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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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## Introduction

In modern theory the properties of the scattering matrix are formulated on the basis of local theory. However, the physical meaning of the scattering matrix, as was pointed out long ago by Heisenberg /1/, can essentially be beyond the narrow framework of local theory x/. This is also seen from the axiomatic approach in which the ambiguity of extrapolating the scattering matrix off the mass surface $p_0^2 = \vec{p}^2 + m^2$ /3/ is clearly displayed.

Whatever this extrapolation is, a direct physical meaning is kept only by the scattering matrix $S$ on the mass surface. Therefore we apply the causality condition only to this quantity which is physically defined, and we call it the condition of macroscopic causality unlike the condition of microscopic causality associated with the notion of local field.

The application of the causality condition to the scattering matrix meets with a difficulty that the scattering matrix transforms the states for $t = -T$ into the states for $t = +T$ at $T \to \infty$. During the time $2T$ the waves fill the whole space. Therefore a stationary state arises which, in its very essence, excludes the conditions necessary for the causal connection to be formulated.

In 2 we will show that it is still possible to construct wave packets which allow a reasonable formulation of the macrocausality conditions and which are compatible with an interpretation of the limit $T \to \infty$ such that the terms of the order $\frac{1}{R}$ ( $R = vT$, , $v$ - is the packet velocity) are assumed to be still finite while the terms of the order $1/R^2$ and higher are neglected. By means of such packets we may formulate the conditions of macrocausality which is thought of as the usual causal connection characteristic of the relativistic metric; events at the points $\mathcal{P}(x')$ and $\mathcal{P}(x'')$ may be causally connected provided only that a) the interval $(x'' - x')^2$ is a time interval, i.e. $(x'' - x')^2 \geq 0$ and b) the event at $\mathcal{P}(x')$ (cause) precedes the event at $\mathcal{P}(x'')$ (consequence) so that $t'' > t'$.

In 3 an example of the unitary acausal scattering matrix satisfying this macrocausality condition is given.

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x/ See also /2/.

## 2. Formulation of the Macrocausality Conditions

Let us consider the two wave packets a and b which at $t_1 = -T$ are going out of the diaphragms A and B (see Fig. 1). Such a description of the packet "creation" simulates most closely the real situation in experiment. Somewhat later a collision of the packets can occur, but for $t_2 = +T$ they fly apart. Let for $t_1 = -T$ the packets be at a distance R which is much larger than the size of the packets $L (R \gg L)$ . We shall assume that the packets have a sufficiently definite momentum p , so that $p \gg \Delta p = \hbar/L$ . Now we require that the packets would not spread considerably during the time 2T i.e. an increase of the packet width $\Delta L$ must not be large as compared with the initial one L . The dispersion of the packet velocity $\Delta v$ is

$$\Delta v = \frac{\partial^2 E}{\partial p^2} \Delta p = \frac{\Delta p}{E} \frac{m^2}{E^2} \tag{1}$$

( m is the particle mass). So, we have $\Delta L = \Delta v \cdot T = \frac{\Delta p}{p} \frac{m^2}{E^2} R$ . From the condition $L \gg \Delta L$ we get

$$L > \frac{m}{E} \sqrt{R \lambda\!\!\!^{-}} , \tag{2}$$

where $\lambda\!\!\!^{-} = \hbar / p$ is the wave length. The condition (2) is compatible with the condition

$$R \gg L \gg \lambda\!\!\!^{-} \tag{3}$$

if $R \gg \frac{\Lambda_0^2}{\lambda\!\!\!^{-2}} \lambda\!\!\!^{-}$ at $\lambda\!\!\!^{-} < \Lambda_0$ , or if $R \gg \frac{\lambda\!\!\!^{-2}}{\Lambda_0^2} \lambda\!\!\!^{-}$ at $\lambda\!\!\!^{-} > \Lambda_0$ here $\Lambda_0 = \hbar/mc$

Thus, there are packets which can be used as in-states transformable to out-states by the S -matrix:

$$\langle f|S|i\rangle = \delta_{fi} - (2\pi)^4 i \delta^4 (p_f - p_i) \langle f|T|i\rangle , \tag{4}$$

where, as usual (i) denote the quantum numbers of the in-state, and (f) are those for the out-state. The matrix element $\langle f|T|i\rangle$ can be represented a more detailed form:

$$\langle f|T|i\rangle = \frac{\langle p_m, p_{m-1}, \dots p_{n+1} |I| p_n, \dots p_1 \rangle}{\sqrt{2p_{0m} \, 2p_{0m-1} \dots 2p_{01}}} , \tag{5}$$

where $\langle p_m, p_{m-1}, \dots p_{n+1} |I| p_n, \dots p_1\rangle$ is the invariant function of the four-momenta $p_m, p_{m-1}, \dots p_1$ and $p_{0m}, p_{0m-1}, \dots p_{01}$ are their fourth components. In what follows, we shall restrict ourselves to the simplest case of the pairing collision of two



particles, when in the initial state (i) there are only two particles described by the wave packets $u_1(x_1)$ and $u_2(x_2)$ of the above considered type. These packets can be represented in the form of the integrals:

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int \tilde{u}(\vec{p}) e^{ipx} \frac{d^3 p}{2p_0} , \tag{6}$$

where $p_0 = + \sqrt{\vec{p}^2 + m^2}$ . The wave function of the initial state in the momentum representation will be of the form:

$$\Phi_{in}(p_2 p_1) = \frac{\tilde{u}_2(\vec{p}_2)}{\sqrt{2p_{02}}} \frac{\tilde{u}_1(\vec{p}_1)}{\sqrt{2p_{01}}} . \tag{7}$$

From (4), (5), (7) we get:

$$\Phi_{out}(p_m, p_{m-1}, \dots p_3) = - (2\pi)^4 i \int \delta^4 (p_m + p_{m+1} + \dots + p_3 - p_2 - p_1) \times \tag{8}$$

$$\times \frac{\langle p_m, p_{m-1}, \dots p_3 |I| p_2, p_1 \rangle}{\sqrt{2p_{0m} 2p_{0m-1} \dots 2p_{03}}} \tilde{u}_2(\vec{p}_2) \tilde{u}_1(\vec{p}_1) \frac{d^3 p_2 d^3 p_1}{2p_{02} 2p_{01}} .$$

and for m = 4 (the elastic collision):

$$\Phi_{out}(p_4, p_3) = \Phi_{in}(p_4, p_3) - (2\pi)^4 i \int \delta^4 (p_4 + p_3 - p_2 - p_1) \times$$

$$\times \frac{\langle p_4 p_3 |I| p_2 p_1 \rangle}{\sqrt{2p_{04} \, 2p_{03}}} \tilde{U}_2(\vec{p}_2) \tilde{U}_1(\vec{p}_1) \frac{d^3 p_2 \, d^3 p_1}{2p_{02} \, 2p_{01}} . \tag{8'}$$

Now we go over to the coordinate representation. For this we multiply the left-hand side of (8) by

$$\frac{1}{(2\pi)^{3/2 (m-2)}} \frac{\exp i(p_m x_m + p_{m-1} x_{m-1} + \dots + p_3 x_3)}{\sqrt{2p_{0m} \, 2p_{0m-1} \dots 2p_{03}}}$$

and integrate over $d^3 p_m, d^3 p_{m-1} \dots d^3 p_3$ . Further by (6) we express $\tilde{u}(\vec{p})$ in terms of $u(x)$:

$$\frac{\tilde{u}(\vec{p})}{2p_0} = \frac{1}{(2\pi)^{3/2}} \int u(x) e^{-ipx} d^3 x . \tag{5'}$$

Then from (8) we get:

$$\Phi_{out}(x_m, x_{m-1}, \dots, x_3) = -(2\pi)^4 i \int g(x_m, x_{m-1}, \dots, x_3 | x_2, x_1) \, u_2(x_2) u_1(x_1) d^3 x_2 \, d^3 x_1 \tag{9}$$

and in a similar way from (8')

$$\Phi_{out}(x_4, x_3) = \Phi_{in}(x_4, x_3) - (2\pi)^4 i \int g(x_4, x_3 | x_2, x_1) \quad u_1(x_1) u_2(x_2) d^4x_2 d^4x_1 \quad (9')$$

In this case we have

$$g(x_m, x_{m-1}, \cdots, x_3 | x_2, x_1) = \tag{10}$$
$$= \int \delta^4 (p_m + p_{m-1} + \cdots + p_3 - p_2 - p_1) < p_m, p_{m-1} \cdots p_3 | I | p_2, p_1 > \times$$
$$\frac{\exp i(p_m x_m + \cdots + p_3 x_3 - p_2 x_2 - p_1 x_1)}{2p_{0m} 2p_{0m-1} \cdots 2p_{03}} d^3p_m \; d^3p_{m-1} \cdots d^3p_1$$

or

$$g(x_m, x_{m-1}, \cdots, x_3 | x_2, x_1) = - \frac{4\partial^2 g_0(x_m, x_{m-1}, \cdots x_3 | x_2, x_1)}{\partial t_2 \partial t_1}, \tag{11}$$

where $g_0$ is the invariant function of the coordinates

$$g_0(x_m, x_{m-1}, \cdots, x_3 | x_2, x_1) = \int \delta^4 (p_m + p_{m-1} + \cdots + p_3 - p_2 - p_1) \times \tag{12}$$
$$<p_m, p_{m-1}, \cdots p_3 | I | p_2, p_1> \exp i(p_m x_m + \cdots + p_3 x_3 - p_2 x_2 - p_1 x_1) \times$$
$$\frac{d^3p_m d^3p_{m-1} \cdots d^3p_1}{2p_{0m} 2p_{0m-1} \cdots 2p_{01}}.$$

We notice that due to the presence of the $\delta$ function under the integral in $g$ and $g_0$ these functions are translation-invariant and depend only on the difference of the variables $x_m, x_{m-1}, \cdots x_1$. No we may formulate the principle of macrocausality: a) the wave packets $u_2(x_2)(\Delta x_2 \sim L)$ and $u_1(x_1)(\Delta x_1 \sim L)$ removed apart at the distance

$$|\vec{x}_2 - \vec{x}_1| = |\vec{x}| = R > L >> \lambda \tag{13}$$

contribute to $\Phi_{out}$ provided only that

$$x^2 = (t_2 - t_1)^2 - (\vec{x}_2 - \vec{x}_1)^2 > 0. \tag{14}$$

b) Further $\Phi_{out} = 0$ if the coordinates of the particles $x_m, x_{m-1}, \cdots x_3$ created in the collision lie out of the future light cone with respect to the points $x_2, x_1$:

$$(x_s - x_2)^2 > 0, \quad (x_s - x_1)^2 > 0, \tag{15}$$



$$t_s > t_2, \quad t_s > t_1, \tag{15'}$$

$s = m, m-1, \cdots 3$. Thus the function $g(x_m, x_{m-1}, \cdots x_3 | x_2, x_1)$ must consequently vanish outside the above-mentioned space-time regions, however, only asymptotically, i.e. for

$$R \to \infty, \quad (t_s - t_2), (t_s - t_1) \to \infty. \tag{16}$$

From the physical point of view these conditions are identical with the requirements of classical macroscopic causality and imply the assumption that all the particles in the final state $\Phi_{out}$ can be produced (or change their state) later than the initial packets exchange the field quanta (see Fig. 2).

The usual local theory satisfies, of course, the above stated requirement of macrocausality (for example see Appendix A). This requirement will be satisfied also by any scattering matrix in which the macrocausality is violated only in a small localized space-time region.

In 3 we give an example of the acausal unitary scattering matrix obeying the requirements of macroscopic causality.

## 3. Acausal Scattering Matrix

Now we turn to a formal construction of the nonlocal scattering matrix, obeying the requirements of unitarity and macroscopic causality.

We represent the scattering matrix $S$ in the form:

$$S = \frac{1 - (i/2)K}{1 + (i/2)K}, \tag{17}$$

where $K$ is the Hermitean matrix, i.e.

$$K = K^+. \tag{18}$$

This provides the unitarity of the considered matrix. To study the structure of the $S$-matrix we shall assume that there is a small parameter which permits to expand our scattering matrix in a power series in this parameter

$$S = \sum_{n=0}^{\infty} a_n (iK)^n, \tag{19}$$

where $a_n$ are the real numbers. After singling out the invariant functions the matrix $\tilde{F}$ can be written in the form

$$K = \frac{\tilde{F}(p_1,\cdots p_n | p_{n+1},\cdots p_m)}{\sqrt{2p_{01} 2p_{02} \cdots 2p_{0m}}} , \qquad (20)$$

where $p_{0i} = +\sqrt{\vec{p}_i^{\,2} + m^2}$. We shall consider only scalar particles. Owing to the CPT theorem, we have for the matrix elements $K$:

$$K(p_1,\cdots p_n | p_{n+1},\cdots p_m) = K(p_{n+1},\cdots p_m | p_1,\cdots p_n) \qquad (21)$$

and taking into account (18):

$$K(p_1,\cdots p_n | p_{n+1},\cdots p_m) = K^*(p_1,\cdots p_n | p_{n+1},\cdots p_m) . \qquad (22)$$

This means that the function must be real. Now we note the following properties of the functions

$$\tilde{F}(p_1,\cdots p_m) = \frac{1}{(2\pi)^{3/2m}} \int F(x_1, x_2,\cdots x_m) \exp(i \sum_{j=1}^{m} p_j x_j) \prod_{j=1}^{m} d^4 x_j \qquad (23)$$

in this case

$$F(x_1,\cdots x_m) = F(-x_1, -x_2, \cdots -x_m) . \qquad (24)$$

Further $F(x_1,\cdots x_m)$ is translation - invariant. In particular, it can be represented as a function of the variables $\xi_j = x_j - x_{j+1}$, $j = 1,2,\cdots m-1$. Then we may write down (23) in the form:

$$\tilde{F}(p_1, p_2,\cdots p_m) = \delta^4(p_1 + p_2 + \cdots + p_m) \times \qquad (23)$$

$$\frac{1}{(2\pi)^{3/2(m-1)}} \int F(\xi_1, \xi_2,\cdots \xi_{m-1}) \exp(i \sum_{j=1}^{m-1} Q_j \xi_j) \prod_{j=1}^{m-1} d^4 \xi_j .$$

Now we turn to the macrocausality condition and, for the sake of definitness, we restrict ourselves to the simplest case of the elastic collision. Basing on (19) we have:

$$\langle p_1, p_2 | S | p_3, p_4 \rangle = I - i \frac{\tilde{F}(p_1, p_2, p_3, p_4)}{\sqrt{2p_{01} 2p_{02} 2p_{03} 2p_{04}}} +$$

$$+ \int \frac{\tilde{F}(p_1, p_2, p', p'') \tilde{F}(p', p'', p_3, p_4)}{\sqrt{2p_{01} \cdots 2p_{04}}} \times \qquad (25)$$

$$\times \delta(p'^2 - m^2) \theta(p'_0) \delta(p''^2 - m^2) \theta(p''_0) d^4 p' d^4 p'' .$$



In the coordinate representation we have:

$$S(x_1, \cdots, x_4) = 1 - i F(x_1, \cdots, x_4) +$$

$$+ \iint F(x_1, x_2, x', x'') D^+(x'-y') D^+(x''-y'') \times \qquad (26)$$

$$\times F(y', y'', x_3, x_4) d^4x' d^4x'' d^4y' d^4y'' + \cdots$$

In the ordinary local theory the function $S(x_1,\cdots x_4)$ being based on the microcausality, satisfies the requirements of macrocausality (14,15) (cf. Appendix A). It has singularities on the light cone with respect to the variables $\xi_1 = x_1 - x_2$, $\xi_2 = x_2 - x_3$, $\xi_3 = x_3 - x_4$; the nature of these singularities is essentially related to causality. The same may be said about the functions $S$ with a larger number of arguments. We denote the corresponding functions of the local theory by $F_0(\xi_1, \xi_2, \cdots \xi_m)$.

We do not violate the macrocausality condition (14), (15) if, instead of the causal functions $F_0(\xi_1, \xi_2, \cdots, \xi_m)$, we introduce the acausal ones $F_a(\xi_1, \xi_2, \cdots, \xi_m)$ which will differ from $F_0(\xi_1, \xi_2, \cdots \xi_m)$ only in a small space-time region near the vertex of the light cone $\xi_j^2 = 0$, $\Omega(\xi_j) \sim a^4$. The quantity $a$ plays the role of an "elementary length". The functions $F_a(\xi)$ possessing such properties, as was shown in ref./4/, can be constructed by averaging the possible singularities of the function $F_0(\xi)$ near the light cone vertex $\xi^2 = 0$:

$$F_a(\xi) = \int F_0(\xi - \xi') \rho(\xi', n) d^4 \xi' , \qquad (27)$$

where $\rho(\xi', n)$ is the weighting function (formfactor), by means of which we average the singularities in that space-time region where the usual causality and the usual geometry may be violated.

The weighting function $\rho(\xi, n)$ depends on some time-like vector $n$, by means of which the domain $\Omega(\xi) \sim a^4$ is determined in an invariant manner. In particular, $\rho$ may be assumed to be a function of the invariant $R$:

$$R^2 = 2(\xi n)^2 - \xi^2 \geq 0 \qquad (28)$$

and $\rho(R) \to 0$ for $R \gg a$ (cf./4/).

The physical meaning of the vector $n$ may be different and is discussed in detail in ref./5/. In principle, two types of the vectors $n$ are conceived: the first one, when the vector $n$ is connected with a system of interacting particles ("internal" vector $n$)x/.

x/ For the connection of our scheme with the usual nonlocal theory, see Appendix B.

In this case a violation of geometry occurs only inside the system of interacting particles for extremely small distances and time intervals. Another possibility is that the vector $a$ is related to the physical vacuum ("external" vector $a$). In this case one of the frames of reference, namely the frame of reference of the "physical vacuum", turns out to be singled out (cf./6,7/).

To summarize, it should be stressed that we consider the introduced averaging of the singularity near the vertices of the light cones only as a tool of a formal description of the situation at small scales which may be very different from the well-known one in contemporary theory.

## APPENDIX A

For the simplest case of the point interaction $W = \lambda \phi^4$ the function $\tilde{F}(p_1, \cdots p_4)$ (see (25)) is simply equal to $\delta^4(p_1 + p_2 - p_3 - p_4)$. Therefore in the first approximation the function $S(x_1, \cdots x_4)$ is

$$S(x_1, \cdots x_4) = \lambda \int \exp i(p_1 x_1 + p_2 x_2 - p_3 x_3 - p_4 x_4) \times \qquad (1)$$

$$\times \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3p_1 \, d^3p_2 \, d^3p_3 \, d^3p_4}{2p_{01} 2p_{02} 2p_{03} 2p_{04}} .$$

We introduce the variables

$$k = p_1, \quad u = p_1 + p_2 - p_3, \quad \xi = x_1 - x_2, \quad \xi_3 = x_3 - x_4, \qquad (2)$$

$$q = p_1 + p_2, \quad k_4 = p_1 + p_2 - p_3 - p_4, \quad \xi_2 = x_2 - x_3$$

then

$$S(x_1, \cdots x_4) = \int \exp i(\xi k + \xi_2 q + \xi_3 u + x_4 k_4)\theta(k_0)\delta(k^2 - m^2) \times$$

$$\times \theta(q_0 - k_0)\delta[(q-k)^2 - m^2]\theta(q_0 - u_0)\delta[(q-u)^2 - m^2] \cdot \times \qquad (3)$$

$$\times \theta(u_0 - k_{04})\delta[(u - k_4)^2 - m^2]\delta^4(k_4) d^4k \, d^4u \, d^4q \, d^4k_4$$

By integrating we get

$$S(\xi, \xi_2, \xi_3) = \int D^+(\xi - x) D^-(x) \int D^+(x + \xi_2 - y) D^+(\xi_3 + y) d^4y \, d^4x . \qquad (4)$$

Since we are interested in the dependence of $S$ on the variable $\xi$, then making the replacement

$$\xi_3 + y = \alpha$$

$$x + \xi_2 + \xi_3 = \beta \qquad (5)$$

$$\xi_2 + \xi_3 = \epsilon$$



we obtain

$$S(\xi) = \int D^+(\xi + \epsilon - \beta) D^-(\beta - \epsilon) \int D^+(\beta - \alpha) D^+(\alpha) d^4\beta \, d^4\alpha . \qquad (6)$$

From eq.(6) it is seen that outside the forward light cone the function $S(\xi)$ exponentially decreases and turns out to be important only along the Compton wave length.

## APPENDIX B

If we assume the weighting function $\rho(\xi, a)$ being independent of the vector $a$ then $\rho$ will be a function of only $\xi^2$ : $\rho = \rho(\xi^2)$ and coincides with the form factor of nonlocal theory/8/.

The role of the vector $a$ (which is necessary for the localization of acausality) is now played by the momentum vector or by a set of such vectors connected with the wave packets; i.e. the vector $a$ is in this case taken from the original data (i.e. from $\Phi_{in}$). As 0was shown in ref./9/, in doing so, we may ensure macrocausality only for sufficiently smooth wave packets. For very narrow wave packets macroscopic causality will be violated. Indeed, we consider a wave packet which corresponds to the quantum transition from the state $\psi_p(x,t) = e^{ipx} \phi_p(x - vt)$ to the state $\psi_{p'}(x,t) = e^{ip'x} \phi_{p'}(x - vt)$.
The current density for the transition $p \to p'$ is

$$J_{4e} = \exp[i(p - p', x)] \phi_p(x - vt) \phi^*_{p'}(x - vt) . \qquad (1)$$

Its nonlocal image is

$$J_{4a} = \int \rho(\tilde{s}^2) J_{4e}(x', t') dx' dt' , \quad \text{where} \quad \tilde{s}^2 = (t - t')^2 - (x - x')^2 . \qquad (2)$$

If the wave packets $\psi_p(x,t)$ , $\psi_{p'}(x,t)$ are very sharp (and $\delta$ - shaped in the limiting case) then $J_{4e}(x,t)$ is nonzero only at the point $x = 0$, $t = 0$ (the point of collision of the packets). So, we may assume:

$$J_{4e}(x,t) = \exp[i(p - p', x)] \delta(x) \delta(t) . \qquad (1')$$

Then from (2) we have:

$$J_{4a}(x,t) = \rho(t^2 - x^2) \qquad (3)$$

and causality is essentially violated because $\rho(t^2 - x^2) \neq 0$ for $t = \pm |x|$ i.e. for any whatever large $|x|$.

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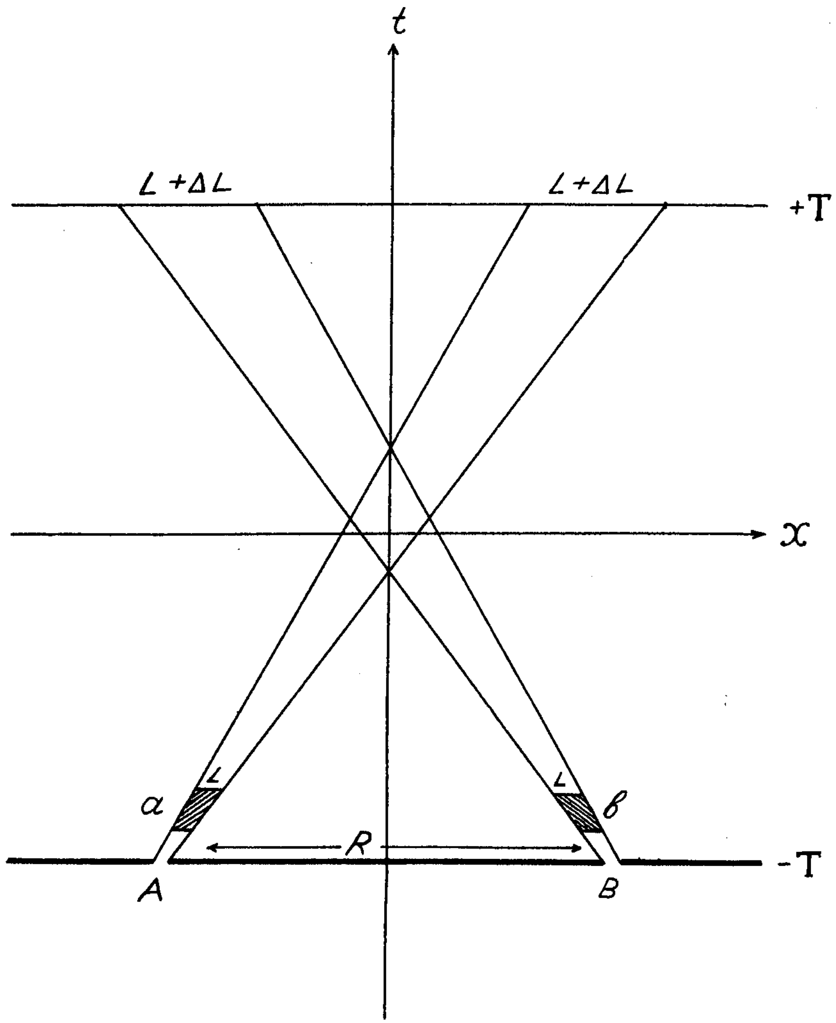
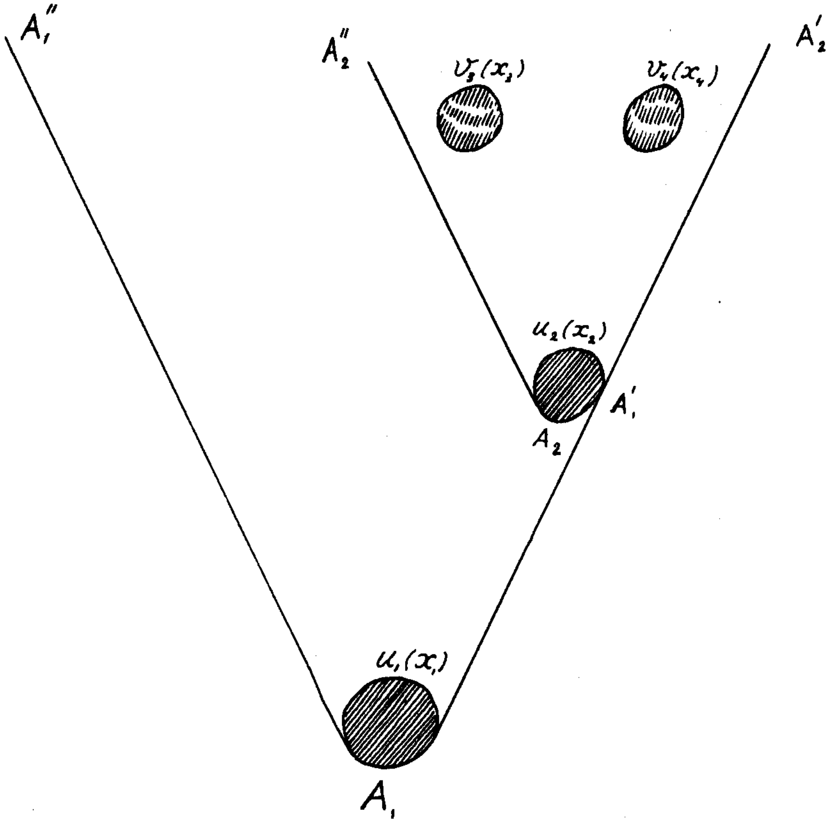



Fig. 1. A and B are the diaphragms; a and b are the wave packets of the initial state (the in-state). L is the initial size of the packets, R is the distance between them at the momentum $t = -T$ , $\Delta L$ is the increase of the dimensions of these packets during the time 2T.




F i g. 2. Relations of causality: $u_1(x_1)$, $u_2(x_2)$ are the initial packets (the in-state) $v_3(x_3)$, $v_4(x_4)$ are the scattered waves of the out-state $A_1' A_1 A_1''$, $A_2' A_2 A_2''$ are the light cones.