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ON THE GENERAL STRUCTURE OF AN OPERATOR WHICH IS MULTIPLICATIVE WITH RESPECT TO A SYMMETRY GROUP

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I. Introduction

How should an operator A look like, that is to denote by the phrase "multiplicative with respect to a group G"? The answer is: Assume one has vectors $\dot{\omega}_1 \dot{\omega}_2 \dot{\omega}_3$ belonging to representations N_1, N_2, N_3 of the group such that N_3 is a part of the Kronecker product $N_1 \times N_2$. We call A a multiplicative one, if A_{ω_3} coincides (up to an overall constant factor) with the restriction of the operator

$$A(\omega_1 \times \omega_2) = A\omega_1 \times A\,\omega_2$$

on N_3.

Now we will find the general form of such an operator, for the indecomposable representations of an arbitrary local bicompact group which occur in its regular representation. There is no restriction for the compact Lie groups, and this covers the main series of the semi-simple non-compact Lie groups also.

2. The regular representation H

Let G be a local bicompact topological group /1/. We denote by H the Hilbert space of all measureable, square integrable and complex-valued functions with respect to the left-invariant integration /2/ on the group. If $f_1, f_2 \subset H$ then

$$(f_1, f_2) = \int_{x \subset G} \bar{f}_1(x)\, f_2(x)\, dx\,.$$

Here dx denotes the left-invariant measure on G. The map

$$f(x) \rightarrow f(x_0 x) \equiv V(x_0) f\,, \qquad x_0 \subset G$$

defines a unitary operator in the Hilbert space H and

$$y \rightarrow V(y)\,, \qquad y \subset G$$

defines a unitary representation of G in H that is called "the regular representation" of G.

3. The Continuous Functions of the Group

In H there is a special dense subset F_o which consists of all such functions $f(x)$ of H being (up to a point set of measure zero) continuous on G. On the other hand, F_o is a subset of the commutative symmetric algebra/3/ F consisting of all continuous functions on G. Identifying functions that differ only on point sets of measure zero, we have

$$F_o = H \cap F$$

The multiplication

$$(f_1 f_2)(x) = f_1(x) . f_2(x)$$

is indeed the Kronecker product of the regular representation of G by itself. This is true because of the following general definition. The Kronecker product of two representations $x \to U_1(x)$ and $x \to U_2(x)$ with $x \subset G$ is the restriction of the direct product representation

$$x \times y \to U_1(x) \times U_2(y)$$

of $G \times G$ on the elements $x \times x$. Finally the last mentioned elements of $G \times G$ will be identified with the elements of the group G.

4. Multiplicative operators

Let A be a bounded operator of the Hilbert space H. The operator A is said to be a multiplicative one, if and only if there exists an operator $\tilde{A}$ which maps F into F having the properties

a) $A f = \tilde{A} f$ if $f \subset F_o = H \cap F$

b) $\tilde{A} (f_1 + f_2) = \tilde{A} f_1 + \tilde{A} f_2$,
$\tilde{A} (f_1 f_2) = \tilde{A} f_1 . \tilde{A} f_2$, for all $f_1, f_2 \subset F$

Note that property b) implies: Given A on some subsets of the regular representation, we can extend A with the help of the Kronecker product.

Fortunately it is possible to give the general structure of multiplicative operators more explicitly. First, because F has a unit element, $\tilde{A}$ induces an automorphism of the field of the complex numbers. An automorphism of the field of complex numbers is either the identity or the complex conjugation. Therefore $\tilde{A}$ is either an automorphism or an involution of the algebra F. From the theory of symmetric commutative algebras one knows, that every automorphism (involution) is induced by a continuous mapping r of the carrier space

$$x \to x^r, \qquad x \subset G$$

Hence we have

Theorem 1: Let A be a multiplicative operator from H into H. There exists a continuous mapping r from G into G with either $A f(x) = f(x^r)$ (all $x \subset G$) or $A f(x) = \overline{f(x^r)}$ (all $x \subset G$).

(The bar stands for complex conjugation). Clearly in the first case A is linear and in the second-antilinear. From theorem 1 and the form of the scalar product in H we further conclude:

Theorem 2: The multiplicative operator A transforming H into H has an inverse, if the inducing transformation r maps G one-to-one onto G. If furthermore the left-invariant measure is stable under r

$$d x^r = d x ,$$

A is a unitary or antiunitary operator.

Finally we add the following remark. In general, a unitary multiplicative operator will not-- map an irreducible representation onto another representation. In general, such an operator maps an irreducible representation onto a part of a reducible one. The requirement, that a multiplicative operator only permutes the representations, thus mapping every irreducible representation onto another representation, is a very restrictive condition. For the compact, simple and simply connected Lie groups of the four main types these multiplicative operators have been given by Biedenharn, Nuyts und Ruegg/4/, however, they used the point of view of group extensions.

References

1. See i.g. Pontrjagin. Topologische Gruppen, Leipzing, 1957. Every Lie goup is local bicompact.
2. See e.g. Neumark. Normierte Algebren, Berlin, 1960. On a local bicompact group there is (up to constant factor) one and only one left invariant integral.
3. See ref. 2).
4. Biedenharn, Nuyets and Ruegg, CERN, 65-3.

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