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Zygmunt Galasiewicz

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* Permanent adress: Institute of Theoretical Physics, University of Wroclaw, Wroclaw, Poland.

A b s t r a c t

The thermodynamics of the "anomalous" state of a Fermi-system (in which the correlation of the pairs of particles with parallel spins is taken into account) is investigated. The transition temperature T_c to the "anomalous" state is found and the temperature dependence of the specific heat for $T \sim 0$ and $T \sim T_c$. For $T = T_c$ the specific heat has a jump. In addition the formulae for paramagnetic susceptibility for $T \sim 0$ and $T \sim T_c$ are obtained.

1. Introduction

In the paper[1] (referred to as part I) the possibility of an "anomalous" (non - superconducting) state of Fermi-system is studied. This state is connected with creation of pairs with parallel spins. The creation occurs if the electron-electron interaction is attractive; we find that in the expansion of the interaction term in spherical harmonics only the coefficients with odd indices give a contribution.

If in the expansion of the interaction in spherical harmonics retained terms with even indices $\mathcal{J}_{2n}$ only (for instance in the paper[2] - $\mathcal{J}_0$) the compensation equation for the pairs with antiparallel spins has two solutions: one trivial and one nontrivial (giving the superconducting state). However the compensation equation for the pairs with parallel spins has the trivial solution only.

If in the expansion of the interaction in spherical harmonics are retained terms with odd indices $\mathcal{J}_{2n+1}$ (paper[1] - $\mathcal{J}_1$), the compensation equation for the pairs of particles with antiparallel spins has the trivial solution only. On the other hand the compensation equation for pairs with parallel spins has two solutions: trivial and nontrivial (giving the "anomalous" state).

The solutions of the compensation equation and of the secular equation for the collective oscillations are in the case $\mathcal{J}_{2n}$ even functions of $\vec{p}$ [3),4)] (the superconducting state, longitudinal oscillations) in the case $\mathcal{J}_{2n+1}$ odd functions of $\vec{p}$ [1)5)] (the "anomalous" state the oscillations of the type of the transversal oscillations).

In the papers [4),5)] is proved that for the general form of the interaction term the collective oscillations split into two branches: for the pairs of particles with antiparallel spins and for the pairs of particles with parallel spins. The terms of the type $\mathcal{J}_{2n}$ give contribution to the oscillations of the pairs with antiparallel spins only (the solution of the secular equation the even function of $\vec{p}$, therefore the longitudinal oscillations), the terms of the type of $\mathcal{J}_{2n+1}$ give a contribution to the oscillations of the pairs with parallel spins (the solution of the secular equations the odd function of $\vec{p}$ therefore the oscillations of the type of the transversal oscillations). In the paper[1] is shown that if $3\mathcal{J}_1 > \mathcal{J}_0$ the energy of the "anomalous" state is lower than the energy of superconducting state and certain properties of this state are studied (for $T=0$).

Now we want to examine the properties of this state for temperature $T \neq 0$.

1. The compensation equation.

The compensation equation has the form[1)]

$$2\,\xi(p)\,\Phi(\vec{p},\sigma) - \frac{1-2F(\vec{p})}{V}\,C(\vec{p},\sigma) = 0 \tag{1}$$

where

$$C(\vec{p},\sigma) = -\frac{1}{V}\sum_{\vec{p}'} \mathcal{J}(\vec{p},-\vec{p};-\vec{p}',\vec{p}')\,\Phi(\vec{p}',\sigma) \tag{2}$$

and

$$\Phi(\vec{p},\sigma) = u(\vec{p})\,v(\vec{p},\sigma)(1-2n_{\vec{p}\sigma}),$$
$$F(\vec{p}) = v^2(\vec{p},\sigma)(1-n_{\vec{p}\sigma}) + u^2(\vec{p})\,n_{\vec{p}\sigma}, \tag{3}$$

$$\xi(p) = \varepsilon(p) + \frac{1}{V}\sum_{\vec{p}'}\left[2\mathcal{J}(\vec{p},\vec{p}';\vec{p}',\vec{p}) - \mathcal{J}(\vec{p},\vec{p}';\vec{p},\vec{p}')\right] \tag{3a}$$

since the expectation value of $\alpha^+_{\vec{p}\sigma}\alpha_{\vec{p}\sigma}$ is not zero but $n_{\vec{p}\sigma}$[4)]. The functions $\{u,v\}$ are the coefficients of the Bogolubov transformation for transition from the Fermi-operators $a_{\vec{p}\sigma}, a^+_{\vec{p}\sigma}$ to $\alpha_{\vec{p}\sigma}, \alpha^+_{\vec{p}\sigma}$; these $\{u,v\}$ satisfy the condition

$$u^2(\vec{p}) + v^2(\vec{p},\sigma) = 1 \tag{4}$$

Similiary as in the paper[1)] we putt

$$\mathcal{J}(\vec{p},-\vec{p};-\vec{p}',\vec{p}') = \sum_n \mathcal{J}_n(p,p')P_n(\cos\gamma) \cong \mathcal{J}_1(p,p')\cos\gamma = \mathcal{J}_1(p_F)(\cos\theta\cos\theta' + \cos\varphi\sin\theta\sin\theta'), \tag{5}$$

$$C(\vec{p},\sigma) = \sigma\cos\theta\,\psi(p)$$

where θ, θ' are the angles between vectors $\vec{p}$ and $\vec{p}'$ and the polar axis, and φ the angle round this axis.

The equation (2) after taking into account (1),(3),(4) can be write in integral form

$$1 = -\frac{1}{2}\frac{p_F^2}{(2\pi)^2}\mathcal{J}_1 \int_{p_F-\Delta}^{p_F+\Delta}\int_0^{\pi} \frac{\cos^2\theta' \sin\theta' d\theta' dp'}{\Omega(p', \cos\theta')}\left[1 - 2n(p', \cos\theta')\right] \tag{6}$$

where

$$n_{\vec{p}\sigma} = n_{\vec{p}} = \frac{1}{e^{\Omega(\vec{p})/T}+1}, \qquad \Omega(\vec{p}) = \sqrt{\psi^2\cos^2\theta + \xi^2(p)} \tag{7}$$

Hence we finally get for ψ as the function of temperature the equation

$$1 = \varrho_1 \int_0^1 x^2 dx \int_0^{\omega} \frac{\operatorname{th}\frac{\sqrt{\psi^2(T)x^2+\xi^2}}{2T}}{\sqrt{\psi^2(T)x^2+\xi^2}} d\xi =$$

$$= \varrho_1 \int_0^1 x^2 dx \int_{\psi x}^{\omega} \frac{\operatorname{th}\frac{\Omega}{2T}}{\sqrt{\Omega^2 - \psi^2 x^2}} d\Omega \tag{8}$$

where

$$\varrho_1 = -\frac{dn}{dE}\mathcal{J}_1 > 0, \qquad \frac{dn}{dE} = \frac{p_F^2}{2\pi E'} \tag{9}$$

this means we assume that the interaction is attractive.

3. The dependence of ψ on the temperature and the transition temperature to the "anomalous" state.

After a change of variables $\Omega = \psi x \operatorname{ch}\varphi$ we get similiary as in [6]

$$\ln\frac{\psi(0)}{\psi(T)} = 6\int_0^1 x^2 dx \int_0^{\infty} \frac{d\varphi}{e^{\psi(T)x\operatorname{ch}\varphi/T}+1} = 6\int_0^1 x^2 dx \sum_{m/1}^{\infty}(-1)^{m+1} K_0\left(\frac{\psi x m}{T}\right) \tag{10}$$

Hence for $T \sim 0$ making use of the asymptotic formulae for Bessel-functions for great argument we obtain

$$\ln\frac{\psi(0)}{\psi(T)} \cong \sqrt{\frac{18\pi T}{\psi(0)}}\int_0^1 x^{1/2} e^{x\psi(0)/T} dx = 9\sqrt{\frac{\pi}{2}}\,\Phi\left(\sqrt{\frac{\psi(0)}{T}}\right)\left(\frac{T}{\psi(0)}\right)^3 \cong 10\left(\frac{T}{\psi(0)}\right)^3, \tag{11}$$

$$\psi(T) = \psi(0)\, e^{-10(T/\psi(0))^3} \tag{11a}$$

where $\Phi(z)$ is the error function.

Now we want to find the critical temperature T_c (for $T \geqslant T_c$ must be $\psi(T)=0$). Moreover we want to get $\psi(T)$ for $T \sim T_c$. For this purpose we make use of a suitable representation of Bessel function $K_0(z)$ 7),6). We obtain

$$\ln\frac{\psi(0)}{\psi(T)} = \ln\frac{e^{1/3}\pi}{\gamma}\frac{T}{\psi(T)} + 3\int_0^1 x^2 dx\left\{2\pi\sum_{l=1}^{\infty}\left[\frac{1}{(2l-1)\pi} - \frac{1}{\sqrt{\frac{\psi^2x^2}{T^2}+(2l-1)^2\pi^2}}\right]\right\} \tag{12}$$

where $\ln\gamma = 0{,}577$, $\gamma = 1{,}8$.

Putting in (12) $\psi = 0$ we find

$$\psi(0) = 2\omega e^{1/3}\left(e^{-1/g_1}\right)^3 = e^{1/3}\frac{\pi T_c}{\gamma} \tag{13}$$

Hence

$$\frac{\psi(0)}{T_c} = 2{,}45 \tag{13a}$$

For $T \sim T_c$

$$\ln\frac{T}{T_c} = \frac{1}{2}\frac{\zeta(3)}{\pi^2}\left(\frac{\psi(T)}{T}\right)^2 + \frac{5}{16}\frac{\zeta(5)}{\pi^4}\left(\frac{\psi(T)}{T}\right)^4 \tag{14}$$

where $\zeta(n)$ zeta function.

Therefore for $T \sim T_c$

$$\frac{\psi(T)}{T_c} = 4{,}1\sqrt{\frac{T_c - T}{T_c}} \tag{15}$$

From (13a) and (15) results that the function $\psi(T)/T_c$ decreases from 2,45 to 0 in the temperature interval $(0, T_c)$.

4. Entropy and specific heat.

Now we want to obtain the formulae for the temperature dependence of the specific heat, especially for $T \sim T_c$. Therefore we must first get the formula for the entropy.

We start from the expression

$$S = -2\sum_{\vec{p}}\left[n_{\vec{p}}\ln n_{\vec{p}} + (1-n_{\vec{p}})\ln(1-n_{\vec{p}})\right] \tag{16}$$

Putting (7) in (16) and passing to the integral form we obtain

$$s = \frac{S}{V} = -4\,\frac{dn}{dE}\,\frac{1}{T}\int_0^1 dx \int_{\psi x}^{\infty} \sqrt{\Omega^2 - \psi^2 x^2}\;\Omega\,\frac{\partial n}{\partial \Omega}\,d\Omega \quad . \tag{17}$$

After changing variables similiary as for (10) we have

$$s = 4\,\frac{dn}{dE}\,\frac{\psi^2}{T}\int_0^1 x^2 dx \sum_{m/1}^{\infty} (-1)^{m+1} K_2\left(\frac{\psi x m}{T}\right) \quad . \tag{18}$$

Hence for $T \sim 0$, making use for the asymptotic formulae for Bessel-functions for great arguments

$$s = 3\sqrt{2\pi}\,\frac{dn}{dE}\,\tilde{\Phi}\left(\sqrt{\frac{\psi(0)}{T}}\right)\psi(0)\left(\frac{T}{\psi(0)}\right)^2 \cong 6{,}65\,\psi(0)\,\frac{dn}{dE}\left(\frac{T}{\psi(0)}\right)^2 \tag{19}$$

Now we want to find $s(T)$ for $T \sim T_c$. Since for $T \to T_c$, $s(T) \to 0$ we put (17) in the form

$$\begin{aligned} s(\psi) &= s(0) + \frac{s'(0)}{1!}\psi + \frac{s''(0)}{2!}\psi^2 + \frac{s'''(0)}{3!}\psi^3 + \frac{s^{IV}(0)}{4!}\psi^4 = \\ &= \tfrac{2}{3}\pi^2\frac{dn}{dE}T\left[1 - \frac{1}{2\pi^2}\left(\frac{\psi}{T}\right)^2 + \frac{21}{80}\,\frac{\zeta(3)}{\pi^4}\left(\frac{\psi}{T}\right)^4\right] \quad . \end{aligned} \tag{20}$$

Having found $s(T)$ we can obtain the temperature dependence of specific heat $C(T)$ for $T \sim 0$ and $T \sim T_c$. The specific heat in normal state for $T = T_c$ we denote by

$$C_n(T_c) = \tfrac{2}{3}\pi^2\frac{dn}{dE}T_c$$

For temperature heat $T = 0$ we have

$$\frac{C_a(T)}{C_n(T_c)} = \frac{9e^{1/3}}{\gamma\sqrt{2}}\left(\frac{T}{\psi(0)}\right)^2 = 4{,}95\left(\frac{T}{\psi(0)}\right)^2 \tag{21}$$

however for temperatures $T \sim T_c$ $(T < T_c)$

$$\frac{C_a(T)}{C_n(T_c)} = 1{,}83 + 2{,}5\,\frac{T - T_c}{T_c} \tag{22}$$

From (22) we see that for $T = T_c$ the specific heat has a jump since $C_a(T_c)/C_n(T_c) = 1{,}83$.

5. The total momentum of elementary excitations.

Let us consider the total momentum $\vec{P}$ of elementary excitations in the case of macroscopic motion with constant velocity $\vec{u}$

$$\vec{P} = \sum_{\vec{k}} \vec{k}\left(n_{\vec{k}\,\frac{1}{2}} + n_{\vec{k}\,-\frac{1}{2}}\right) \tag{23}$$

Now $n = n(\Omega - \vec{k}\vec{u})$. For small velocities

$$\vec{P} = -2\sum_{\vec{k}} \vec{k}(\vec{k}\vec{u})\,\frac{\partial n}{\partial \Omega(\vec{k})} \tag{24}$$

In the integral form we have finally

$$\vec{p} = \frac{\vec{P}}{V} = -p_F^2\frac{dn}{dE}\int_0^1\left[\vec{u}_\perp(1-x^2)+\vec{u}_\parallel 2x^2\right]dx\int_{-\infty}^{+\infty}\frac{\partial n}{\partial\Omega(x,z)}\,dz \tag{25}$$

where $\vec{u}_\perp$ is the component of vector $\vec{u}$ perpendicular to the quantization axis of electron spin and $\vec{u}_\parallel$ the parallel component of $\vec{u}$. We see from (25) that vectors $\vec{p}$ and $\vec{u}$ are not parallel and we cannot write the dependence $\vec{p} = N_n m\vec{u}$ and define the number N_n of normal electrons in "anomalous" state.

Consider $\vec{p}$ for $T\sim 0$ and for $T\sim T_c$. After changing variables $z = \psi x\,\mathrm{sh}\,\varphi$ we get

$$\vec{p} = p_F^2\frac{dn}{dE}\,\frac{2\psi}{T}\int_0^1\left[\vec{u}_\perp(1-x^2)+\vec{u}_\parallel 2x^2\right]x\,dx\int_0^\infty\frac{\mathrm{ch}\,\varphi\,d\varphi}{\left(e^{\psi x\,\mathrm{ch}\varphi/T}+1\right)\left(e^{-\psi x\,\mathrm{ch}\varphi/T}+1\right)} \tag{26}$$

Hence for $T\sim 0$

$$\vec{p} = \vec{p}_\perp + \vec{p}_\parallel \cong \vec{p}_\perp \tag{27}$$

where

$$\begin{aligned}\vec{p}_\perp &= p_F^2\frac{dn}{dE}\sqrt{\pi}\left(\frac{T}{\psi(0)}\right)^{1/2}\vec{u}_\perp = N\,\tfrac{3}{2}\sqrt{\pi}\left(\frac{T}{\psi(0)}\right)^{1/2} m\,\vec{u}_\perp\,,\\ \vec{p}_\parallel &= \tfrac{3}{4}p_F^2\frac{dn}{dE}\sqrt{\pi}\left(\frac{T}{\psi(0)}\right)^{3/2}\vec{u}_\parallel = N\,\tfrac{9}{2}\sqrt{\pi}\left(\frac{T}{\psi(0)}\right)^{3/2} m\,\vec{u}_\parallel\end{aligned} \tag{28}$$

m is the mass of the electron, N the number of electrons in unite volume

$$N = \frac{2}{3}\frac{p_F^2}{m}\frac{dn}{dE} = \frac{8\pi p_F^2}{3(2\pi)^3} \tag{29}$$

since in the formula (9) $E' \sim p_F/m$.

From (28) we see that the perpendicular component of $\vec{p}$ dominates for $T\sim 0$, it is three orders of magnitude smaller than the parallel component.

In order to obtain (26) for $T\sim T_c$ we must use the following identities

$$\frac{\psi}{T}\int_0^1 x^3dx\int_0^\infty \frac{ch\varphi\, d\varphi}{\left(e^{\psi x\, ch\varphi/T}+1\right)\left(e^{-\psi x\, ch\varphi/T}+1\right)} = \frac{1}{6}\,\frac{\frac{1}{T}\frac{\partial\psi}{\partial T}}{\frac{\partial}{\partial T}\left(\frac{\psi}{T}\right)} \tag{29}$$

and

$$A = \int_0^1 x\,dx\int_0^\infty \frac{ch\varphi\, d\varphi}{\left(e^{\psi x\, ch\varphi/T}+1\right)\left(e^{-\psi x\, ch\varphi/T}+1\right)} = -\frac{\frac{\partial f(T)}{\partial T}}{2\frac{\partial}{\partial T}\left(\frac{\psi}{T}\right)} \tag{30}$$

where $f(T)$

$$f(T) = 2\int_0^1 dx\int_0^\infty \frac{d\varphi}{e^{\psi x\, ch\varphi/T}+1} \tag{31}$$

For $T\sim T_c$

$$f(T) = \ln\left(\frac{e\pi}{\gamma}\frac{T}{\psi}\right) - \frac{7}{24}\frac{\zeta(3)}{\pi^2}\left(\frac{\psi}{T}\right)^2 + \frac{93}{640}\frac{\zeta(5)}{\pi^4}\left(\frac{\psi}{T}\right)^4 \tag{32}$$

Hence A for $T\sim T_c$

$$A(T) = \frac{1}{2}\frac{T}{\psi} + \frac{7}{24}\frac{\zeta(3)}{\pi^2}\left(\frac{\psi}{T}\right) - \frac{93}{320}\frac{\zeta(5)}{\pi^4}\left(\frac{\psi}{T}\right)^3 \tag{33}$$

Finally $\vec{p}$ for $T\sim T_c$ $(T<T_c)$

$$\vec{p} = Nm\left[\vec{u} + \frac{1}{4}\left(11\vec{u}_\perp - 8\vec{u}_\parallel\right)\frac{T_c-T}{T_c}\right] \tag{34}$$

We see that even for $T\sim T_c$ the parallel component of $\vec{p}$ is smaller than the perpendicular one.

6. Paramagnetic susceptibility.

From the paper [1] we have for the paramagnetic susceptibility χ

$$\chi(T) = \frac{2e^2}{m^2}\frac{dn}{dE}\left[1 - \left(\frac{\psi(T)}{2\omega}\right)^2\right] \tag{35}$$

where $\psi(T)$ we get from (10).

For the temperatures $T \sim 0$ we obtain from (11a)

$$\chi(T) = \frac{2e^2}{m^2}\frac{dn}{dE}\left[1 - 1{,}96\, e^{-6/\rho_1} e^{-20(T/\psi(0))^3}\right]. \tag{36}$$

For the temperatures $T \sim T_c$ we get from (15)

$$\chi(T) = \frac{2e^2}{m^2}\frac{dn}{dE}\left[1 - 3{,}25\, e^{-6/\rho_1}\sqrt{\frac{T_c - T}{T_c}}\right]. \tag{37}$$

With the elementary excitations considered here (type J_{2n+1}) we could explain the Reif experiment[8] which gives the dependence of paramagnetic susceptibility, if we put $\rho_1 \sim 3{,}5$.

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