrix $\mathcal{U}_\Lambda$ is diagonal with respect to the eigenvalues of Σ and the state $|\vec{p}, m\rangle$ under transformation Λ transforms into

$$\mathcal{U}_\Lambda\, |\vec{p}, m\rangle = \sum_{m'} |\Lambda\vec{p}, m'\rangle\, Q_{m',m}(\vec{p}, \Lambda) = |\Lambda\vec{p}, m\rangle \cdot q(m, \vec{p}) \tag{A.1}$$

Let us show that the diagonal elements $q(m, \vec{p})$ of the spin part of the transformation $\mathcal{U}_\Lambda$ depend only upon $|\vec{p}|$. $\mathcal{U}_\Lambda$ generator is an operator proportional to ($\vec{p}\cdot\vec{N}$) so that $\mathcal{U}_\Lambda = exp\, i\{-Arth\,\beta(\vec{p}\cdot\vec{N})/_{|\vec{p}|}\}$. The operator ($\vec{p}\cdot\vec{N}$) is three-dimensional scalar and commutes therefore with the operators of three-dimensional rotations $\mathcal{U}_R$. Hence

$$\mathcal{U}_R\, \mathcal{U}_\Lambda\, |\vec{p}, m\rangle = \mathcal{U}_\Lambda \mathcal{U}_R\, |\vec{p}, m\rangle$$ or [3])

$$|R\Lambda\vec{p}, m\rangle\, q(m, \vec{p}) = |\Lambda R\vec{p}, m\rangle\, q(m, R\vec{p}) \tag{A.2}$$

It follows that $q(m, \vec{p}) = q(m, R\vec{p}) \equiv q(m, |\vec{p}|)$.

In conclusion we note that the proof of the equivalence of irreducible representations of the inhomogeneous Lorentz group with the same $\mathcal{K}^2$ and Γ^2 which has been presented by Wigner/17/ can be considered apparently true not only for unitary transformations. Wigner has showed that any representation is equivalent to the representation $\mathcal{U}_0$ which can be represented as the product of a representation of a certain rotation in the space of the "little group" (in the case of particle with finite mass such a group contains three-dimensional rotations in the space of wave functions with $\vec{p}_0 = 0$) and a representation of a Lorentz transformation Λ which concerns only the momentum variables (see (67) and (67a) in /17/). More exactly, under the transformation $\Lambda(\vec{p})$ from the rest system into other system where the particle momentum is $\vec{p}$, $Q^{0}_{m\,m'}(\vec{p}_0, \Lambda(\vec{p})) = \delta_{m,m'}$. If some representation $\mathcal{U}'$ does not satisfy this requirement, then we may point out an equivalent representation which will satisfy it. This equivalent representation may be obtained by means of the transformation carrying function $\varphi(\vec{p}, m)$ into $\sum_{m'} Q_{m,m'}(\vec{p}_0, \Lambda^{-1}(\vec{p}))\, \varphi(\vec{p}, m')$. Emphasize that this transformation is contained among operators (which represent the transformations Λ) of the representation $\mathcal{U}'$. So, for any representation of the inhomogeneous Lorentz group we can indicate such an equivalence transformation (which may be an ununitary one) which allows one to reduce the given representation to the form (67a)/17/, which is the same for all the representations.

3) *We fixe the phase-shifts of the states* $|\vec{p}, m\rangle$ *with different* m *defining* $|\vec{p}, m\rangle$ *as* $|\vec{p}, m\rangle = \sum_n |\vec{p}, n\rangle\, \mathcal{D}_{n,m}(\vec{p})$. $\mathcal{D}(\vec{p})$ *is the spin part of* $\mathcal{U}_R$ *which depends upon Euler rotation angles* $\{-\pi, \vartheta, \pi-\varphi\}$, *where* ϑ *and* φ *are spherical angles of the momentum* $\vec{p}$ *in a certain fixed system of axes (the projections* n *are also quantized with respect to these axes).*

Then $$\mathcal{U}_R\, |\vec{p}, m\rangle = \sum_{n',n} |R\vec{p}, n'\rangle\, \mathcal{D}_{n',n}(R)\, \mathcal{D}_{n,m}(\vec{p}) = \sum_{n'} |R\vec{p}, n'\rangle\, \mathcal{D}_{n',m}(R\vec{p}) = |R\vec{p}, m\rangle.$$

References

1. Chou Kuang-chao and M.I. Shirokov , JETF 34, 1230 (1958).
2. Ju.M. Shirokov, JETF 35, 1005 (1958).
3. Ju.M. Shirokov , Dokl. Akad. Nauk USSR 94, 857 (1954). L.L. Foldy. Phys.Rev., 102, 568 (1956).
4. M.H.L. Pryce, Proc . Roy . Soc. Al 95, 82 (1948).
5. Ju.M. Shirokov , JETF, 21, 748 (1951).
6. H.P. Stapp, Phys.Rev., 103, 425 (1957).
7. T.D. Newton and E.P. Wigner, Rev.Mod.Phys., 21, 400 (1949).
8. I.M. Shirokov, JETF, 32, 1022 (1957).
9. M. Jacob and G.C. Wick. Annals of Phys. 7, 404 (1959).
10. Chou Kuang-chao , JETF, 36, 909 (1959).
11. M.I. Shirokov, JETF, 36, 1525 (1959).
12. C. Moller, The Theory of Relativity. Oxford, 1952.
13. Ju. M.Shirokov, JETF, 33, 861 (1957).
14. G.Ja. Lubarsky. The Theory of Groups and its Application in Physics. Moscow, 1957.
15. I.M. Gelfand, R.A. Minlos and Z. Ja. Shapiro, Representations of Rotation Group and the Lorentz Group. Moscow, 1958.
16. Pauli V. General Principles of the wave mechanics. Moscow, 1947, p. 251, Handbuch der Physik, 24,(1933).
17. E. Wigner. Annals of Math. 40, 149 (1939), Section 6C.