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SU(6) AND THE OPERATORS CPT AND C

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1. Generalities

The structure of the charge conjugation operator G for the main types of simple compact Lie groups has been discussed by Biedenharn Nuyts and Ruegg[1]. Further investigations are due to Okubo and Mukunda[2]. Here we shall show in some detail, how the operators CPT, C and PT are to be determined in the SU(6) scheme by the following requirements:

a) They are multiplicative[3] unitary or antiunitary operators with respect to SU(6) and they map group representations onto group representations.

b). The squared operators are in the centre of SU(6). We construct the charge conjugation operator with respect to a SU(2) subgroup of SU(6) and look for the remaining ambiguities. No such ambiguities with experimental consequences are found for the electromagnetic case, using L-spin as the leading subgroup. Turning to the iso-spin, there remains some freedom for states with $B-Y\neq 0$ only. This freedom in the 35-plet just consists in a "rotation" of the K-meson system. This may be of some interest in connection with the recent conjucture of T.D.Lee[4].

Let us mention still another point. From the requirements mentioned above we get $C^2 = \pm 1$. It turns out, that the desired $C\pi^0 = \pi^0$ for the π^0-meson comes out for

$$C^2 = +1 \qquad \text{if} \qquad q\bar{q} = -\bar{q}q \tag{1-1}$$

$$C^2 = -1 \qquad \text{if} \qquad q\bar{q} = \bar{q}q \tag{1-2}$$

Of course, no symmetry requirements have to be imposed for pure quark (antiquark) states. We shall stick to the more natural possibility (1-1) here.

2. Short Review of SU(n)

Before considering more closely SU(6) let us review the general features of SU(n). Denoting the elements (matrices) of SU(n) by x and by $\bar{x}$ the complex conjugation of x, we list the totality of all operators on the regular representation of SU(n) satisfying the requirement (a) of section 1:

$$A f(x) = f(x_1 x) \quad (2-1)$$

$$A f(x) = f(x_2 \bar{x}) \quad (2-2)$$

$$A f(x) = \bar{f}(x_3 x) \quad (2-3)$$

$$A f(x) = \bar{f}(x_4 \bar{x}) \quad (2-4)$$

Here $f(x)$ denotes a square integrable function on the group manifold (square integrable with respect to the invariant integral on the group) and x_1, x_2, x_3, x_4 are arbitrary but fixed elements of the group. The proof runs as follows. First we note, that a mulitplicative operator is completely determined, whence its action on one of the lowest dimensional representation has been given. This follows from the well known fact, that every representation may be reached by reducing Kronecker products of the n-plet. Now we make use of theorems 1 and 2 of reference[4]. A continuous map of SU(n) onto SU(n), that induces a unitary transformation of the n-plet onto itself is clearly of the form (2-1).

Because the operator A should map every representation onto another one, A may transform, besides the just mentioned case, the n-plet onto the n^*-plet. These transformations are given by (2-2). The same arguments apply to the antiunitary operators (2-3) and (2-4). Now the following is to be seen. The operator CPT (antiunitary, particles into antiparticles) should belong to category (2-3) with x_3 - identity. Operator C (unitary, particles into antiparticles) is a member of the family (2-2) and, last but not least, PT (antiunitary, particles into particles) will belong to the operators (1-4). It is CPT= C.(PT) and hence $x_2 \bar{x}_4 =$ identity. Therefore only x_2 is to be determined by further conditions and besides this we have to fix phases of vectors of the representations and between a representation and its conjugate one. Now we turn to do this for SU(6) having in mind, that it is sufficient to give all definitions for quark and antiquark states. Thanks to its multiplicative nature, the calculation of the operator's action on other representations is completely fixed and may be computed straightforwardly.

Remark: We do not consider P and T separately but only in the combination PT.

3. The Phase Convention and the CPT-Operator

We consider the following operators of SU(6): Iso-spin $\vec{I}$ L-spin (electromagnetic spin) $\vec{L}$, strange quark spin $\vec{S}$ and normal quark spin $\vec{N}$.

These operators already generate the total SU(6) Lie algebra. Therefore it is sufficient to restrict ourselves to them.

We first fix the phases of the quark states up to an overall phase by choosing a weight-ordered and normalized set

$$q_1 > q_2 > \dots > q_6 \qquad (3.1)$$

of quark states demanding

$$I_+ q_2 = q_1 , \quad N_+ q_4 = q_1 , \quad S_+ q_6 = q_3 , \quad L_+ q_3 = q_2 , \quad N_+ q_5 = q_2 , \qquad (3\text{-}2)$$

Now turning to the antiquarks we get a weight-ordered set

$$q'_1 > q'_2 > \dots > q'_6 \qquad (3\text{-}3)$$

of quark states by the requirement

$$I_+ q'_5 = q'_2 , \quad N_+ q'_6 = q'_3 , \quad S_+ q'_4 = q'_1 , \quad L_+ q'_2 = q'_1 , \quad N_+ q'_5 = q'_2 , \qquad (3\text{-}4)$$

Now we construct the operator

$$\theta = CPT \qquad (3\text{-}5)$$

in the following way:

θ is an antilinear multiplicative mapping with the properties

$$\theta q_k = (-1)^k q'_{7-k} , \qquad \theta q'_k = (-1)^k q_{7-k} . \qquad (3\text{-}6)$$

Remark 1: By explicit calculation one sees

$$\{\theta, \vec{I}\} = \{\theta, \vec{L}\} = \{\theta, \vec{N}\} = \{\theta, \vec{S}\} = 0$$

Hence the operator θ commutes with the SU(6) group and (because of its antilinearity) anticommutes with all of its generators. By the by we convince ourselves, that (3-4) is an antiquark representation.

Remark 2: The usual rule

$$\theta^2 = -1 \qquad (3\text{-}7)$$

is valid and appears to be a result of particle-antiparticle symmetry.

Remark 3: The definition (3-6) fixes the relative phases of quarks and antiquark and as consequence, those of particles and antiparticles.

We now denote by Λ the skew-symmetric product and use the following abbreviations

$$r_n = q_1 \Lambda q_2 \Lambda \ldots \Lambda q_{n-1} \Lambda q_{n+1} \Lambda \ldots \Lambda q_6 \qquad (3\text{-}8)$$
$$s_n = q'_1 \Lambda q'_2 \Lambda \ldots \Lambda q'_{n-1} \Lambda q'_{n+1} \Lambda \ldots \Lambda q'_6$$

From the point of SU(6) structures, the states (3-8) are found to be isomorphic to the antiquark and quark representation. We get therefore a consistency condition by setting

$$q'_n = a\, r_{7-n}$$
$$q_n = b\, s_{7-n} \qquad (3\text{-}9)$$

Applying the operator θ to equation (3-9) we see

$$ab = -1 \qquad (3\text{-}10)$$

Finally we mention

$$\theta\, q_1 \Lambda \ldots \Lambda q_6 = q'_1 \Lambda \ldots \Lambda q'_6 \;. \qquad (3\text{-}11)$$

4. Charge-Conjugation

We shall carry out the construction of the charge conjugation operator with respect to the iso-spin subgroup of SU(6). The result will indicate, that this covers the general case already. Our first assertion is as usual

$$[C, \vec{J}\,] = [C, I_3] = 0 \,, \qquad \{C, I_1\} = \{C, I_3\} = 0 \qquad (4\text{-}1)$$

Here $\vec{J} = \vec{N} + \vec{S}$ Our furst conclusion is

$$C\, q_1 = \mu\, q'_3 \,, \qquad C\, q_3 = \nu\, q'_1 \qquad (4\text{-}2)$$
$$C\, q_2 = -\mu\, q'_2 \,,$$

Because of the commutativity of C with $\vec{J}$ it is $C\, q_4 = \mu\, q'_6$ and so on. Now let us consider the equations (3-9) and let us apply (4-2) in order to calculate $C\, q'_k$. The result is

$$C\, q_1 = a\, b^{-1} \mu^4 \nu$$
$$C\, q_2 = a\, b^{-1} \mu^3 \nu^2 \qquad (4\text{-}3)$$
$$C\, q_3 = a\, b^{-1} \mu^3 \nu^2$$

Now the quark system has been fixed up to an overall phase. We can therefore change it by a phase factor such that

$$\mu = 1 \qquad (4-4)$$

holds. Now we calculate the square of C and find the quarks and antiquarks to have eigenvalue $ab^{-1}\nu^2$. However, C^2 should belong to the centre of SU(6). There are only two central elements of SU(6) giving equal values in the quark and in the antiquark representation and these are ± 1. Hence for quarks and antiquarks

$$C^2 = a b^{-1} \nu^2 = \pm 1 \qquad (4-5)$$

Now we have to decide between the two possibilities of (4-5). Assuming

$$q q' = - q' q \qquad (4-6)$$

for quark-antiquark states, we get for the π^0 -meson

$$\pi^0 = q_1 q'_6 + q_2 q'_5 - q_5 q'_2 - q_4 q'_8 \qquad (4-7)$$

of the 35-plet

$$C \pi^0 = a b^{-1} \nu^2 \pi^0 \qquad (4-8)$$

Hence we choose

$$a b^{-1} \nu^2 = 1 , \qquad C^2 = 1 . \qquad (4-9)$$

Now let us define C_0 to be the operator just constructed with $\nu = 1$ i.e.

$$C_0 q_8 = q'_1 , \; C_0 q'_1 = q_8 \; \ldots$$

Denoting by B the baryon number we have obtained the <u>result:</u> The most general charge-conjugation operator C satisfying the requirements a) and b) as well as equations (4-1) is

$$C = e^{i\pi\lambda(B-Y)} \, C_0 \qquad (4-10)$$

The real number λ satisfies

$$e^{2\pi i\lambda} \, a = b . \qquad (4-11)$$

5. Application to the 35-plet

Now we may calculate the behaviour under C of various SU(6) representations. Here we give the results for the 35-plet only (calculations of the 189-plet have been used /5/) to test the fit of the plus-partity mesons into this representation).

The states with vanishing hypercharge give the experimental established assignments: The $Q = Y = 0$ members have $C = +1$ for spin zero and $C = -1$ for the vector mesons. Fixing the relative phases of the K- mesons according to

$$I_+ K^- = \bar{K}^0 , \qquad I_+ K^0 = K^+ , \qquad \tfrac{1}{2} L_+ L_+ \bar{K}^0 = K^0 \tag{5-1}$$

it follows

$$C K^0 = -\bar{\nu} \bar{K}^0 , \qquad C K^+ = \bar{\nu} K^- , \qquad C \bar{K}^0 = -\nu K^0 , \qquad C K^- = \nu K^+ \tag{5-2}$$

For the spin one K- mesons we have to reverse the signs of the right hand side of (5-2) if the same phase conventions are used for them.

6. Charge Conjugation with Respect to other Subgroups

Let $\vec{T}$ be the generators of a SU(2) subgroup of SU(6) that commutes with $\vec{J}$. If the quark representation decomposes under $\vec{T}$ into doublets and singlets, there exists an element W of SU(6) that commutes with $\vec{J}$ and that obeys

$$\vec{T} = W \vec{I} W^{-1} \tag{6-1}$$

Hence we define the charge conjugation operator with respect to $\vec{T}$ to be

$$\tilde{C} = W C W^{-1} \tag{6-2}$$

Defining by

$$X = W Y W^{-1} \tag{6-3}$$

the $\vec{T}$ analogue of hypercharge, we get the general solution

$$\tilde{C} = e^{i\pi\tilde{\lambda}(B-X)} \cdot \tilde{C}_0 , \qquad \tilde{C}_0 = W C_0 W^{-1} . \tag{6-4}$$

Now let us consider θC_0. Within our phase conventions we have $(k = 1,2,3)$

$$\theta C_0 q_k = -q_{k+3} , \quad \theta C_0 q_{k+3} = q_k \tag{6-5}$$

However, W will commute with C_0, if it commutes with θC_0, for θ commutes with the group representation. So we get:

Lemma: Assume the possibility to choose W such that Wq_k is a real (real within our phase conventions) linear combination of the q_j. It follows

$$\tilde{\tilde{C}}_o = C_o.$$

As an application we calculate $\tilde{\tilde{C}}_{em}$ using the e.m. spin $\vec{L}$ and not the iso-spin. We may choose W $q_8 = q_1$, $W q_1 = q_2$ and $W q_2 = q_8$ and hence the lemma applies. Because the transformed of Y is $-$ Q, we have

$$C_{em} = e^{i\pi \tilde{\lambda}(B+Q)} C_o \qquad (6-6)$$

However, B and Q satisfy strict conservation laws and are central observables (i.e., they commute with all observables). Therefore the extra factor in (6-6) is of no importance at all. We get the usual definition by choosing

$$C_{em} = C_o \qquad (6-7)$$

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