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A NEW DEFINITION OF TRANSLATIONS IN THE INHOMOGENEOUS SL(6) GROUP

1. The inhomogeneous 106-parametrical SL(6) group, further referred to as ISL(6), which contains 36-dimensional translations was treated recently by a number of authors[1-10] as possible relativistic extension of SU(6) symmetry. It turned out however, that in the framework of this group one cannot introduce a reflection operation in a natural way and the invariant equations for the scalar and spinor fields contain the derivatives of the sixth and the fifth order, respectively. In order to avoid this difficulty, Rühl[5] suggested to extend the group by considering the translations in the two dual 36-dimensional spaces simultaneously. In this letter we propose another solution of the problem. Though our scheme includes now a larger group of translations, it is, in our opinion, more natural.

2. Let $(t^{\alpha\dot{\beta}})$ be the 6x6 matrix of the translation generators in the group ISL(6):

$$t^{\alpha\dot{\beta}} = \sum_{\substack{0 \leq \mu \leq 3 \\ 0 \leq s \leq 8}} (\sigma_\mu \times \lambda_s)^{\alpha\dot{\beta}} \, t^\mu_s \equiv (\Lambda^s_\mu)^{\alpha\dot{\beta}} \, t^\mu_s \tag{1}$$

(t^μ_s are the Hermitian operators commuting with each other) and let $\tilde{t}_{\dot{\alpha}\beta}$ be the algebraic adjunct of the element $t^{\beta\dot{\alpha}}$ in the matrix t. The analogs of the Klein-Gordon equations and Dirac equations in the scheme ISL(6)[4,5]

$$\det t \cdot \phi = \kappa^6 \phi . \tag{2}$$

$$t^{\alpha\dot{\beta}} \chi_{\dot{\beta}} = \kappa \phi^\alpha$$

$$\tilde{t}_{\dot{\alpha}\beta} \phi^\beta = \kappa^5 \chi_{\dot{\alpha}} \tag{3}$$

have the disadvantages we mentioned above. It seems to us that the system (3) (consisting of six equations of the first order and six equations of the fifth order) is non-symmetrical because the "square root from the D'Alembertain" det t is not correctly extracted. A much more symmetrical possibility of the "linearization" of Eq. (2) is prompted by the Laplace formula which allows to write det t as a sum of products of the third order minors of t.

We define the unitary momentum as a 400-component tensor with the elements

$$P^{\alpha_1\alpha_2\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} = \begin{vmatrix} t^{\alpha_1\dot\beta_1} & t^{\alpha_1\dot\beta_2} & t^{\alpha_1\dot\beta_3} \\ t^{\alpha_2\dot\beta_1} & t^{\alpha_2\dot\beta_2} & t^{\alpha_2\dot\beta_3} \\ t^{\alpha_3\dot\beta_1} & t^{\alpha_3\dot\beta_2} & t^{\alpha_3\dot\beta_3} \end{vmatrix} . \tag{4}$$

It is seen from the definition that the tensor P is anti-symmetrical both with respect to the dotted and undotted indices separately. It satisfies the hermiticity condition

$$P^{\alpha_1\alpha_2\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} = \bar{P}^{\dot\beta_1\dot\beta_2\dot\beta_3\alpha_1\alpha_2\alpha_3} \tag{5}$$

and transforms under the representation [20,20] x) of the group SL(6). The conjugate momentum $\tilde{P}_{\dot\beta_1\dot\beta_2\dot\beta_3\alpha_1\alpha_2\alpha_3}$ defined as the algebraic adjunct of the minor (4) of t is expressed linearly by P :

$$\tilde{P}_{\dot\beta_1\dot\beta_2\dot\beta_3\alpha_1\alpha_2\alpha_3} = \frac{1}{36}\, \epsilon_{\dot\beta_1\dot\beta_2\dot\beta_3\dot\beta'_1\dot\beta'_2\dot\beta'_3}\, \epsilon_{\alpha_1\alpha_2\alpha_3\alpha'_1\alpha'_2\alpha'_3}\, \bar{P}^{\alpha'_1\alpha'_2\alpha'_3\dot\beta'_1\dot\beta'_2\dot\beta'_3} \tag{6}$$

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x) In the notation [n, m] the number n corresponds to the dimensionality with respect to the undotted indices, and m - with respect to the dotted ones. A possibility of using the representation [20, 20] to give a new definition of translations in the inhomogeneous SL(6) group is also indicated in preprints by Bacry et al. [4,9].

(three indices may be lowered and raised with the help of the invariant tensor ϵ). Let det $t = m^2$ (we assume $\kappa^3 = m$). Then

$$\frac{1}{6} P^{\alpha_1 \alpha_2 \alpha_3 \dot{\sigma}_1 \dot{\sigma}_2 \dot{\sigma}_3} \tilde{P}_{\dot{\sigma}_1 \dot{\sigma}_2 \dot{\sigma}_3 \beta_1 \beta_2 \beta_3} = m^2 I\binom{\alpha_1 \alpha_2 \alpha_3}{\beta_1 \beta_2 \beta_3} = m^2 \begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \delta^{\alpha_1}_{\beta_2} & \delta^{\alpha_1}_{\beta_3} \\ \delta^{\alpha_2}_{\beta_1} & \delta^{\alpha_2}_{\beta_2} & \delta^{\alpha_2}_{\beta_3} \\ \delta^{\alpha_3}_{\beta_1} & \delta^{\alpha_3}_{\beta_2} & \delta^{\alpha_3}_{\beta_3} \end{vmatrix} . \quad (7)$$

The decomposition of the representation [20, 20] into irreducible representations of the group SU(3) x SL(2) x)

$$[20,20] = (1;\ 2,2) + 2(8;\ 2,2) + (10;\ 2,2) + (\overline{10};\ 2,2) + \\ + (27;2,2) + (8;\ 2,4) + (8;\ 4,2) + (1;4,4) \quad (8)$$

contains only one Lorentz 4-vector which is at the same time a unitary singlet: (1;2, 2). We will identify this 4-vector with the usual physical momentum.

3. Starting from the commutation relations between the generators of the group ISL(6), one can easily become convinced (see, e.g. [4,11]) that the operators P commute, and their commutators with generators M of the homogeneous group SL(6) are expressed linearly in terms of P . The operators P and M generate the Lie algebra of the 470-parametrical group $\mathcal{G}_u$ which can be defined as a semidirect product

$$\mathcal{G}_u = \frac{SL(6)}{Z_3} \cdot T_{400} . \quad (9)$$

Here Z_3 is the cyclic group of the cube roots of unity which is the normal divisor of SL(6), T_{400} is the 400-parametrical invariant Abelian subgroup $\mathcal{G}_u$ xx)

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x) The first figure in each bracket in the right-hand side of (8) denotes the dimensionality of the representation under the group SU(3), the second pair of figures refers to the representation of the group SL(2).

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xx) A more detailed characteristic of the group $\mathcal{G}_u$ was given in preprint[11].

Single-valued representations of the group $\mathcal{P}_n$ describe only the particles with integer charges. The lowest non-trivial representation of the homogeneous group $SL(6)\,/\,Z_3$ is twenty-dimensional. There are two types of twenty-component spinors transforming under representations $[20, 1]$ and $[1, \overline{20}]$ which we shall join together in one bispinor

$$\Psi = \begin{pmatrix} \phi^{a_1 a_2 a_3} \\ \chi_{\dot\beta_1 \dot\beta_2 \dot\beta_3} \end{pmatrix} \tag{10}$$

(ϕ and χ are antisymmetrical with respect to their indices).

The analog of the Dirac equation for this bispinor is written down as

$$\sum_{\dot\beta_1<\dot\beta_2<\dot\beta_3} P^{a_1 a_2 a_3 \dot\beta_1 \dot\beta_2 \dot\beta_3} \chi_{\dot\beta_1 \dot\beta_2 \dot\beta_3} = m\phi^{a_1 a_2 a_3}$$

$$\sum_{a_1<a_2<a_3} \tilde{P}_{\dot\beta_1 \dot\beta_2 \dot\beta_3 a_1 a_2 a_3} \phi^{a_1 a_2 a_3} = m\chi_{\dot\beta_1 \dot\beta_2 \dot\beta_3} \tag{11}$$

or in a more compact form

$$\hat{P}\Psi = m\Psi, \quad \hat{P} = \begin{pmatrix} 0 & P \\ \tilde{P} & 0 \end{pmatrix}. \tag{12}$$

Equations (11) or (12) are invariant not only under the proper group P_n but also under the "reflection" I_s defined by the equalities

$$I_s\Psi = \eta \begin{pmatrix} \chi_{\dot\beta_1 \dot\beta_2 \dot\beta_3} \\ \phi^{a_1 a_2 a_3} \end{pmatrix} = \eta\Gamma_0\Psi$$

$$I_s\hat{P}I_s^{-1} = \Gamma_0\hat{P}\Gamma_0^{-1} = \begin{pmatrix} 0 & \tilde{P} \\ P & 0 \end{pmatrix}. \tag{13}$$

4. The Lie algebra of the group $\mathcal{P}_n$ has 6 polynomial invariants (Casimir operators). One of them is a function of the momenta only and is given by (7). The remaining ones are the generators of the "little group" and are determined in the following manner.

Let K^{μ}_{a} and $K^{*\mu}_{a}$ be nonhermitian generators of the homogeneous SL(6) group which satisfy the usual commutation relations (see [11])

$$[K^{\mu}_{a}, K^{\nu}_{u}] = i(\delta_{\mu\nu\rho} f_{auv} + \epsilon_{0\mu\nu\rho} d_{auv}) K^{\rho}_{v}$$
$$[K^{*\mu}_{a}, K^{*\nu}_{u}] = i(\delta_{\mu\nu\rho} f_{auv} + \epsilon_{0\mu\nu\rho} d_{auv}) K^{*\rho}_{v} \qquad (14)$$

$$[K^{\mu}_{a}, K^{*\nu}_{u}] = 0 \qquad (15)$$

where the structure constants are determined from the identities

$$[\sigma_{\mu}, \sigma_{\nu}] = 2i\epsilon_{0\mu\nu\rho}\, \sigma_{\rho}, \quad \{\sigma_{\mu}, \sigma_{\nu}\} = 2\delta_{\mu\nu\rho}\, \sigma_{\rho}, \quad \mu,\nu,\rho = 0,1,2,3$$
$$[\lambda_{a}, \lambda_{u}] = 2if_{auv}\, \lambda_{v}, \quad \{\lambda_{a}, \lambda_{u}\} = 2d_{auv}\, \lambda_{v}, \quad a,u,v = 0,1,\dots,8 . \qquad (16)$$

We put then (like in (1))

$$K^{\dot\alpha}_{\dot\beta} = (\Lambda^{a}_{\mu})^{\dot\alpha}_{\dot\beta} K^{\mu}_{a}, \quad K^{*\alpha}_{\beta} = (\Lambda^{a}_{\mu})^{\alpha}_{\beta} K^{*\mu}_{a} \qquad (17)$$

and introduce the 400-component tensor

$$W^{\alpha_1\alpha_2\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} = \frac{1}{6}\Big(P^{\alpha_1\alpha_2\alpha_3\dot\beta_1\dot\beta_2\dot\sigma} K^{\dot\beta_3}_{\dot\sigma} + P^{\alpha_1\alpha_2\alpha_3\dot\beta_1\dot\sigma\dot\beta_3} K^{\dot\beta_2}_{\dot\sigma} + P^{\alpha_1\alpha_2\alpha_3\dot\sigma\dot\beta_2\dot\beta_3} K^{\dot\beta_1}_{\dot\sigma} + P^{\alpha_1\alpha_2\sigma\dot\beta_1\dot\beta_2\dot\beta_3} K^{*\alpha_3}_{\sigma} + P^{\alpha_1\sigma\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} K^{*\alpha_2}_{\sigma} + P^{\sigma\alpha_2\alpha_3\dot\beta_1\dot\beta_2\dot\beta_3} K^{*\alpha_1}_{\sigma} \Big). \qquad (18)$$

The tensor W is invariant under traslations and is an analog of the 4-dimensional vector of Pauli-Lubanski-Bargmann. Using it, the matrix of 35 generators of the little group is defined by the formula

$$V^{\alpha}_{\beta} = \frac{1}{36} W^{\alpha_1\alpha_2\alpha\dot\sigma_1\dot\sigma_2\dot\sigma_3} P_{\dot\sigma_1\dot\sigma_2\dot\sigma_3\alpha_1\alpha_2\beta} \quad (\mathrm{Sp}V = 0) \qquad (19)$$

The Casimir operators characterizing the irreducible representations of the little group are given by

$$C_\ell = Sp\, V^\ell \quad , \quad \ell = 2,3,4,5,6 . \tag{20}$$

Since the operators (20) are Hermitian we would arrive at the same expressions for them if instead of the tensor V^α_β we would make use of the Hermitian conjugate tensor $\overset{*}{V}{}^\alpha_\beta$

5. As far as the group $\mathcal{P}_u$ contains SU(6) as a subgroup and SU(6) symmetry is not rigorous, then $\mathcal{P}_u$ symmetry should be inevitably broken down either. Following[3] it is reasonable to introduce the breakdown of the symmetry under the group $\mathcal{P}_u$ as a supplementary condition on the state vectors. In other words, we postulate that the physical states are the eigenvectors with zero eigenvalues of all those 400-momentum components which do not commute with the hypercharge and isotopic spin operators. Using this assumption we hope to obtain some formulae for particle masses and relations among form-factors (cf.[3]) [x)].

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x) One may hope that in the theory of broken $\mathcal{P}_u$ symmetry the mass dependence on the spin will be obtained in a natural way since in the decomposition (8) of the 400-momentum in irreducible representations SU(3) x SL(2) the unitary singlet appears twice with different values for the spin.

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