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# TIME AND ENERGY IN QUANTUM MECHANICS

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## Introduction

The present paper is devoted to some problems related to the role of time in quantum mechanics:

1. Is time an operator or a parameter?

2. Is energy conserved in time?

3. What is the meaning of the uncertainty relation $\Delta E \Delta t \approx h$ ?

As far as these problems are concerned there is a great deal of opinions often contradictory. In order to sketch the situation we shall give some of them.

From the Plank-DeBroglie relation $E = h\omega$ and the fact that a finite time is needed to measure the frequency there follows the relation $\Delta E \Delta t \approx h$ /1,2/. Suitable gedanken experiments are known /3/. In generalizing relativistically the commutation relations $[x_i, x_j] = i\delta_{ij}$ ; $i, j = 1, 2, 3$; we obtain also the relation $[x_0, p_0] = -ih$.

However, in spite of this, predominant is the opinion that time is not an operator, but a parameter. It is based on the fact that the energy operator in quantum mechanics is not $ih\, \partial/\partial t$ but a Hamiltonian which is a function of only momenta and coordinates commuting with $t$ /2,4/. Such an operator may have a discrete eigenvalue spectrum (in accordance with our experience), while from $[E, t] = ih$ it follows that the energy must have only a continuous spectrum /5/ (in a similar way as from $[x, p_x] = ih$ it follows the continuity of the $p_x$ spectrum /6/). Our interpretation of the wave function implies that time may be only a parameter: $\int |\Psi(x, t)|^2 d^3x$ may not be considered as the probability for the system to be at the point $t$. On the contrary, the probability $\int |\Psi(x, t)|^2 d^3x$ must be a time-independent constant (conservation of the normalization).

In the present paper a general point of view is suggested which naturally includes the listed aspects of the problem (as well as some others not mentioned yet). It allows one to discuss the role of time from apparently more general point of view than this is done in the papers by Mandelstam and Tamm[2], Fock and Krylov[7], Aharonov and Bohm[8].

## I. Time as an Operator and as a Parameter

First, we shall consider a system consisting of one free particle. The all possible states of it can be obtained from any one state by displacements, rotations and Lorentz transformations (as well as by superposition of the obtained states). This fact underlies the application of the theory of the inhomogeneous Lorentz group representations (further we call it the Poincaré group PG) for describing the state of a particle [9]. In particular, the particle physical operators are the PG generators (or can be constructed from them). The displacement generators $P_\mu$ and the four-dimensional rotation ones $M_{\mu\nu}$ must obey the well-known commutation relations (the method of their derivation is presented, e.g., in ref.[10]):

$$[P_\mu , P_\nu] = 0; \quad [M_{\mu\nu}, P_\lambda] = i(P_\nu \delta_{\mu\lambda} - P_\mu \delta_{\nu\lambda});$$

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i(\delta_{\mu\sigma} M_{\lambda\nu} + \delta_{\mu\lambda} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\lambda} + \delta_{\nu\lambda} M_{\sigma\mu}). \tag{1}$$

Here and in what follows we take the system of units in which $\hbar = 1$ and $c = 1$; $\mu, \nu = 1, 2, 3, 4$.

The commutation relations (1) do not determine unambiguously $P_\mu$ and $M_{\mu\nu}$; different representations for them are possible. The ordinary representation in the simplest case (spinless particle) is got by introducing the operators $x_\mu = (\vec{x}, it)$ and $p_\mu = (\vec{p}, iE)$ satisfying the commutation relations

$$[x_\mu , p_\nu] = i\delta_{\mu\nu} \tag{2}$$

If now we put

$$P_\mu = p_\mu ; \quad M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu , \tag{3}$$

then (1) are satisfied. But the representation (3) is, generally speaking, reducible. This statement will be clear somewhat below. As for the particle (more exactly, the elementary system[9]), it must be described by an irreducible PG representation with definite value of $\sum_\mu p_\mu p_\mu = p^2$ (and by the spin value in the general case). It is most easy to consider this statement as a postulate[1] (further we shall see that it is equivalent to the requirement that the wave function should obey the Schrödinger equation).

Therefore the particle wave function must belong to the space $M$ of functions obeying the equation

$$(p^2 + m^2)\Psi_m = 0 . \qquad (4)$$

The condition for the norm to be positive (or the condition for the particle energy to be positive) gives in fact that the wave function obeys the equation of the first order in time (for example, $i\partial\Psi_m/\partial t = \sqrt{\vec{p}^{\,2} + m^2}\,\Psi_m$ for the scalar case see[11], (3.10); or the Dirac equation $i\partial\Psi_m/\partial t = (\vec{\alpha}\vec{p} + \beta m)\Psi_m$ for spin 1/2), i.e. the relativistic Schrödinger equation.

In the space $M$ it is possible to indicate another representation of $P_\mu$ and $M_{\mu\nu}$, namely:

$$P_i = p_i \; ; \quad P_0 = \sqrt{\vec{p}^{\,2} + m^2}$$
$$M_{ij} = x_i p_j - x_j p_i \; ; \quad M_{i,0} = \tfrac{1}{2}(x_i\sqrt{\vec{p}^{\,2} + m^2} + \sqrt{\vec{p}^{\,2} + m^2}\, x_i) - tp_i \qquad (5)$$

It can be verified that eq. (1) is satisfied as before, but now all the operators (5) commute with t. Since they exhaust all the physical operators, according to our fundamental assumption, t may be considered as a parameter,. Note that the function $t\Psi_m$ does not belong to $M$ and therefore $[t, E]\Psi_m$ does not reduce to $[t, \sqrt{\vec{p}^{\,2} + m^2}]\Psi_m = 0$.

In his paper on various forms of dynamics[12] Dirac has formulated and solved an inverse problem: how to change the representation (3) for $P_\mu$ and $M_{\mu\nu}$ so that new generators would commute with t. The solution of this problem leads again to the condition (4) and is of the same form (5). This approach together with the previous one makes the condition (4) necessary and sufficient for time to become a parameter.

In the same paper Dirac indicated that one can obtain that the proper time

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[1] The postulate can be commented as follows. As is known, the wave functions of representations with different $p^2$ must be orthogonal: $(\Psi_{m_1}, \Psi_{m_2}) = 0$ where $m_1^2$ and $m_2^2$ are different eigenvalues of $-p^2$. As far as all the PG generators commute with $p^2$, we have also $(\Psi_{m_1}, P_\mu\Psi_{m_2}) = 0$ and $(\Psi_{m_1}, M_{\mu\nu}\Psi_{m_2}) = 0$. If there is not a single particle operator which could not be constructed from $P_\mu$ and $M_{\mu\nu}$ (to this postulate reduces our original postulate) then the wave function $\Psi = \Sigma a_m \Psi_m$ of the reducible PG representation will behave as a mixed assembly of particles with different masses[9]. Namely the norm of $\Psi$ is broken up into the sum of the norms $\Sigma|a_m|^2(\Psi_m, \Psi_m)$; the matrix elements of any physical operator are broken up in the same way, so that the state in fact is described by the density matrix of the type $\Sigma|a_m|^2\Psi_m\Psi_m^*$.

$\tau$ (instead of $t$), $\tau^2 = t^2 - x^2$, will be a parameter. In the same way as $t$ numbers the planes parallel to $xyz$, $\tau$ numbers the hyperboloids $t^2 - x^2 = \tau^2$ which at $t^2 - x^2 > 0$ are space-like surfaces[2]. This remark is given here because it can illustrate additionally the general viewpoint of the present paper and some possible modifications of it.

So, time is a parameter in the $M$ - space and an operator outside it. In the latter case it is possible to get formally from $[t, E] = -i$ the corresponding uncertainty relation. As is known, to do this it is necessary only to define the norm or the scalar product $(\Psi_1, \Psi_2)$ of wave functions not belonging to $M$. We may assume the invariant expressions

$$\int d^4x\, \Psi_1^*(x_\mu)\Psi_2(x_\mu) \qquad \text{and} \qquad \int d^4p\, \phi_1^*(p_\mu)\phi_2(p_\mu) \tag{6}$$

in the coordinate and momentum representations respectively[3]. Let us stress that the interpretation of the wave function $\Psi(x,y,z,t)$ not obeying equation (4) is unclear. Indeed, $\int |\Psi(x,y,z,t|^2 d^3x$ may even vanish for some values of $t$, so the usual interpretation (see Introduction) is not suitable.

Note that the concept of function spaces wider than $M$ is necessary in the apparatus of the theory. For example, before writing eq. (4) it is necessary to indicate the class of functions in which the action of each operator $p_\mu^2$ (including $\partial^2/\partial t^2$) is defined.

Physical systems consisting of interacting particles or fields are characterized not only by the quantities $P_\mu$ and $M_{\mu\nu}$ (total momentum and angular momentum) but also by other operators. Following Dirac /12/ the construction of the relativistic dynamics for a system is generally reduced to finding $P_\mu$ and $M_{\mu\nu}$ such that they satisfy (1) and at the same time commute with t. To show that other operators commute with t, a concrete consideration is needed.

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[2] In ref. /12/ the consideration is made in the framework of the classic (nonquantum) mechanics. The author has made the corresponding consideration in a quantum case. In contrast to (5) it turns out to be necessary to change the representation of the operators $P_\mu$ ; $M_{\mu\nu}$ may be left the same as in (3). The commutation relations (1) must again be fulfilled in the sense $[\tilde{P}_\mu, \tilde{P}_\nu]\Psi_m = 0$ and so on. The form of new representatives $\tilde{P}_\mu$ of the displacement generators is rather cumbersome.

[3] It may be of some interest to note that if $\Psi$ obeys (4) then (6) reduce in some sense to the corresponding known expressions. For example, $\phi(p_\mu)$ then must be of the form $\delta(p^2 + m^2)\theta(E) f(\vec{p})$ and therefore we have

$$\int d^4p\, \phi_{m_1}^*(p_\mu)\phi_{m_2}(p_\mu) \sim \delta(m_1 - m_2)\int f_1^*(\vec{p}) f_2(\vec{p}) d^3p/\sqrt{\vec{p}^2 + m^2}.$$

As is seen, the known definition of the scalar product for the solutions of eqs. (4)/11/ appears.

But in any case, in the constructed dynamics the total energy $P_0$ will commute with t.

The above consideration is a relativistic one. A presentation based on the Galilean group would be more difficult, see [13].

## 2. Energy Conservation in quantum Field Theory

Systems with interaction are usually described in the framework of field theory. As for this theory we note only that in the available formulation of the quantum field theory time is a parameter [4] and discuss only one problem: conservation of energy with time.

All the physical quantities of the second quantized theory may be referred to a definite time including the total energy operator

$$H = -\int T_{44}(\vec{x}, t)\, d^3 x \quad . \tag{7}$$

As is known, this integral is independent of time (owing to $\partial T_{\mu\nu} / \partial x_\nu = 0$ ; the system is assumed to be closed).

Thus, in field theory the expressions: "total energy in moment $t$ ", " total energy is conserved in time" have an exact operator meaning. Let us show that this operator law of conservation means the conservation of the probability amplitude of distribution over the eigenvalues of the operator H.

The general solution of the Schrödinger equation

$$i\, \partial \Phi(t) / \partial t = H \Phi(t) \tag{8}$$

can be written in the form

$$\Phi(t) = S_\nu\, f_\nu\, \phi_\nu\, e^{-iW_\nu t} \quad , \tag{9}$$

where $S_\nu$ means a summation or integration over some variables, including the number $\nu$ of the eigenfunction $\phi_\nu$ of the H operator (which belongs

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[4] The coordinates $x, y, z$ in field theory are also parameters numbering the degrees of freedom of the field. However, in quantized theory they can simultaneously play the role of the particle coordinate. Indeed, the Fock one-particle amplitude $\Phi_1(\vec{x}, t)$ is connected with the field operator $\Phi_1(\vec{x}, t) = \langle 0 | \phi(\vec{x}, t) | \Phi \rangle$ so that its arguments are the same $\vec{x}, t$ as for the field operator $\phi(\vec{x}, t)$. On the other hand, it must be interpreted as the wave function of a particle in coordinate representation. Therefore $\vec{x}$ may be interpreted as eigenvalues of the operator of the particle coordinate (see [11], ch.7 §3; we notice that $x$ is an Hermitean operator, if (6) is used).

to the eigenvalue $W_\nu$ ); $f_\nu$ does not depend on $t$ . Such a form is possible just because $H$ is independent of $t$ (hence, it follows that it is possible to separate the variables in (8) and to write the complete orthonormal system of the solutions of eq. (8) in the form $\phi_\nu(t) = \phi_\nu \exp(-iW_\nu t)$). The probability amplitude for finding the state $\phi_\mu(t)$ with the definite energy $W_\mu$ in the state (9) at the moment $t$ is $(\phi_\mu(t), \Phi(t)) = f_\mu$. Notice that the law of energy conservation is just expressed in the fact that $f_\nu$ is time independent but not in the fact that the corresponding $\delta$ - function is present anywhere in formulas.

The deduced law is in an apparent contradiction with the well-known result of non-stationary perturbation theory; energy is not conserved during a finite interval of time. Sometimes this result is considered as an illustration for the uncertainty relaiion $\Delta E \Delta t \approx h$ [5], see, e.g.[14] § 29. Show that this contradiction is due only to a wrong physical interpretation of the formulas of nonstationary perturbation theory. Let us obtain them in such a way which automatically garantees the fulfilment of the law of conservation of total energy. Namely, to calculate $\Phi(t)$ we take the expression (9) and make use of the ordinary stationary perturbation theory for finding the $\phi_\nu$ and $W_\nu$ . The coefficients $f_\nu$ as usual are determined by the initial wave function $\Phi(0)$ . In the usual representation (the Dirac method of variation of constants[14] § 29) $\Phi(t)$ is presented in the form:

$$\Phi(t) = S_n a_n(t) u_n e^{-iE_n t}. \qquad (10)$$

Here $u_n$ is the complete system of eigenfunctions of $H_0 = H - H'$ ( $H'$ is perturbation energy) with eigenvalues $E_n$ ; $a_n(t)$ obeys the Schrödinger equation in the interaction picture and is found according to perturbation theory.

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[5] Mandelstam and Tamm[2] has already stressed that such a treatment does not follow, strictly speaking, from the perturbation formulas. The transition probability to states with energy $E$ different from the initial one $E_0$ does not, in fact, decrease at all with increasing $t$ , but it oscillates: it is proportional to $\sin^2(t/2)(E - E_0)/(E - E_0)^2$ . It is only the transition probability <u>per unit of time</u> that contains the multiplier $\sin^2(t/2)(E - E_0)/(E - E_0)^2 t$ which at $t \to \infty$ turns into $\delta(E - E_0)$.

For our purpose it is sufficient to get from (9) the formula of the first approximation. In the Schiff notations (see [14] § 25) we have

$$\phi_\nu = u_\nu + S'_\mu \frac{H'_{\mu\nu} u_\mu}{E_\nu - E_\mu} \; ; \quad W_\nu = E_\nu + H'_{\nu\nu} \; . \tag{11}$$

Let the initial state be described by the function $u_m$ . From

$$u_m = S_\nu \phi_\nu f_\nu \tag{12}$$

we find

$$f_\nu = (\phi_\nu , u_m) = \delta_{\nu m} + \frac{H'^*_{m\nu}}{E_\nu - E_m} \Big|_{m \neq \nu} \; . \tag{13}$$

The transition amplitude to the state $u_k \exp(-iE_k t)$, $k \neq m$ , at the moment $t$ is (in the first order in $H'$ ):

$$(u_k e^{-iE_k t}, \Phi(t)) = e^{iE_k t} S_\nu e^{-iW_\nu t} (u_k , u_\nu + S'_\mu \frac{H'_{\mu\nu} u_\mu}{E_\nu - E_\mu}). \tag{14}$$

$$(\delta_{\nu m} + \frac{H'_{\nu m}}{E_\nu - E_m}) = \frac{H'_{km}}{E_k - E_m} (e^{i(E_k - E_m)t} - 1).$$

As is known, just such an expression for this amplitude (it is equal to $a_k(t)$, see (10)) is obtained in the Dirac's way, see (29.9) in [14].

Note that since the initial state $u_m$ is not the eigenstate of $H$ we have no definite total energy initially. In the superposition (12) there are states with energies $W_k$ close to $E_k \neq E_m$ . Therefore the fact that at a moment $t > 0$ in $\Phi(t)$ there may be another state $u_k$ with $E_k \neq E_m$ does not contradict the law of conservation of energy (by the way, total energy has no definite value in the state $u_k$ also).

The fact that the transition amplitude to the state $u_k$ with $E_k = E_m$ increases linearly with time while the amplitudes with $E_k \neq E_m$ only oscillate should be considered simply as a phenomenon of the resonance type. In this case the qualitative relation $(E - E)t \approx h$ gives the upper boundary for times $t$ when the oscillating amplitudes are still comparable with the increasing one, comp. [2].

## Discussion

The above opinion on the role of time allows us to answer the questions put in Introduction as follows:

1. Time is a parameter when the considered physical system obeys some equation of motion (the Schrödinger equation in quantum mechanics). In the framework of a set of quantal postulates (preceding the writing of the equation) time is an operator Are there physical situations when time is an operator? It is usually believed that during the measurement the Schrodinger equation does not work. However it is possible that there are some equations which describe this process. If so, time may be a parameter even in the measurement. Therefore the author thinks that the above question is still to be studied.

2. Energy must be conserved in most details $E(t_0) = E(t_1) = E(t_2)...$ when the system is freely extended according to the Schrödinger equation. The widely spread opinion that such a conservation is not obtained by perturbation theory is based on a wrong interpretation of the formulas (the identification of the total energy with its zero approximation). Analogously the expression "the energy of an isolated system at the time moment $t$ "has the meaning in quantum mechanics with Schrödinger equation.

The problem of the meaning of the law of conservation of energy in measurement process should be considered still unsolved as well as the problem of a physical existence of the time operator.

3. If time is a parameter then the relation $\Delta E \Delta t \approx h$ may have only the meaning of the Mandelstam-Tamm relation /2/ or of the relation between the life-time and the level width /15/. In the sense of the Bohr-Heisenberg uncertainty relation /7/ it may be understood only in those situations when time is an operator. Since the question about the existence of such situations was left open then this statement has yet no physical content. But it has a quite definite theoretical meaning, especially in connection with the available discussion on the relation $\Delta E \Delta t \approx h$ /7,8,15-17/. The point is that in this discussion both sides consider time as a parameter while measuring. Form the point of view of this paper such a position means that the abovementioned open question is assumed to be solved in favor of time-parameter. It is the following problem that is discussed in fact 6/: Is the Bohr relation $\Delta E \Delta t \approx h$ valid for measurement process in

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6/ As to the Landau and Peierls' point of view /18/ (repeated in ref. /10 and /8/) we notice only that it is entirely based on the above wrong interpretation of the formulas of perturbation theory (see also the Fock's criticism /1/).

spite of the absence of the corresponding commutation relation $\Delta E \Delta t \approx h$ ?

Aharonov and Bohm[8] stressed that in such an interpretation the relation loses the base which underlies other uncertainty relations: it cannot be deduced with the help of the apparatus of quantum mechanics. It may be assumed only as an additional postulate (we call it the "Fock's postulate") but then it is necessary to prove its consistency with other quantal postulates. The present-day state of the discussion can be described as follows: Aharonov and Bohm suppose that they have indicated the (gedanken) method of measuring accurately the energy during an arbitrary small time in contradiction with the Fock's postulate. While Fock states that they take into account his postulate not everywhere and there is no proof of the contradiction.

The following solution of the problem is suggested: The Fock's postulate is unnecessary. In fact, the purpose for which it has been introduced is achieved in the present paper: there is a place in the apparatus of quantum mechanics for the relation $\Delta E \Delta t \approx h$ in the Bohr-Heisenberg's sense. No other theoretical or experimental difficulties are solved by the postulate. As to the gedanken experiments it is possibly difficult to disprove the postulate if its general applicability is stated. But if this has not be done then other quantal postulates do not forbid to measure the energy exactly at the moment $t$. This has a clear meaning from the point of view of the present paper: if time is a parameter then the energy of a free particle is expressed in terms of its momentum. Nothing forbids to measure the momentum exactly at the moment $t$. The gedanken Aharonov and Bohm's experiment is an illustration of such a possibility for measuring the energy during an arbitrary small time interval. Moreover, the Fock's postulate leads to an essential difficulty in the theoretical apparatus (which is otherwise absent). One of the dynamics problems is the prediction of the future state of the system knowing the initial one (the Cauchy problem). Among the state characteristics there is energy which cannot be referred to a definite time following Fock. Hence, we are not allowed to consider a definite state of the system at the initial moment $t_0$ or at the subsequent ones $t_1$, $t_2$ and so on.

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## References


1. Л. де Бройль. Магнитный электрон, Харьков, 1936.
   D. Bohm. Quantum Theory, New York, 1952.
2. Л.И.Мандельштам, И.Е.Тамм. Изв. АН СССР, сер. физ. 1945,т.9 № 1-2, 122.
   L.Mandelstam, I. Tamm, J. Phys. (USSR) 9 , 249 (1945).
3. Н.Бор, УФН, 66, 580 (1958).
   N. Bohr. "Discussion with Einstein on ... " The Library of Living Philosophers, A. Einstein (1949).
4. Д.И.Блохинцев. Основы квантовой механики, § 113. Высшая школа, Москва, 1963 г.
5. В.Паули. Общие принципы волновой механики, ГИТТЛ, Москва, 1947, часть 1, §8.
6. П.А.М.Дирак. Основы квантовой механики, изд.1, § 19. ГТТИ, Москва, 1932.
7. Н.С.Крылов, В.А.Фок, ЖЭТФ, 17, 93 (1947).
   V. Fock, N. Krylov, J. Phys. (USSR) 11, 112 (1947).
8. Y. Aharonov, D. Bohm. Phys.Rev., 122 , 1649 (1961).
9. E.P. Wigner. Nuovo Cim., 3 , 517 (1956), sec. 5,
   T.D. Newton, E.P.Wigner. Rev.Mod.Phys., 21 , 400 (1949). Introd.
10. И.Г.Гельфанд, Р.А.Минлос, З.Я.Шапиро. Представления группы вращений и группы Лоренца, ГИФМЛ, Москва, 1958.
11. S.S.Schweber. An Introduction to Relativistic Quantum Field Theory, New York, 1961.
12. P.A.Dirac. Rev.Mod.Phys., 21 , 392 (1949).
13. J.M. Levy- Leblond, Journ.Math.Phys., 4 , 776 (1963).
14. L.I.Schiff. Quantum Mechanics, sec.ed. New York-London, 1955.
15. В.А.Фок, ЖЭТФ 42, 1135 (1962).
16. Y.Aharonov. D.Bohm. Phys.Rev. 134B, 1416 (1964).
17. В.А.Фок, УФН, 86, 363 (1965).
18. L.Landau, R.Peierls, Z.Physik, 69, 56 (1931).
19. Л.Д.Ландау, Е.М.Лифшиц. Квантовая механика, § 44, ГИФМЛ, Москва 1963.



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