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ON THE ASYMPTOTIC AND CAUSALITY CONDITIONS IN QUANTUM FIELD THEORY II

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The conclusions reached in a recent paper on the connection between the interacting field operator approach to quantum field theory of Lehmann, Symanzik and Zimmermann and the functional derivative approach to quantum field theory of Bogolubov are established in more details. Especially it is shown that Bogolubov's causality condition is a necessary integrability condition for the retarded and advanced solutions of the inhomogeneous Klein-Gordon equation. Associated with this the fact is investigated in some detail (related to Haag's theorem) that the transformation operator which connects the interacting field operator with the incoming or outgoing free field operator can only be unitary up to a positive renormalization constant smaller than one for real interactions. Further we discuss the differences existing between both approaches with respect to the extrapolation of the reduced S-matrix elements off the mass shell. Concluding we show that it is without any importance for the analytic behaviour of the reduced S-matrix elements in the interacting field operator approach considered in the theory of dispersion relations wether the causality condition for the interacting field operator in the commutator form is fulfilled or not.

INTRODUCTION

In a recent paper/1/ we have studied (on a preliminary stage) the connection between the interacting field operator approach to quantum field theory as worked out by Lehmann, Symanzik and Zimmermann/2/ and the functional derivative approach to quantum field theory as proposed by Bogolubov/3/. The equivalence between these two is in general not obvious since both approaches start from the consideration of a general field theory (S-matrix theory) in which are derived reduction formulae for the S-matrix elements using quite different mathematical tools: the former makes use of interacting field operators as retarded or advanced solutions of the inhomogeneous Klein-Gordon equation together with the employment of the asymptotic condition whereas the latter avoids these completely and works only with a given S-matrix as a functional of incoming or outgoing free field operators and its functional derivatives with respect to these free field operators. The causal field theories are then considered as special cases of the general theory restricted by the causality requirement. However, the causality condition assumed in the first approach in the conventional form that the commutator of the interacting field operator is to vanish for space-like distances of the corresponding space-time points is formulated in the second approach as follows: the current operator (2) constructed from the S-matrix and its functional derivative with respect to the incoming or outgoing free field operator is to have the retarded or advanced properties (8), (8') as for its functional derivative with respect to the incoming or outgoing free field operator respectively. The latter condition (in contrast to the former) involves obviously also a time distinghuishing causality condition for time-like distances of the corresponding space-time points which should be required for a definition of 'causality' from a physical point of view. On the other hand, the conventional causality condition for the interacting field operator in the commutator form presumes the existence of the latter as retarded or advanced solutions of the inhomogeneous Klein-Gordon equation, and the explicit absence of a time distinghuishing causality condition for time-like distances might be overcome by the fact that the interacting field operator constructed in the above manner exists only for field theories which are causal in the Bogolubov sense (at least, if*

** In the following quoted as 1.*

we require the commutator condition for the interacting field operator). However, it will be come clear from the investigations of the present paper that really the commutator condition for the interacting field operator – in contrast to Bogolubov's condition – cannot be considered as a condition on the reduced S-matrix elements which is important for their analytic behaviour as studied in the theory of dispersion relations (as it is hitherto generally believed).

The present paper is devoted to a thorough investigation of the conclusions reached in I that 1) the interacting field operator approach holds only for causal field theories (in contrast to the assumption in [2]) and 2) the causality condition for the interacting field operator in the commutator form might not be sufficient for a general approach to quantum field theory as needed in the theory of dispersion relations. First we show that Bogolubov's causality condition is a necessary integrability condition for the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation (section 2). In this sense the interacting field operator approach presumes Bogolubov's causality condition and, if the latter is taken into account, we expect also no principal difficulties for an application of the asymptotic condition (compare also [4] and section 4). In connection with this we discuss in some detail the fact (related to Haag's theorem) that the transformation operator which connects the interacting field operator $\varphi(x)$ with the incoming or outgoing free field operator $\varphi_{in/out}(x)$ can only be unitary up to a positive renormalization constant smaller than one for real interactions of the conventional local type. Its consequences for the commutation relations of the free field operators are pointed out. Further we discuss the differences existing between the interacting field operator approach and the functional derivative approach with respect to the extrapolation of the reduced S-matrix elements off the mass shell (as considered in the theory of dispersion relations; section 3) Concluding we show that it is quite unimportant for the analytic behaviour of the reduced S-matrix elements in the interacting field operator approach considered in the theory of dispersion relations wether the causality condition for the interacting field operator in the commutator form is fulfilled or not (section 4). This means that the commutator condition for the interacting field operator cannot be interpreted as a condition on the reduced S-matrix elements which has some analytic consequences in the theory of dispersion relations but that the causal properties of the interacting field operator are a priori assumed in the derivation of the reduction formulae (obviously by the use of the asymptotic condition). In this sense we have to understand the indication 2 in 1.

2. CAUSALITY CONDITION AS INTEGRABILITY CONDITION OF THE KLEIN-GORDON EQUATION

We show (after some introducing remarks) that Bogolubov's causality condition is a necessary integrability condition for the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation

$$(\Box - m^2)\varphi(x) = j(x) \qquad (1)$$

where /1(12)/

$$j(x) = i S^{+} \frac{\delta S}{\delta \varphi_{in}(x)} = i \frac{\delta S}{\delta \varphi_{out}(x)} S^{+} \qquad (2)$$

is the current operator. The S-matrix is considered as a general operator in the Hilbert space of the incoming or outgoing free particle states and allows therefore the representation /1 (4), 1 (11) /

$$S = \sum_{n=0}^{\infty} \int dx_1 \dots dx_n f(x_1, \dots, x_n) : \varphi_{\substack{in \\ out}}(x_1) \dots \varphi_{\substack{in \\ out}}(x_n) : \qquad (3)$$

with

$$S S^{+} = S^{+} S = 1 \qquad (4)$$

$$(\Box - m^2)\, \varphi_{\substack{in \\ out}}(x) = 0 \quad ; \quad [\varphi_{\substack{in \\ out}}(x), \varphi_{\substack{in \\ out}}(y)] = i\Delta(x-y) \qquad (5)$$

and

$$\varphi_{out}(x) = S^{+} \varphi_{in}(x) S \qquad (6)$$

It is evident (compare also the discussion in /2/) that the functional derivatives in (2) cannot be determined from the S-matrix (3) in a unique manner since the expansion functions $f_n(x_1, \dots, x_n)$ are not completely determined by the expansion (3). The reason is that because of (5) the Fourier transforms of $f_n(x_1, \dots, x_n)$ contribute only on the mass shell in (3) in contrast to the situation in (2), i.e. in general (2) depends on the extrapolation off the mass shell in a completely arbitrary manner*. Of course, as to their contributions on the mass shell the expansion functions $f_n(x_1, \dots, x_n)$ have

* We remark that the performance of the functional derivation of the S-matrix (3) with respect to the free field operators $\varphi_{in}(x)$ or $\varphi_{out}(x)$ in the conventional manner makes a priori necessary an extension in the definition of the S-matrix since no regard is paid at this to the fact that the free field operators have really to satisfy the equations (5). Thus to obtain (2) we have to go beyond the 'physical' S-matrix (3) where the fields $\varphi_{in}(x)$ or $\varphi_{out}(x)$ obey (5) to an 'extrapolated' S-matrix which is a functional of the fields $\varphi_{in}(x)$ or $\varphi_{out}(x)$ considered as arbitrary classical functions (see also /3/, especially the footnote on p. 180 in the German translation). It is obvious that such an extrapolation is arbitrary since the expansion functions $f_n(x_1, \dots, x_n)$ are not uniquely determined by (3) and (5) and the same is then true for the result of the functional derivation which finally is considered as a functional of the field operators $\varphi_{in}(x)$ or $\varphi_{out}(x)$ obeying (5) and ordered in normal product form.

to fulfil some conditions, especially the unitarity condition (4) and the requirement of invariance with respect to the inhomogeneous Lorentz group. Beyond it in a causal field theory they have to obey the causality condition which will be discussed in the following. Generally they may be expressed by vacuum expectation values of the functional derivatives of the S-matrix and if we are only interested in matrix elements of S between states where all momenta of the outgoing particles differ from those of the incoming (which we also assume for (9), (10) and (10')) we may use the simple representation

$$f_n(x_1,\dots,x_n)=\frac{1}{n!}\langle 0|\frac{\delta^n S}{\delta\varphi_{\substack{in\\out}}(x_1)\dots\delta\varphi_{\substack{in\\out}}(x_n)}|0\rangle$$

$$=\frac{1}{n!}\langle 0|S^+\frac{\delta^n S}{\delta\varphi_{\substack{in\\out}}(x_1)\dots\delta\varphi_{\substack{in\\out}}(x_n)}|0\rangle=\frac{1}{n!}\langle 0|\frac{\delta^n S}{\delta\varphi_{out}(x_1)\dots\delta\varphi_{out}(x_n)}S^+|0\rangle \qquad (7)$$

(7) may easily be checked on the ground of the expansion (3). In the second line of (7) we have made use of the stability of the vacuum (putting the arbitrary phase factor equal to unity).

If we define a causal field theory by / 1 (17), 1 (17') /

$$\frac{\delta j(x)}{\delta\varphi_{in}(y)}=0 \qquad \text{if } y \gtrsim x \qquad (8)$$

$$\frac{\delta j(x)}{\delta\varphi_{out}}=0 \qquad \text{if } y \lesssim x \qquad (8')$$

and use the definition (2) then it is possible to write (7) in the form

$$f_n(x_1,\dots,x_n)=\frac{(-i)^n}{n!}\langle 0|Tj(x_1)\dots j(x_n)|0\rangle \qquad (9)$$

(compare 1 (34) /. Since (8), (8') involves no conditions for $x_0=y_0$ the expressions (9),*

* *This formula, however, contains a misprint: the first expression is missing a minus sign.*

(11) and (11') are only determined up to contributions arising from quasilocal operators. We remark*

* *For the definition of the latter see/5/; they lead to terms of the form*

$$P\left(\frac{\partial}{\partial x_{i,\mu}}\right)\delta(x_1-x_2)\dots\delta(x_1-x_n)$$

where $P\left(\frac{\partial}{\partial x_{i,\mu}}\right)$ is a covariant polynomial with respect to $\frac{\partial}{\partial x_{i,\mu}}$ with constant coefficients.

the important point (compare also the discussion in/2/) that (8), (8') is in general a condition for the Fourier transforms of the expansion functions $f_n(x_1,\dots,x_n)$ on the mass shell in (3) as well as on their extrapolation off the mass shell in (2). However, the last cannot be of principal importance in an S-matrix theory: every causality condition is equivalent to (8), (8') provided it has the same consequences for the 'physical' S-matrix (3), i.e. for the contributions of the expansion functions $f_n(x_1,\dots,x_n)$ on the mass shell.

If one is concerned only with the case of two incoming or two outgoing particles in the S-matrix elements then the expansion functions $f_n(x_1,\dots,x_n)$ can be represented as vacuum expectation values of advanced or retarded products of the current operator respectively

$$f_n^{(i)}(x_1,\dots,x_n)=\frac{-i}{n!}\langle 0|A(x_1;x_2,\dots,x_n)|0\rangle \qquad (10)$$

$$f_n^{(f)}(x_1,\dots,x_n) = \frac{-i}{n!} < 0 / R(x_1; x_2, \dots x_n) / 0 > \qquad (10')$$

where (compare also/2/ (27))

$$A(x_1\, x_2,\dots,x_n) = \frac{\delta^{n-1} j(x_1)}{\delta\varphi_{out}(x_2)\dots\delta\varphi_{out}(x_n)} = (-i)^{n-1} \sum \Theta(x_n - x_{n-1})\dots\Theta(x_2 - x_1)\cdot \cdot [j(x_n)\dots[j(x_2), j(x_1)]\dots] . \qquad (11)$$

$$R(x_1\, x_2,\dots,x_n) = \frac{\delta^{n-1} j(x_1)}{\delta\varphi_{in}(x_2)\dots\delta\varphi_{in}(x_n)} = (i)^{n-1} \sum \Theta(x_1 - x_2)\dots\Theta(x_{n-1} - x_n)[\dots[j(x_1), j(x_2)]\dots j(x_n)] \qquad (11')$$

(10) corresponds to the case of two incoming particles and (10') to that of two outgoing particles and the summation in (11) and (11') is taken over all permutations of the (n − 1) coordinates $x_2, \dots, x_n$ /, These formulae may be easily proved by the performances made in sections 2 (relations (80), (81)) together with an employment of the causality condition (8') or (8) in (11) or (11') respectively.

Now we proceed to the discussion of the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation (1). We write them in the form

$$\varphi(x) = \varphi_t(x) - \int_t^{\infty} \Delta_{ret}(x-y) j(y)\, dy \quad \text{for } x_0 > t \qquad (12)$$

$$\varphi(x) = \varphi_t(x) - \int_{-\infty}^{t} \Delta_{adv}(x-y) j(y)\, dy \quad \text{for } x_0 < t \qquad (12')$$

where $\varphi_t(x)$ is the solution of the homogeneous Klein-Gordon equation

$$(\Box - m^2)\, \varphi_t(x) = 0 \qquad (13)$$

which coincides with $\varphi(x)$ for $x_0 = t$

$$\varphi(x) = \varphi_t(x) \qquad \text{for } x_0 = t \qquad (14)$$

We shall be especially interested in the limits $t \to -\infty$ in (12) and $t \to +\infty$ in (12')

$$\varphi(x) = \varphi_{in}(x) - \int_{-\infty}^{+\infty} \Delta_{ret}(x-y) j(y)\, dy, \quad \varphi_{in}(x) = \lim_{t \to -\infty} \varphi_t(x) \qquad (15)$$

$$\varphi(x) = \varphi_{out}(x) - \int_{-\infty}^{+\infty} \Delta_{adv}(x-y) j(y)\, dy, \quad \varphi_{out}(x) = \lim_{t \to +\infty} \varphi_t(x) \qquad (15')$$

which, of course, make necessary the implication of an adiabatic conception. On the other hand, we can make use of the invariance properties of the wave equation (1) assuming that the extrapolation off the mass shell in (2) is performed in a Lorentz invariant (and, of course, finite) manner. Expecially from the translation invariance follows the existence of the energy-momentum operator P_μ as a displacement operator in the sense

$$\frac{\partial}{\partial x_\mu} \varphi(x) = i\left[P_\mu, \varphi(x)\right], \quad \left[P_\mu, P_\nu\right] = 0 \qquad (16)$$

so that there exists with respect to the time-coordinate the relation

$$\varphi(x) = e^{iP_0(x_0 - x_0')} \varphi(\vec{x}, x_0') e^{-iP_0(x_0 - x_0')}. \qquad (17)$$

For the free field operator (13) we have in a quite analogous manner

$$\frac{\partial}{\partial x_\mu} \varphi_t(x) = i\left[P^\circ_\mu(t), \varphi_t(x)\right], \left[P^\circ_\mu(t), P^\circ_\nu(t)\right] = 0 \qquad (18)$$

where $P^\circ_\mu(t)$ is the energy-momentum operator of a free particle system corresponding to the initial condition at time t and expecially it is

$$\varphi_t(x) = e^{iP^\circ_0(t)(x_0 - x_0')} \varphi_t(\vec{x}, x_0') e^{-iP^\circ_0(t)(x_0 - x_0')}. \qquad (19)$$

For $t \to \mp\infty$ (18) and (19) go over to the corresponding relations for the incoming or outgoing fields respectively (compare (15), (15')), for instance,

$$\varphi_{\substack{in\\out}}(x) = e^{iP^\circ_{0,\substack{in\\out}}(x_0 - x_0')} \varphi_{\substack{in\\out}}(\vec{x}, x_0') e^{-iP^\circ_{0,\substack{in\\out}}(x_0 - x_0')}, \quad P^\circ_{0,\substack{in\\out}} = \lim_{t \to \mp\infty} P^\circ_0(t) \qquad (19')$$

For the following we assume, for simplicity, the equality between the eigenvalues of P_μ and $P^\circ_\mu(t)$ *

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* Thereby assuming that no bound states appear (for the possibility of a conventional treatment of the latter case combined with a modified adiabatic conception see /6/).

$$P_\mu \Psi_n = P_\mu^{(n)} \Psi_n \qquad (20)$$

$$P^\circ_\mu(t) \Psi^\circ_n(t) = P_\mu^{(n)} \Psi^\circ_n(t) \qquad (21)$$

and require the existence of a state with lowest energy-eigenvalue, the vacuum Ψ_0 or $\Psi^\circ_0(t)$ respectively ($P^{(0)}_0 = 0$). Equation (21) is indeed possible for arbitrary t because there exists a unitary transformation between $P^\circ_\mu(t)$ and $P^\circ_\mu(t')$ which will be shown in the following (see (37)). (20) and (21) together imply the existence of a unitary transformation which connects Ψ_n with $\Psi_n^\circ(t)$

$$\Psi_n = \mathcal{U}(t)\,\Psi_n^\circ(t), \qquad \mathcal{U}(t)\,\mathcal{U}^+(t) = \mathcal{U}^+(t)\,\mathcal{U}(t) = 1. \tag{22}$$

The operator $\mathcal{U}(t)$ which operates in the Hilbert space of the free particle states $\Psi_n^\circ(t)$ may be represented as an expansion with respect to the normal products of the free field operator $\varphi_t(x)$ (compare (3))

$$\mathcal{U}(t) = \sum_{n=0}^{\infty} \int dx_1 \ldots dx_n \; g_n(x_1,\ldots,x_n;t) : \varphi_t(x_1)\ldots\varphi_t(x_n): \tag{23}$$

If one takes into account the well known fact that the free particle states $\Psi_n^\circ(t)$ may be built up from the vacuum state $\Psi_0^\circ(t)$ by means of repeated application of the creation operator

$$a_t^*(\vec{q}) = \frac{i}{(2\pi)^{3/2}} \int d\vec{x} \left\{ \varphi_t(x) \frac{\partial}{\partial x_0} \frac{e^{-iqx}}{\sqrt{2q_0}} - \frac{\partial}{\partial x_0} \varphi_t(x) \frac{e^{-iqx}}{\sqrt{2q_0}} \right\} \tag{24}$$

where (compare 1 (6), 1 (7))

$$\varphi_t(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{q}}{\sqrt{2q_0}} \left\{ a_t(\vec{q})\,e^{-iqx} + a_t^*(\vec{q})\,e^{iqx} \right\}, \quad q_0 = +\sqrt{m^2 + \vec{q}^{\,2}} \tag{25}$$

and

$$[a_t(\vec{q}), a_t^*(\vec{q}')] = \delta(\vec{q} - \vec{q}'), \quad [a_t(\vec{q}), a_t(\vec{q}')] = 0$$

$$a_t(\vec{q})\,\Psi_0^\circ(t) = 0 \tag{26}$$

it is obvious in view of (23) that the left expression (22) represents then the expansion of the states Ψ_n with respect to the free particle states $\Psi_n^\circ(t)$. For $t \to \mp\infty$ we get from the above relations the corresponding for the incoming or outgoing free field operators. We remark already the important fact (see appendix) that the operators $P^\circ_{0,in}$ and $P^\circ_{0,out}$ have to be identified with itself

$$P^\circ_{0,in} = P^\circ_{0,out} = P_0 \tag{27}$$

so that we have (up to some phase factor)

$$\mathcal{U}_{in} = \mathcal{U}_{out} = 1. \tag{28}$$

Now we perform (17) with $x_0' = t$ in the following way

$$\varphi(x) = \mathcal{U}^+(x_0,t)\,e^{iP_0^\circ(t)(x_0-t)}\,\varphi(\vec{x},t)\,e^{-iP_0^\circ(t)(x_0-t)}\,\mathcal{U}(x_0,t) \tag{29}$$

with

$$\mathcal{U}\,(x_0,t) = e^{iP_0^0(t)(x_0-t)}e^{-iP_0(x_0-t)} \tag{30}$$

where obviously

$$\mathcal{U}\,(x_0,t)\,\mathcal{U}^{+}(x_0,t) = \mathcal{U}^{+}(x_0,t)\,\mathcal{U}\,(x_0,t) = 1\,. \tag{31}$$

Using (14) and (19) we may write (29) in the form

$$\varphi\,(x) = \mathcal{U}^{+}(x_0,t)\,\varphi_t\,(x)\,\mathcal{U}\,(x_0,t) = \varphi_t(x) + \mathcal{U}^{+}(x_0,t)\big[\varphi_t\,(x),\; \mathcal{U}\,(x_0,t)\big]. \tag{32}$$

From (30) it follows

$$e^{-iP_0^0(t)(x_0-t)}\,\mathcal{U}\,(x_0,t) = \mathcal{U}^{+}(t,x_0)\,e^{-iP_0^0(x_0)(x_0-t)}\left(= e^{-iP_0(x_0-t)}\right) \tag{33}$$

and from this using (31)

$$P_0^0\,(x_0) = \mathcal{U}\,(t,x_0)\,P_0^0\,(t)\,\mathcal{U}\,(x_0,t). \tag{34}$$

Interchanging x_0 *and* t *in (33) yields on the other hand*

$$P_0^0\,(t) = \mathcal{U}\,(x_0,t)\,P_0^0\,(x_0)\,\mathcal{U}\,(t,x_0) \tag{35}$$

(34) and (35) together with (31) imply

$$\mathcal{U}^{+}(x_0,t) = \mathcal{U}\,(t,x_0) \qquad \mathcal{U}^{+}(t,x_0) = \mathcal{U}\,(x_0,t). \tag{36}$$

Thus (34) may also be written in the form

$$P_0^0\,(x_0) = \mathcal{U}^{+}(x_0,t)\,P_0^0\,(t)\,\mathcal{U}\,(x_0,t) \tag{37}$$

which gives the connection for the equation (21) for different t.

Using (19), (14), (32), (37) and (31) we may now further conclude

$$\begin{aligned} \varphi_t\,(x) &= e^{iP_0^0(t)(x_0-t)}\,\varphi_t\,(\vec{x},t)\,e^{-iP_0^0(t)(x_0-t)} \\ &= e^{iP_0^0(t)(x_0-t)}\,\mathcal{U}^{+}(t,t')\,\varphi_{t'}(\vec{x},t)\,\mathcal{U}\,(t,t')\,e^{-iP_0^0(t)(x_0-t)} \\ &= \mathcal{U}^{+}(t,t')\,e^{iP_0^0(t')(x_0-t)}\,\varphi_{t'}\,(\vec{x},t)\,e^{-iP_0^0(t')(x_0-t)}\,\mathcal{U}\,(t,t') \\ &= \mathcal{U}^{+}(t,t')\,\varphi_{t'}\,(x)\,\mathcal{U}\,(t,t') \end{aligned} \tag{38}$$

and applying this formula a second time we find

$$\mathcal{U}(t,t'') = \mathcal{U}(t',t'')\,\mathcal{U}(t,t'). \tag{39}$$

Using the well known addition theorems for exponential operators we may write (30) in the form

$$\mathcal{U}(x_0,t) = \mathcal{U}_t(x_0,t) = \exp_{+}\left\{-i\int_t^{t_0} P_0^i(t',t)\,dt'\right\} \tag{40}$$

where

$$P_0^i(t',t) = e^{iP_0^\circ(t)(t'-t)}\,P_0^i(t)\,e^{-iP_0^\circ(t)(t'-t)}, \quad P_0^i(t) = P_0 - P_0^\circ(t). \tag{41}$$

The $+$-symbol prescribes the chronological ordering of the operators in an expansion with respect to $P_0^i(t',t)$. The index t in $\mathcal{U}_t(x_0,t)$ is in the following to relate to the t -dependence of $P_0^i(t',t)$ and the argument t to that of the integral limit (see also (43)). In general $\mathcal{U}_t(x_0,t)$ is the solution of the differential equations

$$\frac{\partial\,\mathcal{U}_t(x_0,t)}{\partial x_0} = -i\,P_0^i(x_0,t)\,\mathcal{U}_t(x_0,t), \quad \frac{\partial\,\mathcal{U}_t(x_0,t)}{\partial t} = i\,\mathcal{U}_t(x_0,t)\,P_0^i(t) \tag{42}$$

with the initial condition

$$\mathcal{U}_t(x_0,t) = 1 \qquad \text{for } x_0 = t \tag{42'}$$

or of the integral equations

$$\mathcal{U}_t(x_0,t) = 1 - i\int_t^{x_0} P_0^i(t',t)\,\mathcal{U}_t(t',t)\,dt' = 1 + i\int_{x_0}^{t}\mathcal{U}_t(x_0,t')\,P_0^i(t',t)\,dt'. \tag{43}$$

In (42) only the differentiation with respect to the argument t is meant (compare (43)). The unitarity relation (31) now reads

$$\mathcal{U}_t(x_0,t)\,\mathcal{U}_t^{+}(x_0,t) = \mathcal{U}_t^{+}(x_0,t)\,\mathcal{U}_t(x_0,t) = 1 \tag{44}$$

from which in connection with (43) the important relation follows

$$\mathcal{U}_t(x_0,t) = \mathcal{U}_t^{+}(x_0,t)\,\mathcal{U}_t(x_0,t)\,\mathcal{U}_t(x_0,t) = \mathcal{U}_{x_0}(x_0,t) \tag{45}$$

because of

$$\begin{aligned}\mathcal{U}_t^{+}(x_0,t)\,P_0^i(t',t)\,\mathcal{U}_t(x_0,t) &= \mathcal{U}^{+}(x_0,t)\,e^{iP_0^\circ(t)(t'-t)}\,P_0^i(t)\,e^{-iP_0^\circ(t)(t'-t)}\,\mathcal{U}(x_0,t)\\ &= \mathcal{U}^{+}(x_0,t)\,\mathcal{U}^{+}(t,t')\,P_0\,\mathcal{U}(t,t')\,\mathcal{U}(x_0,t) - P_0^\circ(x_0) = \mathcal{U}^{+}(x_0,t')\,P_0\,\mathcal{U}(x_0,t') - P_0^\circ(x_0)\\ &= e^{iP_0^\circ(x_0)(t'-x_0)}\,P_0^i(x_0)\,e^{-iP_0^\circ(x_0)(t'-x_0)} = P_0^i(t',x_0)\end{aligned} \tag{46}$$

where we have made use of the first equation (40), (41), (30), (36), (37) and (39). Using (45) we may now conclude for $\mathcal{U}_t(x_0,t)$ *from (36)**

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* *The relation* $\mathcal{U}_t^+(x_0,t)=\mathcal{U}_t(t,x_0)$ *may also be concluded from (43).*

$$\mathcal{U}_t^+(x_0,t)=\mathcal{U}_{x_0}(t,x_0)=\mathcal{U}_t(t,x_0) \tag{47}$$

and from (39)

$$\begin{aligned}\mathcal{U}_t(x_0,t) &= \mathcal{U}_t(t',t)\,\mathcal{U}_{t'}(x_0,t')=\mathcal{U}_{t'}(t',t)\,\mathcal{U}_{t'}(x_0,t')\\ &= \mathcal{U}_t(x_0,t')\,\mathcal{U}_{t'}(t',t)=\mathcal{U}_t(x_0,t')\,\mathcal{U}_t(t',t)\end{aligned} \tag{48}$$

where we used in (48) in addition (44) together with (47) and

$$\mathcal{U}_{t'}^+(t,t')\,\mathcal{U}_{t'}(x_0,t')\,\mathcal{U}_{t'}(t,t')=\mathcal{U}_t(x_0,t') \tag{49}$$

which is obvious in view of (43), (44) and (46).

Now going to the limits $t\to\mp\infty$ *in (32) we get**

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* *See for this also the discussion at the end of this section. We remark that because of (45) we may perform the limit* $t\to\mp\infty$ *in (32) also in such a way that* $\mathcal{U}_{\substack{in\\out}}(x_0,\mp\infty)$ *is replaced by* $\mathcal{U}_{x_0}(x_0,\mp\infty)$ *in (50).*

$$\begin{aligned}\varphi(x) &= \mathcal{U}_{\substack{in\\out}}(x_0,\mp\infty)\,\varphi_{\substack{in\\out}}(x)\,\mathcal{U}_{\substack{in\\out}}(x_0,\mp\infty)\\ &= \varphi_{\substack{in\\out}}(x)+\mathcal{U}^+_{\substack{in\\out}}(x_0,\mp\infty)\left[\varphi_{\substack{in\\out}}(x),\,\mathcal{U}_{\substack{in\\out}}(x_0,\mp\infty)\right]\end{aligned} \tag{50}$$

with (compare (19') and (40), (41))

$$\varphi_{\substack{in\\out}}(x)=\lim_{t\to\mp\infty} e^{iP_0^0(t)(x-t)}\,\varphi(\vec{x},t)\,e^{-iP_0^0(t)(x_0-t)} \tag{51}$$

$$\mathcal{U}_{\substack{in\\out}}(x_0,\mp\infty)=\lim_{t\to\mp\infty}\mathcal{U}_t(x_0,t)=\exp_+\left\{-i\int_{\mp\infty}^{x_0}P^i_{0,\substack{in\\out}}(t')\,dt'\right\},\quad P^i_{0,\substack{in\\out}}(t')=\lim_{t\to\mp\infty}P_0^i(t',t) \tag{52}$$

It is easy to see that in such cases where $P_0^i(t)$ *depends only on the field operator* $\varphi_t(\vec{x},t)=\varphi(\vec{x},t)$ *and not on its time-derivatives* $P^i_{0,\substack{in\\out}}(t')$ *is nothing else but* $P_0^i(t)$ *expressed by the incoming or outgoing fields at time* t'. *In general (52) allows expansions with respect to normal products of incoming and outgoing free field operators respectively (compare (3) or (23)) so that, using a performance as in 1(21), we may write (50) in the form*

$$\varphi(x) = \varphi_{\substack{in \\ out}}(x) + i\int dy\, \Delta(x-y)\, \overset{+}{U}_{\substack{in \\ out}}(x_0, \mp\infty)\, \frac{\delta U_{\substack{in \\ out}}(x_0, \mp\infty)}{\delta \varphi_{\substack{in \\ out}}(y)}\,, \tag{53}$$

or

$$\varphi(x) = \varphi_{in}(x) + i\int dy\, \Delta(x-y)\, U^{+}_{in}(x_0, \mp\infty)\, \frac{\delta U_{in}(x_0, -\infty)}{\delta \varphi_{in}(y)} \tag{54}$$

$$\varphi(x) = \varphi_{out}(x) - i\int dy\, \Delta(x-y)\, \frac{\delta U_{out}(+\infty, x_0)}{\delta \varphi_{out}(y)}\, U^{+}_{out}(+\infty, x_0). \tag{54'}$$

Equation (54') which is more appropriate for the following results if one starts (using (47)) from the Hermitian conjugated equation (29).

The comparison of (54), (54') with (15), (15') requires necessarily the retarded or advanced properties

$$\frac{\delta U_{in}(x_0, -\infty)}{\delta \varphi_{in}(y)} = 0 \qquad \text{if} \quad y_0 > x_0 \tag{55}$$

$$\frac{\delta U_{out}(+\infty, x_0)}{\delta \varphi_{out}(y)} \qquad \text{if} \quad y_0 < x_0 \tag{55'}$$

We shall call (55), (55') the 'proper causality condition' which, from the mathematical point of view, has the meaning of a necessary integrability condition for the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation.

From (55), (55') it follows further using the definition (2)

$$i\, \overset{+}{U}_{in}(x_0, -\infty)\, \frac{\delta U_{in}(x_0, -\infty)}{\delta \varphi_{in}(y)} = i\,\Theta(x-y)\, S^{+} \frac{\delta S}{\delta \varphi_{in}(y)} = \Theta(x-y)\, j(y) \tag{56}$$

$$i\, \frac{\delta U_{out}(+\infty, x_0)}{\delta \varphi_{out}(y)}\, \overset{+}{U}_{out}(+\infty, x_0) = i\,\Theta(y-x)\, \frac{\delta S}{\delta \varphi_{out}(y)}\, S^{+} = \Theta(y-x)\, j(y) \tag{56'}$$

with (compare (48) which we also use for infinite t *)

$$S = S_{\substack{in \\ out}} = U_{\substack{in \\ out}}(+\infty, -\infty) = U_{\substack{in \\ out}}(+\infty, x_0)\, U_{\substack{in \\ out}}(x_0, -\infty) \tag{57}$$

where, of course, it is $S = S_{in} = S_{out}$ *according to (49) and (4) (compare also (3)). For we have (using the unitarity relation (44) also for infinite* t *)

* *See for this the discussion at the end of this section.*

$$j(y)=iS^{+}\frac{\delta S}{\delta\varphi_{in}(y)}=i\,U^{+}_{in}(x_0,-\infty)\,U^{+}_{in}(+\infty,x_0)\,U_{in}(+\infty,x_0)\,\frac{\delta U_{in}(x_0,-\infty)}{\delta\varphi_{in}(y)}$$
$$=i\,U^{+}_{in}(x_0,-\infty)\,\frac{\delta U_{in}(x_0,-\infty)}{\delta\varphi_{in}(y)}\qquad \text{for } y<x_0\,; \tag{58}$$

$$j(y)=i\frac{\delta S}{\delta\varphi_{out}(y)}S^{+}=i\,\frac{\delta U_{out}(+\infty,x_0)}{\delta\varphi_{out}(y)}\,U_{out}(x_0,-\infty)\,U^{+}_{out}(x_0,-\infty)\,U^{+}_{out}(+\infty,x_0)$$
$$=i\,\frac{\delta U_{out}(+\infty,x_0)}{\delta\varphi_{out}(y)}\,U^{+}_{out}(+\infty,x_0)\qquad \text{for } y_0>x_0 \tag{58'}$$

Here we have also made use of the fact that $U_{in}(+\infty,x_0)$ cannot depend on $\varphi_{in}(y)$ for $y_0<x_0$ because of (55') and $U_{out}(x_0,-\infty)$ not on $\varphi_{out}(y)$ for $y_0>x_0$ because of (55) (we remark that, for instance $U_{out}(x_0,-\infty)=S^{+}U_{in}(x_0-\infty)$ according to (49) or to (6) and an expansion with respect to normal products of incoming or outgoing fields respectively; from the latter the above statements follow immediately). The Θ -functions in (56), (56') follow from the proper causality condition (55) (55').) .*

* *The equality between the expressions on the left-hand-side and on the right-hand-side of (56), (56') may also be concluded by direct comparison between (54), (54') and (15), (15') and the definition (2) of the current operator is obvious in view of 1 (21).*

Now we get from (56), (56') using the proper causality condition (55), (55')

$$\frac{\delta j(y)}{\delta\varphi_{in}(z)}=0 \qquad \text{if} \quad z_0>x_0>y_0 \tag{59}$$

$$\frac{\delta j(y)}{\delta\varphi_{out}(z)}=0 \qquad \text{if} \quad z_0<x_0<y_0 \tag{59'}$$

From (59), (59') we may now conclude Bogolubov's causality condition (8), (8') since we may choose x_0 very close to y_0 in*

* *Write here only y instead of x and z instead of y*

(59), (59'). The symbol $\sim$ in (8), (8') in addition is a simple consequence from the requirement of covariance. Thus we have shown that the causality condition (8), (8') is a necessary integrability condition for the retarded or advanced solutions of the inhomogeneous Klein-Gordon equation (15),(15') to which the relations (54), (54') are equal under the assumption (55), (55') according to (56), (56'). From these considerations it is also very obvious that we have no prescription for $x_0=y_0$ in (8), (8') which leads to the possibility of adding quasilocal operators to (8), (8').

The above considerations can be generalized to the case where translation invariance need not be assumed (i.e. to the case of open systems). We start from the relations (12), (12') where we quantize the free field operator $\varphi_t(x)$ according to (compare (25), (26))

$$[\varphi_t(x),\varphi_t(y)]=i\Delta(x-y) \tag{60}$$

where the right-hand-side is independent on t, i.e. on the special initial condition (that is a well known property of the solutions of the homogeneous Klein-Gordon equation; compare[7] (22)). But from (60) the existence of a unitary transformation follows, such that

$$\varphi_t(x) = \mathcal{U}^+(t,t')\,\varphi_{t'}(x)\,\mathcal{U}(t,t') \qquad (61)$$

$$\mathcal{U}(t,t')\,\mathcal{U}^+(t,t') = \mathcal{U}^+(t,t')\,\mathcal{U}(t,t') = 1. \qquad (62)$$

According to (14) we get from (61) for $x_0 = t$ writing t instead of t'

$$\varphi(x) = \mathcal{U}^+(x_0,t)\,\varphi_t(x)\,\mathcal{U}(x_0,t). \qquad (63)$$

Thus we arrived at relations of the type (38) and (32) which may be handled as above.

We remark that the causality condition for the interacting field operator

$$[\varphi(x), \varphi(y)] = 0 \quad if \quad x \sim y \qquad (64)$$

is then fulfilled in a trivial manner since we have according to (63), (62) and (60)

$$[\varphi(x), \varphi(y)]_{x_0=y_0} = \mathcal{U}^+(x_0,t)\,[\varphi_t(x), \varphi_t(y)]\,\mathcal{U}(x_0,t)\big/_{x_0=y_0} = 0 \qquad (65)$$

and for the reason of covariance (65) has to hold also for $x \sim y$.

For the above considerations it was assumed that the operator $\mathcal{U}_t(x_0,t)$ (or $\mathcal{U}_{x_0}(x_0,t)$) has a well-defined limit for $t \to \mp\infty$ and that also the unitarity condition (44) holds in these limits (it is obvious that (44) must be right for finite t and x_0 in view of (30) and (40)). However, as it is well known, in going to the limits $t \to \mp\infty$ we have to adopt a special adiabatic conception (in order to get mathematically well-defined results) and the limiting process may influence the unitarity property for the operators (52)* . That this is indeed the case may be seen as follows.

*We do not discuss here the situation where bound states have to be considered (see[6]).

We consider first the operator (compare, (40), (41) and (30))

$$\mathcal{U}_0(x_0,t) = exp_+\left\{-i\int_t^{x_0} P_0^i(t',0)\,dt'\right\} = e^{iP_0^0(0)x_0}\, e^{-iP_0(x_0-t)}\, e^{-iP_0^0(0)t} \qquad (66)$$

where

$$P_0^i(t',0) = e^{iP_0^0(0)t'}\, P_0^i(0)\, e^{-iP_0^0(0)t'}, \quad P_0^i(0) = P_0 - P_0^0(0). \qquad (67)$$

It is obvious in view of (66) that $\mathcal{U}_0(x_0,t)$ is unitary for finite t and x_0 . Now we introduce explicitly the adiabatic conception in the form

$$P_0^i(t',0) \to e^{-\varepsilon|t'|} P_0^i(t',0) \tag{68}$$

where $e^{-\varepsilon|t'|}$ is a damping factor in the sense that, after the calculation is performed, the limit $\varepsilon \to 0$ is to be taken*.

---

*For another possibility of defining the limiting process see[8]

Then we get from the right equation (43)

$$\mathcal{U}_0(x_0,t) = 1 + i\int_{x_0}^{t} \mathcal{U}_0(x_0,t')\, P_0^i(t',0)\, dt' \tag{69}$$

and from this putting $x_0 = 0$ and $t = \mp\infty$

$$\mathcal{U}_0(0,\mp\infty)\,\psi_n^\circ(0) = \left\{1 + i\int_0^{\mp\infty} dt'\, e^{-\varepsilon|t'|}\, e^{i(P_0 - P_0^{(n)})t'} P_0^i(0)\right\}\psi_n^\circ(0) \tag{70}$$

$$= \left\{1 + \frac{1}{P_0^{(n)} - P_0 \pm i\varepsilon}\, P_0^i(0)\right\}\psi_n^\circ(0)$$

where $\psi_n^\circ(0)$ is given by (21) and we have made use of the relation (compare (66), (67)).

$$\mathcal{U}_0(0,t')\, P_0^i(t',0) = e^{iP_0 t'}\, P_0^i(0)\, e^{-iP_0^\circ(0)t'} \tag{71}$$

From (70) we conclude using the right equation (67)

$$(P_0 - P_0^{(n)})\,\mathcal{U}_0(0,\mp\infty)\,\psi_n^\circ(0) = 0 \tag{72}$$

i.e. the states

$$\psi_n^{(\pm)} = \mathcal{U}_0(0,\mp\infty)\,\psi_n^\circ(0) \tag{73}$$

are eigenstates of the energy operator P_0 of the total system (compare (20)) corresponding to the free particle situation $\psi_n^\circ(0)$ at $t = \mp\infty$.

Now we show that either $\mathcal{U}_0(0,\mp\infty)$ is not unitary or there is no interaction between two particles. The proof is in two parts: first we show that a unitary $\mathcal{U}_0(0,\mp\infty)$ cannot have any influence on the stable states of the system and then conclude from this that there cannot be any interaction between two particles.

For the first part we choose $\psi_n^\circ(0)$ as a stable state $\psi_s^\circ(0)$ (vacuum or one-particle state) and get from (70)

$$\psi_s = \mathcal{U}_0(0,\mp\infty)\,\psi_s^\circ(0) = \mathcal{U}_0\,\psi_s^\circ(0) = \left\{1 + \frac{1}{P_0^{(s)} - P_0}\, P_0^i(0)\right\}\psi_s^\circ(0) \tag{74}$$

since the fixing of the singularity is unnecessary in this case. However, we have

$$\frac{1}{P_0^{(s)} - P_0}\, P_0^i(0)\,\psi_s^\circ(0) \approx \Lambda\,\psi_s \tag{75}$$

where the operator Λ *projects off the state* $\psi_S^{\,o}(0)$ *and in view of (74) it is then clear that a unitary operator* U_o *cannot have any influence on the stable states* $\psi_S^{\,o}(0)$ *(otherwise: if* $\psi_S^{\,o}(0)$ *is a normalized state, the state* ψ_S *cannot be a normalized one). To prove (75) we proceed as follows: from (compare (20), (21))*

$$P_o \psi_S = P_o^{(s)} \psi_S \qquad P_o^{\,o}(0)\,\psi_S^{\,o}(0) = P_o^{(s)} \psi_S^{\,o}(0) \tag{76}$$

$$P_o = P_o^{\,o}(0) + P_o^{\,i}(0)$$

and

$$\psi_S = C_S\, \psi_S^{\,o}(0) + \Lambda\, \psi_S \tag{77}$$

where C_S *is a normalization constant, it follows*

$$P_o \psi_S = P_o\,(C_S\, \psi_S^{\,o}(0) + \Lambda\, \psi_S) = C_S\,(P_o^{(s)} + P_o^{\,i}(0))\,\psi_S^{\,o}(0) + P_o \Lambda\, \psi_S \tag{78}$$

$$= P_o^{(s)} \psi_S = P_o^{(s)} (C_S\, \psi_S^{\,o}(0) + \Lambda\, \psi_S)$$

and from this

$$(P_o^{(s)} - P_o)\,\Lambda\, \psi_S = C_S\, P_o^{\,i}(0)\, \psi_S^{\,o}(0) \tag{79}$$

i.e. our statement (75) (because of the stability of $\psi_S^{\,o}(0)$ *the operator* $(P_o^{(s)} - P_o)$ *has a unique inverse in (79)).*

Now we show in the second part of our proof that there cannot be any interaction between two particles provided that a unitary operator $U_o\,(0, \mp\infty)$ *has no influence on the stable states. We consider the general S-matrix element for two incoming particles*

$$\underset{in}{<} n/S/q_1,q_2 \underset{in}{>} = \underset{out}{<} n/S/q_1,q_2 >_{out} = \underset{out}{<} n/q_1,q_2 >_{in} = \delta_{n;q_1 q_2} + \underset{out}{<} n/R/q_1,q_2 >_{out} \tag{80}$$

with

$$\underset{out}{<} n/R/q_1,q_2 >_{out} = \frac{-i}{(2\pi)^{3/2}} \int dx \frac{e^{-iq_1x}}{\sqrt{2q_{1,0}}}\, \underset{out}{<} n/j(x)/q_2 >_{out} = \frac{-i}{(2\pi)^{3/2}} \int dx \frac{e^{-iq_1x}}{\sqrt{2q_{1,0}}} \times$$

$$\times \underset{out}{<} n/a^*_{out}(\vec{q}_2)\, j(x)/0> + \frac{-i}{(2\pi)^3} \int dx\,dy \frac{e^{-iq_1x - iq_2y}}{2\sqrt{q_{1,0}q_{2,0}}}\, \underset{out}{<} n/\frac{\delta j(x)}{\delta \varphi_{out}(y)}/0>;\quad q_{i,0} = +\sqrt{m^2 + \vec{q}_i^{\,2}}\,. \tag{81}$$

reduced in the functional derivative approach to quantum field theore by means of commutation relations of the type 1(8), 1 (32) (where we have chosen in (80) the states $/q_1, q_2>$, $/n>$ *as outgoing states in order to make use of the definition (2) for the current operator in connection with the stability property of the one-particle states after the first reduction;* $/0>$ *is as in (7) the state vector of the incoming or outgoing particle vacuum). Using 1(19) we get for a causal theory up to contributions from the corresponding quasilocal operators*

$$
{}_{out}\langle n|R|q_1,q_2\rangle_{out} = \frac{-i}{(2\pi)^{3/2}}\int dx\,\frac{e^{-iq_1x}}{\sqrt{2q_{1,0}}}\,{}_{out}\langle n|a^*_{out}(\vec{q}_2)j(x)|0\rangle
$$

$$
+\frac{1}{(2\pi)^3}\int dx\,dy\,\frac{e^{-iq_1x-iq_2y}}{2\sqrt{q_{1,0}\,q_{2,0}}}\,\theta(y-x)\,{}_{out}\langle n|[j(x),j(y)]|0\rangle;\; q_{i,0}=+\sqrt{m^2+\vec{q}^{\,2}} \quad (82)
$$

In the interacting field operator approach (82) takes the form (compare also section 4)

$$
{}_{out}\langle n|R|q_1,q_2\rangle_{in} = \frac{-i}{(2\pi)^{3/2}}\int dx\,\frac{e^{-iq_1x}}{\sqrt{2q_{1,0}}}\,(\Box_x-m^2)\,{}_{out}\langle n|a^*_{out}(\vec{q}_2)\varphi(x)|0\rangle \quad (83)
$$

$$
+\frac{1}{(2\pi)^3}\int dx\,dy\,\frac{e^{-iq_1x-iq_2y}}{2\sqrt{q_{1,0}\,q_{2,0}}}\,(\Box_x-m^2)(\Box_y-m^2)\,\theta(y-x)\,{}_{out}\langle n|[\varphi(x),\varphi(y)]|0\rangle;\; q_{i,0}=+\sqrt{m^2+\vec{q}_i^{\,2}}
$$

The first expression on the right-hand-side in (81), (82) and (83) vanishes unless one of the momenta in the final state $|n\rangle_{out}$ is equal to $\vec{q}_2$ and vanishes identically if $|n\rangle_{out}$ is a two-particle state because of (90), (91). First we consider the second expression in (83) which we write in the form

$$
\frac{1}{(2\pi)^3}\int dx\,dy\,\frac{e^{-iq_1x-iq_2y}}{2\sqrt{q_{1,0}\,q_{2,0}}}\Big\{(\Box_x-m^2)\,\theta(y-x)(\Box_y-m^2)\,{}_{out}\langle n|\varphi(x)\varphi(y)|0\rangle
$$

$$
-(\Box_y-m^2)\,\theta(y-x)(\Box_x-m^2)\,{}_{out}\langle n|\varphi(y)\varphi(x)|0\rangle\Big\};\; q_{i,0}=+\sqrt{m^2+\vec{q}_i^{\,2}} \quad (83')
$$

where we have made a partial integration (see, for details, also the next section). We expand the matrix elements of the field operators in (83') with respect to a complete set of incoming or outgoing particle states $|n'\rangle_{in/out}$, for instance,

$$
{}_{out}\langle n|\varphi(x)\varphi(y)|0\rangle = \sum_{n'}\,{}_{out}\langle n|\varphi(x)|n'\rangle_{in/out}\;{}_{in/out}\langle n'|\varphi(y)|0\rangle \quad (84)
$$

and consider the matrix element ${}_{in/out}\langle n'|\varphi(y)|0\rangle$ which we perform as follows

$$
{}_{in/out}\langle n'|\varphi(y)|0\rangle = e^{iq'y}\,{}_{in/out}\langle n'|\varphi(0)|0\rangle
$$

$$
= e^{iq'y}\langle \Psi^{\circ}_{n'}(0)|\,U_0^{+}(0,\mp\infty)\,\varphi_0(0)\,U_0(0,\mp\infty)|\Psi^{\circ}_0(0)\rangle = e^{iq'y}\langle \Psi^{\circ}_{n'}(0)|\varphi_0(0)|\Psi^{\circ}_0(0)\rangle \quad (85)
$$

where we have made use of the translation invariance (q' is the energy-momentum vector of the state $|n'\rangle_{in/out}$), expressed all quantities in the usual interaction representation (which is possible since $U_0(0,\mp\infty)$ is assumed as unitary; compare for details the appendix, especially (A.6), (A.7) and (A.8)) and in the last step used our assumption that $U_0(0,\mp\infty)$ or $U_0^{+}(0,\mp\infty)$ respectively has no influence on the stable states $\Psi^{\circ}_s(0)$ (the fact that with $U_0(0,\mp\infty)$ also $U_0^{+}(0,\mp\infty)$ cannot have any influence on the stable states follows simply from the unitarity condition $U_0^{+}(0,\mp\infty)U_0(0,\mp\infty)\Psi^{\circ}_s(0)=1\,\Psi^{\circ}_s(0)$).
However, from (85) we conclude that (83') vanishes identically since according to (85) only the one-particle states contribute in (83') and the application of the corresponding Klein-Gordon operators yields zero. By the same argument it is immediately seen that also the first term in (83) vanishes identically.

*This completes our proof**.

* *It is easy to show that our proof can be extended to the case of more than two incoming particles (where one has to deal with T-products; compare (9)) which, however, has no immediate physical interest. Mathematically it would then completely prove that an unitary transformation between $\varphi(x)$ and $\varphi_{\substack{in\\out}}(x)$ is only possible in the free field case.*

Thus we have shown that the operator $U_0(0,\mp\infty)$ *cannot be unitary for real interactions and the same must be true for the operator* $U_{\substack{in\\out}}(0,\mp\infty)$ *because we have, for instance, according to (49)* $U_0(0,\mp\infty) = U^t_{\substack{in\\out}}(0\mp\infty)\, U_{\substack{in\\out}}(0,\mp\infty)\, U_{\substack{in\\out}}(0,\mp\infty)$. *Since, however, according to the assumption (20), (21)* ψ_n *and* $\psi^{\circ}_n(0)$ *are connected by a unitary transformation the operator* $U_0(0,\mp\infty)$ *which transforms between them must then correspondingly unitary up to a general finite (re-)normalization constant (related to the constant* Z *of the following).*

It is also very instructive to discuss this situation by studying the commutation relations. From (32) or (63) it follows using (42) and (60)

$$[\varphi(x),\dot\varphi(y)]\big| = [\dot U^t(x_0,t)\,\varphi_t(x)\,U(x_0,t) + U^t(x_0,t)\,\varphi_t(x)\,\dot U(x_0,t),\,\varphi(y)]\big|_{x_0=y_0}$$
$$+\, U^t(x_0,t)\,[\dot\varphi_t(x),\varphi_t(y)]\,U(x_0,t)\big|_{x_0=y_0} = \tag{86}$$
$$= i\,[U^t(x_0,t)\,[P^i_0(x_0,t),\varphi_t(x)]\,U(x_0,t),\varphi(y)]\big|_{x_0=y_0} - i\delta(\vec{x}-\vec{y}) = -i\delta(\vec{x}-\vec{y})$$

if we assume in the last step that $P^i_0(t)$ *depends only on the field operator* $\varphi_t(\vec{x}',t) = \varphi(\vec{x}',t)$ *itself and not on its time-derivatives (according to (41) and (19))* $P^i_0(x_0,t)$ *can then only depend on* $\varphi_t(\vec{x}',x_0)$ *such that*

$$[P^i_0(x_0,t),\varphi_t(x)] = 0$$

is fulfilled). However, (86) cannot hold if* $\varphi(x)$ *is of the form (15), (15') where the free*

* *(86) holds, of course, also in more general cases, for instance, if* $P^i_0(t)$ *depends in addition on the first time-derivative of the field operator in first order. We have always in mind only the conventional local interactions for which (86) can be performed.*

field operators $\varphi_{\substack{in\\out}}(x)$ *obey (5) in connection with the stability requirement for the one-particle states. For then we must have (* $|0\rangle$ *is as in (7) and (81) the state vector of the incoming or outgoing particle vacuum)*

$$\langle 0|[\dot\varphi(x),\varphi(y)]|0\rangle\big|_{x_0=y_0} = -i\,Z^{-1}\,\delta(\vec{x}-\vec{y}) \tag{87}$$

where*

$$Z^{-1} = \int d\mu^2 \varrho(\mu^2) \geq 1 \tag{88}$$

* We exclude the singular case $Z = 0$. It would mean that the solution of the inhomogeneous Klein-Gordon equation cannot exist in the form (12), (12') or (29) in quantized field theory since the quantization (93) for the free field operator $\varphi_t(x)$ makes no sense. On the other hand, (15), (15') is the limiting case of (12), (12') (notice also that the limiting solutions (15), (15') can always be brought into the form (12), (12') with

$$\varphi_t(x) = \begin{cases} \varphi_{in}(x) + \int_{-\infty}^{t} \Delta(x-y) j(y)\, dy & \text{for} \quad x_0 > t \\ \varphi_{out}(x) - \int_{t}^{+\infty} \Delta(x-y) j(y)\, dy & \text{for} \quad x_0 < t \end{cases}$$

which obviously obey (13) and (14)). It would further mean that we cannot exclude the case that because of (87) (which is more singular than a δ -function) the quasilocal operators appearing, for instance, in (108) have infinite coefficients (probably also the explicitly written term in (108) will then not make sense).

$\varrho(\mu^2)$ is the well-known spectral function of Källèn[9] and Lehmann[10] which follows from an expansion of the left-hand-side of (87) with respect to a complete set of incoming or outgoing particle states (compare (84)). For really interacting fields it must be

$$Z^{-1} > 1 \tag{89}$$

because

$$Z^{-1} = 1 \tag{89'}$$

results already from the contribution of the stable one-particle states $(\varrho(\mu^2) = \delta(\mu^2 - m^2))$. In general (89') would be equivalent to the case that $U_{0,\,\substack{in\\out}}(0, \mp\infty)$ has no influence on the stable states (compare the considerations (84) ff) which we have to exclude according to the considerations (84) ff. Only for the contribution of the intermediate one-particle states in (87) the operator $U_{0,\,\substack{in\\out}}(0, \mp\infty)$ is without any influence according to (compare (15), (15'), (25) and (26))

$$\langle 0 | \varphi(x) | \vec{q} \rangle = \langle 0 | \varphi_{\substack{in\\out}}(x) | \vec{q} \rangle = \frac{1}{(2\pi)^{3/2}} \frac{e^{-iqx}}{\sqrt{2q_0}}; \quad |\vec{q}\rangle = a^*_{\substack{in\\out}}(\vec{q})|0\rangle \tag{90}$$

since it is

$$\langle 0 | j(x) | \vec{q} \rangle = 0 \tag{91}$$

(91) follows from the stability condition

$$\langle \vec{q} | S | \vec{q} \rangle = 1 \tag{92}$$

(to show this use the commutation relation 1(8)).

From the above considerations we conclude that the free field operator $\varphi_t(x)$ for finite t has really to be quantized according to *

* We remark that (93) (as well as (60)) yields (61), (62). We have no reason to conclude that $U(t,t')$ is not unitary for finite times. But under the assumption (93) the relations (86) and (87) are compatible for a unitary $U(t,t')$ since now the factor Z^{-1} appears also in (86).

$$[\varphi_t(x), \varphi_t(y)] = i Z^{-1} \Delta(x-y) \tag{93}$$

in contrast to (60) (and consequently we have to put the same factor into (26)) or the limiting value for

$$\left[\varphi_{\substack{in \\ out}} (x), \varphi_{\substack{in \\ out}} (y) \right] = i \Delta (x-y) \qquad (93')$$

(93) and (93') show evidently that the operator $U_{x_0, \substack{in \\ out}} (x_0, \mp \infty)$ *which connects the interacting field operator* $\varphi(x)$ *with the incoming or outgoing free field operators* $\varphi_{\substack{in \\ out}} (x)$ *can only be unitary up the constant* Z *. Thus the employment of the solutions (15), (15') where the free field operators* $\varphi_{\substack{in \\ out}} (x)$ *obey (5) together with the stability condition (92) or, more generally spoken, the requirement of compatibility between the commutation relations (quantization) on the one hand and the properties of the mass spectrum and state vectors (Hilbert space) on the other makes necessary a redefinition of the quantization prescription for the free field operator* $\varphi_t (x)$ *for finite* t *

* *The problem of the existence of a unitary operator* $U_0 (0, -\infty)$ *was first discussed by R. Haag/11/. We have adopted the following point of view: whereas (93) and (93') show that really no unitary transformation between the free field operator* $\varphi_t (x)$ *or – because of* $\varphi_t (x) = \varphi (x)$ *for* $x_0 = t$ *-the interacting field operator* $\varphi (x)$ *and the incoming or outgoing free field operators* $\varphi_{\substack{in \\ out}} (x)$ *can exist it still exists according to (93) and (93') between the unrenormalized field operator* $Z^{1/2} \varphi (x)$ *and* $\varphi_{in, out} (x)$*. It is obvious that the unitarity relation for* $U_{x_0 \substack{in \\ out}} (x_0 \mp \infty)$ *can then correspondingly hold only up to a renormalization factor which, of course, is unimportant for the study of integrability conditions for the solutions (15), (15') of the wave equation (1). However, it should also be remarked that it is quite unclear wether the quantization (93), (93') (which makes the theory consistent only subsequently) can lead to a theory which is consistent at all (and probably it can only be consistent up to the renormalization factor* $Z^{-1/2}$*). For instance: if we take into account (93), (93') in the relations (50) ff there the free wave part must be multiplied by the renormalization factor* $Z^{-1/2}$ *(and also* $j (x)$ *in (58), (58'), both in contrast to our assumption for the discussion in (86) ff) which again has to be modified subsequently by considerations analogously to (86) ff (which can be repeated ad infinitum). Furthermore, it should always possible to replace* $U_{\substack{in \\ out}} (x_0, \mp \infty)$ *by* $U_{x_0} (x_0, \mp \infty)$ *(compare the footnote for (50)) but these two operators cannot be identical according to (45) if one takes into account (93), (93'). We add the remark (which is quite evident from the above considerations) that we are always only concerned with the equivalent ordinary representations of the commutation relations for which vacuum states exist and which are connected by unitary transformations considering the whole question as a renormalization problem with respect to the constant* Z *connected immediately with the adiabatic conception (compare (93), (93')) which yields only weakly convergent results.*

3. THE EXTRAPOLATED S-MATRIX ELEMENTS

We study first the situation connected with the problem of an extrapolation of the reduced S-matrix elements off the mass shell in the functional derivative approach to quantum field theory. If we write 1 (16) in the form

$$\frac{\delta j (x)}{\delta \varphi_{in} (y)} = -i \left[j (x), j (y) \right] + \frac{\delta j (y)}{\delta \varphi_{in} (w)} ; \qquad (94)$$

multiply this equation by $\Theta_1 (x, y)$ *and add on both sides the term* $\Theta_2 (x, y) \dfrac{\delta j (x)}{\delta \varphi_{in} (y)}$ *, where* $\Theta_1 (x, y)$ *and* $\Theta_2 (x, y)$ *obey the relation*

$$\Theta_1 (x-y) + \Theta_2 (x_1, y) = 1 \qquad (95)$$

we obtain

$$\frac{\delta j(x)}{\delta \varphi_{in}(y)} = -i\,\theta_1(x,y)\left[j(x),j(y)\right] + \theta_1(x,y)\frac{\delta j(y)}{\delta \varphi_{in}(x)} + \theta_2(x,y)\frac{\delta j(x)}{\delta \varphi_{in}(y)}; \qquad (96)$$

The commutation relation 1 (32) appears then in the form

$$\left[a_{in}(\vec{q}), j(x)\right] = \frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} \frac{\delta j(x)}{\delta \varphi_{in}(y)} = \qquad (97)$$

$$= \frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} \left\{ -i\,\theta_1(x,y)\left[j(x),j(y)\right] + \theta_1(x,y)\frac{\delta j(y)}{\delta \varphi_{in}(x)} + \theta_2(x,y)\frac{\delta j(x)}{\delta \varphi_{in}(y)} \right\};$$

$$q_0 = +\sqrt{m^2+\vec{q}^{\,2}}$$

If we choose $\theta_1(x,y)$ as the usual step function

$$\theta_1(x,y) = \theta(x-y) \qquad (98)$$

such that according to (95)

$$\theta_2(x,y) = 1 - \theta(x-y) = \theta(y-x) \qquad (99)$$

we get for a causal theory from (97) according to (8)

$$\left[a_{in}(\vec{q}), j(x)\right] = \frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} \left\{ -i\,\theta(x-y)\left[j(x),j(y)\right] \right\}; \quad q_0 = \sqrt{m^2+\vec{q}^{\,2}} \qquad (100)$$

with $\left[j(x),j(y)\right] = 0$ for $x \sim y$ (compare 1 (18)), up to terms resulting from the corresponding quasilocal operators. Of course, we may also work in this case with the more general representation (97) and the important point is now that this is also possible (without introducing any modification) if we extrapolate (97) off the mass shell according to Bogolubov's original idea

$$m^2 \longrightarrow \tau \qquad (101)$$

as the starting point for the analytic continuation of the corresponding matrix elements of (100) with respect to q_0 /3/. The reason is that (96) itself is a simple identity (which is not only valid on the mass shell in (97)), i.e. the choice of the θ -functions has no influence on the extrapolated relations.

In the interacting field operator approach to quantum field theory we meet a quite different situation. Here the causal properties of the field theory are expressed by *

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* We assume here that (64) is really a condition on the formalism which is important for the analytic behaviour of the reduced S-matrix elements.

$$\left[\varphi(x), \varphi(y)\right] = 0 \quad \text{if} \quad x \sim y \qquad (64)$$

and it was shown in 1 that in the reduction formula 1 (29) determined from an application of the asymptotic condition*

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* We write here more generally $\theta_{ret}(x,y)$ instead of $\theta(x-y)$

$$[a_{in}(\vec{q}),j(x)] = \frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} (\Box_y - m^2)\{-i\theta_{ret}(x,y)(\Box_x - m^2)[\varphi(x),\varphi(y)]\}; q_0 = +\sqrt{m^2+\vec{q}^2} \quad (102)$$

the function $\theta_{ret}(x,y)$ can only be determined by the boundary values

$$\lim_{y_0 \to \mp\infty} \theta_{ret}(x,y) = \begin{cases} 1 \\ 0 \end{cases} \quad (103)$$

with vanishing derivatives $\frac{\partial}{\partial y_0}\theta_{ret}(x,y)$ at these limits*. If, however, we extrapolate (102) off the

* We remark that the partial integration in the last step in 1 (27) is not influenced by a dependence of $\theta_{ret}(x,y)$ on the space-coordinates $\vec{y}$. For the dependence on x there is really no condition (as far as (103) is not violated).

mass shell according to (101) there appear additional terms depending on $\theta_{ret}(x,y)$ which make an analytic continuation of the corresponding matrix elements of (102) with respect to q_0 in general impossible.

We may study this as follows. First it is clear that according to (103) we may add to a given function $\theta_{ret}(x,y)$ in (102) an arbitrary additional function $\theta_a(x,y)$ obeying the boundary condition

$$\lim_{y_0 \to \mp\infty} \theta_a(x,y) = 0 \quad (104)$$

with vanishing derivatives $\frac{\partial}{\partial y_0}\theta_a(x,y)$ at these limits. The reason is that on the mass shell the relation

$$(\Box_y - m^2)\, e^{iqy} = 0; \qquad q_0 = +\sqrt{m^2+\vec{q}^2} \quad (105)$$

is valid*. However, if we extrapolate off the mass shell according to (101) the relation (105) takes

* Strictly speaking this is only right if we replace the plane waves in (102) by the corresponding wave group solutions of positive energy; compare the performance 1 (27).

the form

$$(\Box_y - \tau)\, e^{iqy} = 0; \qquad q_0 = +\sqrt{\tau+\vec{q}^2} \quad (106)$$

and the contribution resulting from the function $\theta_a(x,y)$ leads correspondingly to the additional term

$$(\tau - m^2)\frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} \{-i\theta_a(x,y)[j(x),\varphi(y)]\}; q_0 = +\sqrt{\tau+\vec{q}^2} \quad (107)$$

in the extrapolated relation (102).

If we choose in (102) $\theta_{ret}(x,y)$ as the usual step function we obtain

$$[a_{in}(\vec{q}),j(x)] = \frac{1}{(2\pi)^{3/2}} \int dy \frac{e^{iqy}}{\sqrt{2q_0}} \{-i\theta(x-y)[j(x),j(y)]\}; q_0 = +\sqrt{m^2+\vec{q}^2} \quad (108)$$

with $[j(x),j(y)] = 0$ for $x \sim y$ (compare (64)), up to contributions from some quasilocal operators. (108) is equivalent to the relation (100). If the extrapolation off the mass shell is carried out the analytic continuation of the corresponding matrix-elements of (108) can be performed as usually/3/. However, the use of other θ -functions leads to the appearance of additional terms of the form (107) which in view of the condition (104) make in general impossible an analytic continuation of the matrix elements of the extrapolated relation (102) with respect to q_0 *. But since according to (107) these non-

*We remark that the extrapolated relations 1 (28), 1 (29) (between which we cannot distinguish a priori) differ also from each other by terms of the form (107).

analytic contributions lead to zero contributions on the mass shell (and we are only interested in the final result for $\tau \to m^2$) they are really spurious: we have no need to continue them analytically.

Thus we have shown that both approaches give in general rise to different extrapolations of the reduced S-matrix elements off the mass shell: that part which can be analytically continued is essentially determined by Bogolubov's causality condition and corresponds to a special choice of the θ function in the interacting field operator approach.

4. CAUSALITY CONDITION IN THE INTERACTING FIELD OPERATOR APPROACH

Concluding we show that the causality condition for the interacting field operator in the commutator form is without any meaning for the analytic behaviour of the reduced S-matrix elements in the interacting field operator approach as studied in the theory of dispersion relations. For this purpose we consider the S-matrix element for elastic scattering (compare (80), (83))

$$ {}_{out}\langle q_3, q_4 | q_1, q_2 \rangle_{in} = \delta_{q_3,q_4;\, q_1,q_2} + {}_{out}\langle q_3, q_4 | R | q_1, q_2 \rangle_{out} \qquad (109) $$

with

$$ {}_{out}\langle q_3, q_4 | R | q_1, q_2 \rangle_{out} = \frac{1}{(2\pi)^3} \int dx\, dy\, \frac{e^{-iq_1 x - iq_2 x}}{2\sqrt{q_{1,0}\, q_{2,0}}} (\Box_x - m^2)(\Box_y - m^2) $$
$$ \times\, \theta_{adv}(x,y)\, {}_{out}\langle q_3, q_4 | [\varphi(x), \varphi(y)] | 0 \rangle ; \quad q_{i,0} = +\sqrt{m^2 + \vec{q}_i^{\,2}} \qquad (110) $$

where the function $\theta_{adv}(x,y)$ has only to fulfill the condition (103) for interchanged limits (past and future). Let us assume that the field operator $\varphi(x)$ is a non-causal one, i.e.

$$ [\varphi(x), \varphi(y)] \neq 0 \quad \text{if} \quad x \sim y . \qquad (111) $$

Now we choose the function $\theta_{adv}(x,y)$ such that it vanishes for space-like separated points

$$ \theta_{adv}(x,y) = 0 \quad \text{if} \quad x \sim y . \qquad (112) $$

This choice is quite generally compitable with the condition (103) for interchanged limits since the space like region (x – y) vanishes for $y_0 = \mp\infty$ (such points (x – y) which can contribute on the planes $y_0 = \mp\infty$ lie asymptotically on the light cone; they have infinite values of the points $\vec{x}$ or $\vec{y}$ in ordinary space where in addition the field operator $\varphi(x), \varphi(y)$ itself vanishes**). For time-like

** Strictly speaking we have for this to replace always the plane waves by wave packet solutions of positive energy (for $\varphi(x)$ we assume an expansion of the form 1 (25), really understood for wave group solutions). Only in the final results we go then over to the limiting case of plane waves (compare /2/). We remark further for the case of possible singularities in space-like regions that it is as usual assumed that the integral over them with a finite δ -function in (110) is defined.

points (x − y) we choose $\theta_{adv}(x,y)$ such that it is equal to one in the backward light cone and equal to zero in the forward light cone in (x − y)-space which again is in accordance with (103) for interchanged limits. This shows that it is always possible to bring (110) in such a form that it represents the Fourier transform of a function f (x, y) which is different from zero only in the backward light cone of (x − y). But this property involves all what we need from a causality condition in the theory of dispersion relations.

Of course, we do not believe that the relation (110) derived from an application of the asymptotic condition makes any sense in a non-causal field theory * . It seems evident that the causal character of

* Otherwise dispersion relations would be valid for causal as well as non-causal field theories which contradicts the results of the functional derivative approach. The conclusion that the contributions from space-like regions of the commutator (111) cannot make any sense in the reduction formulae of the interacting field operator approach may also be reached from the requirement of covariance (an argumentation used in 1).

the theory is a priori assumed in the derivation of (110). If one makes a first reduction in (109) according to the relation (compare 1 (21))

$$a_{in}(\vec{q}) = a_{out}(\vec{q}) + \frac{i}{(2\pi)^{3/2}} \int dx \frac{e^{iqx}}{\sqrt{2q_0}} j(x) \qquad (114)$$

one sees immediately that one has to solve for a further performance in (110) the wave equation (1). In a second reduction there is assumed that the solution $\varphi(y)$ has a well-defined asymptotic behaviour for $y_0 \to +\infty$ and it seems very unlikely that $\varphi(y)$ can be another solution as an advanced one which assumes Bogolubov's causality condition (8') as a necessary integrability condition according to section 2.

In any case, the causality condition (64) cannot be interpreted as a condition on the reduced S-matrix element (110) which is important for its analytic behaviour as studied in the theory of dispersion relations. If one wants to investigate the consequences of a possible non-causal structure of quantum field theory (which is a very important physical problem also within the theory of dispersion relations) one has to use reduction formulae for the S-matrix elements 'derived' in the functional derivative approach.

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APPENDIX

We derive some relations valid for a representation at arbitrary t. We perform

$$\,_{in}\langle n'|S|n\rangle_{in} = \,_{out}\langle n'|S|n\rangle_{out} = \langle \Psi^0_{n'}(t)|S_t|\Psi^0_n(t)\rangle \qquad (A.1)$$

where

$$\Psi^0_n(t) = \mathcal{U}^{+}_{\substack{in\\out}}(t, \mp\infty)|n\rangle_{\substack{in\\out}} \qquad (A.2)$$

and (compare (3) and (4))

$$S_t = \sum_{n=0}^{\infty} \int d^4x_1 \dots dx_n\, f_n(x_1,\dots,x_n) : \varphi_t(x_1)\dots\varphi_t(x_n) : \;; \quad S_t S_t^+ = S_t^+ S_t = 1 \tag{A.3}$$

with (compare (38))

$$\varphi_t(x) = \mathcal{U}^+_{\substack{in\\out}}(t,\mp\infty)\,\varphi_{\substack{in\\out}}(x)\,\mathcal{U}_{\substack{in\\out}}(t,\mp\infty) \tag{A.4}$$

*We remark that because of (45)****

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* *For $\mathcal{U}_{t,\substack{in\\out}}(t,\mp\infty)$ we may also have in mind representations of the form (3) or (23) for which (45) is trivial because of (38). Here we ignore as usually the fact that $\mathcal{U}_{t,\substack{in\\out}}(t,\mp\infty)$ can only be unitary up to a constant (if one takes into account this fact one has always to replace $\varphi_t(x_i)$ by $Z^{1/2}\varphi_t(x_i)$ or to drop the Z- factor in (93) and to normalize the states $\psi_n^o(t)$; see also the footnote at the end of section 2).*

$$\mathcal{U}_{\substack{in\\out}}(t,\mp\infty) = \mathcal{U}_t(t,\mp\infty). \tag{A.5}$$

For the matrix elements of $\varphi(x)$ we write

$$\langle n'|\varphi(x)|n\rangle_{\substack{in\\out}} = \langle\, \mathcal{U}_t(t,\mp\infty)\psi^o_{n'}(t)|\varphi(x)|\,\mathcal{U}_t(t,\mp\infty)\psi^o_n(t)\rangle \tag{A.6}$$

since according to (A.2) and (A.5)

$$\mathcal{U}_t(t,\mp\infty)\,\psi^o_n(t) = |n\rangle_{\substack{in\\out}} \tag{A.7}$$

and where we may use on the right-hand-side of (A.6) employing (32) and (17)

$$\varphi(x) = \mathcal{U}^+_t(x_0,t)\,\varphi_t(x)\,\mathcal{U}_t(x_0,t) = e^{iP_0(x_0-t)}\varphi(\vec{x},t)\,e^{-iP_0(x_0-t)} \tag{A.8}$$

as the interacting field operator corresponding to the t-representation. According to (21) we have for (A.7)

$$P^o_{o,\substack{in\\out}}\,\mathcal{U}_t(t,\mp\infty)\,\psi^o_n(t) = P_o^{(n)}\,\mathcal{U}_t(t,\mp\infty)\,\psi^o_n(t);\; P^o_o(t)\,\psi^o_n(t) = P_o^{(n)}\,\psi^o_n(t) \tag{A.9}$$

where obviously (compare (37))

$$P^o_o(t) = \mathcal{U}^+_t(t,\mp\infty)\,P^o_{o,\substack{in\\out}}\,\mathcal{U}_t(t,\mp\infty) \tag{A. 10}$$

*For $t=0$ (conventional interaction representation **) (A.9) reads*

** *For this case see also* [12]

$$P_{o,\substack{in\\out}}\,\mathcal{U}_o(0,\mp\infty)\,\psi^o_n(0) = P_o^{(n)}\,\mathcal{U}_o(0,\mp\infty)\,\psi^o_n(0) \tag{A. 11}$$

Comparing this with (72) we arrive at the result that $P^o_{o,\substack{in\\out}}$, have to be identified with P_o itself which was stated in (27).

Concluding we derive the causality conditions valid in the t-representation. For finite t (54), (54') *reads**

* *For a quantization according to (93) the integral terms in (A.12), (A.12') have to be multiplied by Z^{-1}. Since this factor is unimportant for the study of integrability conditions for the solutions (12), (12') of the wave equation (1) we drop it as in section 2.*

$$\varphi(x) = \varphi_t(x) + i\int dy\, \Delta(x-y)\, \mathcal{U}_t^+(x_0,t)\, \frac{\delta \mathcal{U}_t(x_0,t)}{\delta \varphi_t(y)} \quad \text{for } x_0 > t \tag{A.12}$$

$$\varphi(x) = \varphi_t(x) - i\int dy\, \Delta(x-y)\, \frac{\delta \mathcal{U}_t(t,x_0)}{\delta \varphi_t(y)}\, \mathcal{U}_t^+(t,x_0) \quad \text{for } x_0 < t \tag{A.12'}$$

As in section 2 we now conclude the 'proper causality condition'

$$\frac{\delta \mathcal{U}_t(x_0,t)}{\delta \varphi_t(y)} = 0 \qquad \text{if } y_0 > x_0 > t \tag{A. 13}$$

$$\frac{\delta \mathcal{U}_t(t,x_0)}{\delta \varphi_t(y)} = 0 \qquad \text{if } y_0 < x_0 < t \tag{A.13'}$$

and the causality condition

$$\frac{\delta j(y)}{\delta \varphi_t(z)} = \frac{\delta}{\delta \varphi_t(z)} \left\{ i\, \mathcal{U}_t^+(x_0,t)\, \frac{\delta \mathcal{U}_t(x_0,t)}{\delta \varphi_t(y)} \right\} = 0 \qquad \text{if } z_0 > x_0 > y_0 > t \tag{A.14}$$

$$\frac{\delta j(y)}{\delta \varphi_t(z)} = \frac{\delta}{\delta \varphi_t(z)} \left\{ i\, \frac{\delta \mathcal{U}_t(t,x_0)}{\delta \varphi_t(y)}\, \mathcal{U}_t^+(t,x_0) \right\} = 0 \qquad \text{if } z_0 < x_0 < y_0 < t \tag{A. 14'}$$

as necessary integrability conditions for the retarded or advanced solution (12), (12').

We note also the formula

$$\begin{aligned} j(x) &= i\, S^+ \frac{\delta S}{\delta \varphi_{in}(x)} = i\, \frac{\delta S}{\delta \varphi_{out}(x)}\, S^+ \\ &= i\, \mathcal{U}_t(t,-\infty)\, S_t^+ \frac{\delta S_t}{\delta \varphi_t(x)}\, \mathcal{U}_t^+(t,-\infty) = i\, \mathcal{U}_t^+(+\infty,t)\, \frac{\delta S}{\delta \varphi_t(x)}\, S_t^+ \left(\mathcal{U}_t(+\infty,t)\right) \end{aligned} \tag{A. 15}$$

which is evident in view of (3), (A. 3) and (A.4). For the operator

$$j_t(x) = i\, S_t\, \frac{\delta S_t^+}{\delta \varphi_t(x)} = i\, \frac{\delta S_{\bar t}}{\delta \varphi_{\bar t}(x)}\, S_{\bar t}^+ \tag{A.16}$$

where

$$\varphi_{\bar t}(x) = S_t^{+} \varphi_t(x) S_t , \quad S_t = S_{\bar t} \tag{A.17}$$

(compare (A.3)) we conclude from (8), (8') and (A.3) and (A.4) the causality condition

$$\frac{\delta j_t(x)}{\delta \varphi_t(y)} = 0 \qquad \text{if} \quad y \gtrsim x \tag{A. 18}$$

$$\frac{\delta j_t(x)}{\delta \varphi_{\bar t}(y)} = 0 \qquad \text{if} \quad y \lesssim x \tag{A. 18'}$$

which one has to use if one derives reduction formulae for the S-matrix in the t-representation.

REFERENCES

1. F. Kaschluhn. Nuovo Cimento 12, 541, (1959).
2. H. Lehmann, K. Symanzik and W. Zimmermann. Nuovo Cimento 6, 319, (1957).
3. N.N. Bogolubov, B.V. Medvedev and M.K. Polivanov. Problems of the Theory of Dispersion Relations (Moscow 1958); abridged translation in German in Fortschr. d. Phys. 6, 169, (1958).
4. Y. Okabayashi. Suppl. Nuovo Cimento 9, 599, (1958).
5. N.N. Bogolubov and D.V. Shirkov. Usp. Fiz. Nauk 55, 149, (1955); German translation in Fortschr. d. Phys. 3. 439, (1955), p. 458.
6. F. Kaschluhn. Nuclear Physics (in press).
7. C.N. Yang and D. Feldman. Phys.Rev. 91, 972, (1950).
8. M. Gell-Mann and M.L. Goldberger. Phys. Rev. 91, 398, (1953).
9. G. Källen, Helv. Phys. Acta 25, 417, (1952).
10. H. Lehmann. Nuovo Cimento 11, 342, (1954).
11. R. Haag, K. Danske Vidensk Selsk. mat.-fys. Medd. 29, No. 12, (1955), see for refinement D. Hall and A.S. Wightman, K. Danske Vidensk Selsk. mat-fys. Medd. 31, No. 5, (1957) and also O.W. Greenberg. Phys.Rev. (L) 3, 159, (1959).
12. S.S. Schweber. Nuovo Cimento 2, 397, (1955).

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