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ASYMPTOTICS OF THE FEYNMAN GRAPHS III. SPINOR GRAPHS

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1. Introduction

The present paper is devoted to the determination of the asymptotic behaviour of both the convergent and divergent Feynman graphs basing on their topological structure. In many papers this problem is solved for graphs with scalar lines. The most complete and general form of this solution seems to be presented by Zavialov[1] and completed in some sense by Menke[2]. The asymptotics of divergent scalar graphs is carefully studied in ref.[3] where the R - operation in the α - representation is made and general recipe of obtaining (unfortunately with some inaccuracy) of the asymptotics is given. Removing the above inaccuracies we will use this method as applied to graphs with spinor lines. These graphs are, of course, more complicated than the scalar ones, however practically the former are much more interesting, at least, from the point of view of electrodynamics. Spinor graphs have been considered in a number of papers[4,5], but a constructive recipe of plotting the asymptotics for a rather wide class of spinor graphs is absent there. The difficulty of this problem is that, in addition to the usual exponential dependence on momenta, there appears a preexponential depending both on the parameters α and the momenta. This leads to two additional mechanisms of increasing the asymptotics as compared to scalar graphs. First, the dependence of the preexponential can increase the effective index c' essential subgraphs and, second, the involved momenta can be combined in an additional power of an asymptotically large variable. Section 2 is devoted to the structure of the preexponential and its connection with the graph topology. The convergent spinor graph asymptotics is considered in Section 3. Divergent graphs, their regularization and asymptotics are considered in Section 4 and 5. The results of all the previous sections are formulated as a recipe which, together with necessary definitions, is singled out in Section 6. Section 7 deals with the examples of application of this recipe.

2. Contribution of a Spinor Graph in the α -Representation

We recall, first of all, that to each scalar line ν of the graph G in the alpha- representation there corresponds the function $\Delta(\epsilon_{\nu j} x_j) = \frac{1}{(4\pi)} \int \frac{d\alpha}{\alpha^2} e^{-i(m_\nu^2 - i\delta)\alpha_\nu - \frac{i}{4\alpha_\nu}(\epsilon_{\nu j} x_j)^2}$ where $\epsilon_{\nu j}$ is the incidence matrix of the graph G, i.e. $\epsilon_{\nu j} = 1$ if the line ν goes into the vertex j, $\epsilon_{\nu j} = -1$ if it goes out from the vertex and $\epsilon_{\nu j} = 0$ in other cases. To spinor lines of the graph there corresponds the function[6] $S(x) = (i\gamma^n \frac{n\partial}{\partial x^n} + m)\Delta(x)$ i.e. to each spinor line σ in the contribution of graph G according to the usual rules, there corresponds the multiplier

$$S(\epsilon_{\sigma j} x_j) = -\frac{1}{(4\pi)^2}\int \frac{d\alpha_\sigma}{\alpha_\sigma^2}\left(\frac{1}{2\alpha_\sigma}\epsilon_{\sigma j}\hat{x}_j + m_\sigma\right) e^{-i(m_\sigma^2 - i\delta)\alpha_\sigma - \frac{i}{4\alpha_\sigma}(\epsilon_{\sigma j} x_j)^2} \quad (1)$$

Thus, the coefficient function described by the graph G in the x-representation is of the form

$$K(x_1 \dots x_n) = Y\left\{\int_0^\infty \prod_{\nu=1}^{\ell} \frac{d\alpha_\nu}{\alpha_\nu^2} \prod_\sigma \left(\frac{1}{2\alpha_\sigma}\epsilon_{\sigma k}\hat{x}_k + m_\sigma\right) \times\right.$$

$$\left. \times \exp\left[-i\sum_{\nu=1}^{\ell}(m_\nu^2 - i\delta)\alpha_\nu - \frac{i}{4}\sum_{\nu=1}^{\ell}\frac{1}{\alpha_\nu}(\epsilon_{\nu k} x_k)^2\right]\right\}, \quad (2)$$

where the symbol Y denotes a correct arrangement of the γ matrices entering the expressions for spinor lines (and, perhaps, the vertices which we are not dealing with).

To pass to the momentum representation it is necessary as usual to multiply the coefficient function (2) by $\exp\{ip_j x_j\}$ and integrate over all x_j. This means that the external momenta enter each of the vertices. To go over to the real situation some of the "fictitious" momenta should be assumed to be equal to zero, what we are going to do in due time. If now we use the equality

$$\hat{x}_k e^{ip_j x_j} = -i\hat{\partial} p_k e^{ip_j x_j}$$

then integrating and factoring out the conservation of four-momentum $(K(p_1 \dots p_n) = \delta(\Sigma p_j) T(p_1 \dots p_{n-1}))$ we get

$$T(p_1 \dots p_{n-1}) = Y\left\{\int_0^\infty \frac{\prod d\alpha_\nu}{\Delta^2(\alpha)} \prod_\sigma \left(\frac{1}{2i\alpha_\sigma}\epsilon_{\sigma k}\partial p_k + m_\sigma\right) \times\right.$$

$$\left. \times \exp\left[i\sum_{j,k=1}^{n-1} d_{jk}(\alpha) p_j p_k - i\sum_\nu \alpha_\nu (m_\nu^2 - i\delta)\right]\right\}.$$

The functions $\Delta(\alpha)$ and $d_{jk}(\alpha) = \frac{A(jk;n)}{\Delta(\alpha)}$ and their connection with the graph topology were obtained in ref. /7/. We remind in brief what this connection is. To obtain Δ we need to construct a so-called tree of the graph G i.e. a connected subgraph containing all the vertices of the graph G but having no closed cycles, then to write the product of the parameters α corresponding to lines not entering the tree (e.g. to the tree shords) and sum up such expressions for all possible trees of the graphs G. To obtain $A(jk;n)$ we need to construct a 2-tree of the graph (i.e. a subgraph containing all the vertices and consisting of two separate connected components without closed cycles) such that vertices $\{j,k\}$ and vertex $\{n\}$ belong to various components, then to write the product of the parameters α corresponding to the chords

of each of the 2-trees and sum up over all such 2-trees of the graph G. In other words, using all possible ways we must make sections of the graph into two components (by removing some lines) one of which contains the vertex $\{n\}$ and another the vertices $\{j,k\}$, take a product of the corresponding α-parameters and multiply it by the sum over all possible trees of these two components, i.e.

$$A(jk;n) = \sum_{\text{over all sect.}} \Delta' \Delta'' \prod_{\text{sect.}} \alpha .$$

Look now how the operator $\prod_\sigma \left(\frac{1}{2i\alpha_\sigma}\epsilon_{\sigma k}\hat{\partial} p_k + m_\sigma\right)$ acts on the exponential in the expression (3). If the graph contains only one spinor line then the action of this operator leads to a multiplier to be appeared before the exponential

$$P_\sigma = \left(\frac{1}{\alpha_\sigma}\epsilon_{\sigma k} d_{km}\hat{p}_m + m_\sigma\right). \quad (4)$$

Two spinor lines yield the multiplier

$$P_{\sigma_1} P_{\sigma_2} + C_{\sigma_1\sigma_2}$$

where

$$C_{\sigma_1\sigma_2} = i(\gamma^j_{(\sigma_1)}\gamma^j_{(\sigma_2)})\frac{\epsilon_{\sigma_1 k}\epsilon_{\sigma_2 m} d_{km}}{2\alpha_{\sigma_1}\alpha_{\sigma_2}} \quad (5)$$

three lines yield

$$C_{\sigma_1\sigma_2}P_{\sigma_3} + C_{\sigma_1\sigma_3}P_{\sigma_2} + C_{\sigma_2\sigma_3}P_{\sigma_1} + P_{\sigma_1}P_{\sigma_2}P_{\sigma_3}$$

and so on. There is no necessity to write down a general form of this expression. It is very cumbersome and besides not very useful. However, the rule by which the preexponential multiplier is formed is now clear. (It resembles the summation over all the contractions in the Wick theorem).

Now we may assume that all the "fictitous" external multipliers vanish and keep only the real p_1, p_2 and p_3 (p_4 is omitted owing to the conservation law).

We are going to clarify the topological meaning of the α-dependent functions enterning P_σ and $C_{\sigma\sigma'}$. Let the spinor line σ joint the vertices s and q of the graph G then the α-dependent part of P_σ reads

$$\frac{1}{\alpha_\sigma}[(d_{s1} - d_{q1})\hat{p}_1 + (d_{s2} - d_{q2})\hat{p}_2 + (d_{s3} - d_{q3})\hat{p}_3] . \quad (6)$$

At first glance, this function seems to be singular when α_σ tends to zero, but, in fact, this is not the case. Really it is known, that, for example,

$$d_{s1} - d_{q1} = \frac{A(s1;4) - A(q1;4)}{\Delta} \quad (7)$$

However the vertex q enters any of the 2-trees $A(si;4)$ either together with the vertex 4, or with the vertices $\{s,i\}$ i.e.

$$A(si;4) = A(si;q4) + A(siq;4).$$

In just the same way we can write

$$A(qi;4) = A(qi;s4) + A(qis;4).$$

Inserting these results into eq. (7) we find

$$\frac{1}{\alpha_\sigma}(d_{s1} - d_{qi}) = \frac{A(si;q4) - A(qi;s4)}{\alpha_\sigma \Delta} \quad (8)$$

Hence it is seen that for α_σ there is no singularity because the nominator contains only such 2-trees whose vertices s and q enter different components, i.e. the line σ should be a chord and, consequently, the nominator should be proportional to α_σ.

We pass now to the term $C_{\sigma\sigma'}$. Let the spinor lines σ and σ' joint the vertices s,q and s',q', respectively. The α-dependent part can be written in the form

$$\frac{A(ss';4) - A(sq';4) - A(qs';4) + A(qq';4)}{\alpha_\sigma \alpha_{\sigma'} \Delta}$$

But each term of this expression can be expanded taking into account vertices not written down explicitly, e.g.

$$A(ss';4) = A(ss'qq';4) + A(ss'q;4q') + A(ss'q';4q) + A(ss';4qq').$$

In this case many obtained terms cancel out and remaining ones combine themselves as

$$\frac{A(ss';qq') - A(sq';s'q)}{\alpha_\sigma \alpha_{\sigma'} \Delta} \quad (9)$$

It is seen again that for $\alpha_\sigma = 0$, $\alpha_{\sigma'} = 0$ there is no singularity since the nominator contains only such 2-trees in which different ends of both lines enter different components, i.e. the lines σ and σ' are to be chords.

It is known[8] that the expression in the exponential power does not depend on which of the real momenta was omitted due to the conservation law. The same can be proved also for the preexponential factor in the spinor case.

3. Asymptotics of the Spinor Graph

Now we go over to the determination of the asymptotic behaviour of a scattering amplitude defined by a certain graph G. First of all we rewrite eq. (3) as

$$T(s,t,\hat{p}) \approx \int \frac{\Pi d\alpha_\nu}{\Delta^2(\alpha)} f(\alpha,\hat{p},m_\sigma) \exp\left[i\frac{A(\alpha)}{\Delta(\alpha)} S + iB(\alpha,t,m)\right] \quad (10)$$

and make the Mellin transformation

$$T(s,t,\hat{p}) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} d\xi \frac{G(\xi,t,\hat{p})}{\sin \pi\xi} (-S)^\xi \quad (11)$$

in this case

$$\Phi(\xi) = \frac{G(\xi,t,\hat{p})}{\sin \pi\xi} \approx \frac{i^\xi}{\Gamma(\xi+1)} \int \frac{\Pi d\alpha_\nu}{\Delta^2(\alpha)} f(\alpha,\hat{p},m_\sigma)\left(\frac{A(\alpha)}{\Delta(\alpha)}\right)^\xi e^{iB(\alpha,t,m)} \quad (12)$$

where $A = A(12;34) - A(13;24)$ (for a planar graph, i.e. for a graph which has no nonintersecting paths connecting the vertices 1 with 3 and 2 with 4, the second term is zero). Remind what we have for a graph with the scalar lines only (i.e. when $f \equiv 1$), what topological elements define the asymptotic behaviour when $S \to \infty$. From eq. (11) it is seen that it is determined by the extreme right singularity of $\Phi(\xi)$ in the complex ξ-plane. For a planar graph this singularity is due to the vanishing of $\frac{A}{\Delta}$ at the boundary of the integration region, i.e. when some set of the parameters α vanishes (graphically it means a contraction of the corresponding lines into a point such that the obtained graph have no S-section, i.e. no section separating the vertices 1, 2 from 3, 4 and dividing the graph into two connected parts). If a subgraph V contains ℓ' lines and μ' independent cycles ($\mu' = \ell' - n' + i$ where n is the number of vertices and i is the number of connected components of the subgraph) and any S-section increases the number of its components, at least, there is one S-section increasing the number of them by one (let us call such a subgraph the t-subgraph with respect to G; in Appendix it is shown that only the graphs of such a kind yield the main contribution to the asymptotics) then by barycentric transformation $\alpha_\nu \to \lambda\alpha_\nu$ for all $\nu \in V$ and when $\lambda \to 0$ (it is just this region which gives the most right singularity corresponding to the subgraph V) $\Delta \to \lambda^{\mu'}\Delta'$, $\frac{A}{\Delta} \to \lambda\frac{A'}{\Delta'}$ and eq. (11) for $\Phi(\xi)$ takes the form

$$\int \lambda^{k-1} d\lambda \int \frac{\Pi d\alpha_\nu}{\Delta^2(\alpha)} \; \delta(1 - \Sigma\alpha_\nu \in V)\left(\frac{A'}{\Delta'}\right)^{\xi} e^{iB(\lambda=0)} \quad .$$

It is clear that integrating it over λ we are led to the appearance of a pole at the point $\xi = -k$ (and, consequently, to the behaviour S^{-k}). Therefore the main asymptotic term is determined by t - subgraphs with minimal index $k = \ell' - 2\mu'$. Let us look what determines the order of this pole First of all we note that terms corresponding to the S -sections which increase the number of components V more than by one do not enter A′ since after the multiplier λ being factored out from $\frac{A}{\Delta}$ they vanish, at least, as the first power of λ when $\lambda \to 0$. There arises naturally the following generalization of the concept of t -subgraph with respect to G: V is referred to as t - subgraph with respect to V if any S -section increasing the number of V components by one increases also the number of V′ components, at least one of the sections increasing the number of components by one. It happens that if $V_1 \ldots V_r$ is a maximal sequence of independent subgraphs and each of them has, at least, one line not entering any of the subsequent subgraphs) with a minimal index k , then the order of the pole at the point $\xi=-k$ is r therefore when $S \to \infty$ this graph behave like $S^{-k}(\ln S)^{r-1}$ In ref.[1, 3] these subgraphs were assumed to be t -subgraphs with respect to G and this inaccuracy may cost in some cases an additional power of the logarithm. For example, for the graph in Fig. 1a, in addition to the subgraph a, b, c, d, e, f obtaned by the method of ref./1/, according to new rule, one more "broken" subraph Ig is possible which is the t -subgraph not with respect to G , but to the subgraph I c. This leads to the additional power of the logarithm, i.e. $T \approx S^{-2}(\ln S)^6$. The graph considered in ref.[2] may serve as another example of such a type.

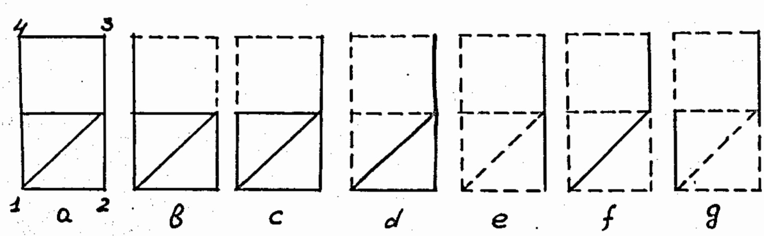


Fig. 1.

What changes in the recipe occur owing to the factor $f(a, \not p, m)$ arising due to spinor lines ? Here two mechanisms act which tend to increase the asymptotics power. First, the dependence of this factor on a can lead to a decrease of the V's index and, second, due to the presence in f of the momenta $\not p$, after expanding by the matrix structures and trace operation there can appear the scalar products of the momenta $(p_1 p_2)$ and $(p_1 p_3)$ which behave like $-S/2$ and $-S/2$ if $S \to \infty$.

First of all we consider how the terms P_σ and $C_{\sigma\sigma'}$ influence the V's index. We make, as usual, the barycentric transformation $a_\nu \to \lambda a_\nu$ for $\nu \in V$ and test whether the preexponential function can lead to the negative power of λ and when this can occur. Let us consider what minimal power of λ may be given by the term P_σ corresponding to the spinor line σ connecting the vertices s and q. It is not difficult to see that in this case $A(sj\,;\,q4)$ transforms into $\lambda^{\mu'+i-1} A'(sj\,;\,q4)$ where i is the smallest number of connected components into which the subgraph V is cut in separating the points $\{s,j\}$ from $\{q,4\}$. Indeed, this means that the subgraph enters the corresponding 2-trees is an i-tree which can be obtained from its tree by removing some $i-1$ lines. At the same time Δ as before, transforms into $\lambda^{\mu'} \Delta'$. Thus there can appear two possibilities:

a. For the line $\sigma \notin V$, P_σ behaves as λ^{i-1}. But the smallest possible value of i will be unity, therefore $i-1$ cannot be negative and the index cannot increase.

b. For the line $\sigma \in V$, in spite of the factor λ^{i-2} is singled out, i cannot be less than two since for any section the point s and q must belong to different components. This does not lead again to any increase of the index. Thus, it turns out that terms P_σ give no desirable effect and can influence the asymptotics only due to involved external momenta. We turn now to terms $C_{\sigma\sigma'}$. Here three cases are possible:

a. If $\sigma, \sigma' \notin V$ then the power of λ cannot be negative for the same reason as before.

b. The same occurs when only one of the lines σ, σ' belongs to V.

c. But if $\sigma \in V$ and $\sigma' \in V$ then the power of λ which is singled out from $C_{\sigma\sigma'}$ is $i-3$. If, in addition, a section will be found separating $\{s,s'\}$ from $\{q,q'\}$ or $\{s,q'\}$ from $\{s',q\}$ so that the subgraph V could be divided into two parts then the effective index will increase by unity. Such pair of spinor lines will be called by us <u>essential</u>.

Thus, when there are spinor lines the effective index of the graph is equal to $\ell - 2\mu - r$, where r is the number of the essential pairs of spinor lines. Now we go over to consideration of the second mechanism. It is possible in unique way to take into account the fact that when $S \to \infty$ both scalar products $(p_1 p_2)$ and $(p_1 p_3)$ are large enough. For this we introduce new variables

$$\zeta = \frac{p_2 + p_3}{2} \quad \text{and} \quad \eta = \frac{p_2 - p_3}{2}.$$

It is seen that the only large product will be $(p_1 \eta) \sim S/2$ From eqs. (8) we have already known the coefficient of $\hat{p}_1$ in P_σ. Let us determine the coefficient of $\hat{\eta}$. Using eqs. (6) and (8) it can be easily found:

$$\frac{1}{a_\sigma \Delta}\,[A(s2;q4) - A(q2;s4) - A(s3;q4) + A(q3;s4)].$$

Expanding the 2-trees of the denominator with account of the vertex 3 in the two first terms and the vertex 4 in the two last ones, we find after combining and calcelling out that the coefficient of $\hat{\eta}$ is simply

$$\frac{A(s2;q3) - A(s3;q2)}{a_\sigma \Delta}$$

i.e. with account of only asymptotically large momenta, P_σ can be written in the form

$$\frac{1}{a_\sigma \Delta}\,\{[A(s1;q4) - A(s4;q1)]\,\hat{p} + [A(s2;q3) - A(s3;q2)]\,\hat{\eta}\}.$$

What now happens if any t - subgraph is contracted into the point? The graph G became "tied" in the middle graph G'. If several subgraphs are contracted then the number of knotes will be larger (see Fig. 2)

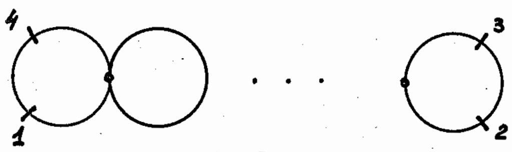


Fig. 2

First of all we consider spinor lines not entering the tied subgraph. If the contraction of such a graph leads to the vertices 1 and 4 or 2 and 3 being joinedl then the coefficients of $\hat{p}_1$ or $\hat{\eta}$ in P_σ for all such lines vanish, since it is impossible to cut the graph so that these points should lie in different parts. If the knot occured everywhere in the middle then in spinor lines to the right of the knot all the coefficients of $\hat{p}_1$ vanish, and in spinor lines to the left of it the coefficients for $\hat{\eta}$ vanish. Indeed, all paths joining the vertices s,q with 1,4 pass through this knot and hence cannot be separated. From the analytical point of view this means that after the barycentric transformation $a_\nu \to \lambda a_\nu$ for

$v \in V$ the indicated coefficients will be proportional to the positive power of λ. As an example, let us take the coefficient of $\hat{p}_1$ in a line to the right of V. Any section separating the points $\{s,1\}$ from $\{q,4\}$ e.g., must obligatorily cut V at least into two parts and as far as for $\lambda \to 0$, $A(s1;q4) \to \lambda^{\mu'+i-1} A(s1;q4)$ and $i \geq 2$ then $\frac{A(s1;q4)}{a_6 \Delta} \to \lambda^{i-1} \frac{A'(s1;q4)}{a_6 \Delta'}$. Besides it is clear that if the line σ belongs to V and there is at least one section separating the vertices $\{s,1\}$ from $\{q,4\}$ or $\{s,4\}$ from $\{q,1\}$ which divides V into two parts, then after the contraction of this subgraph into a point the coefficient of $\hat{p}_1$ does not vanish. The similar conclusion may be drawn about the coefficient of $\hat{\eta}$.

Now we consider a collective effect of the spinor lines forming a chain. Such spinor chains in the graph may be of two kinds: these are spinor cycles and so-called spinor polygons (i.e. unclosed spinor chains beginning and ending by external spinor lines). Consider a spinor cycle. We have just obtained the following result: in terms P_σ for spinor lines entering V_L (the most left of the contracted graphs) or lying to the left of it only momenta $\hat{p}_1$ "survives" and for lines entering V_R (the most right of the contracted subgraphs) or lying to the right of it only momenta $\hat{\eta}$ that "survives". For all the intermediate lines the momenta $\hat{p}$ and $\hat{\eta}$ fall out at all. Further, owing to $\hat{a}\hat{a} = a^2$ the contribution of each piece of the spinor cycle belonging to one of the above groups reduces to single $\hat{p}$ or $\hat{\eta}$ (It may be always assumed that each of these groups contains an odd number of lines, otherwise, we may "remove" one of them taking instead of $\hat{p}_1$ or $\hat{\eta}$ in P_σ the term with ζ or m_σ. Thus an additional power from the cycle will determined by $Sp(\hat{p}_1 \hat{\eta} \hat{p}_1 \hat{\eta} \ldots \hat{p}_1 \hat{\eta})$, i.e. by the number of subsequent pairs of intercepts of the spinor cycle one of which is placed to the left of V_L or enters it and another is placed to the right of V_R or enters it. Of course, if each of such intercepts contains at least one P_σ i.e. at least one line not entering the number of essential pairs of lines. Such a pair of intercepts will be called by us an <u>increasing</u> one.

In contrast to the spinor cycle, an intercept contiguous to the external spinor line should not be included in the number of increasing intercepts of the spinor polygon. Indeed, let, e. g., the polygon begins in the vertex I. Then for the intercept of this polygon lying to the left of V_L and entering it, after contraction only a term with $\hat{p}_1$ survives. But if now we remind that the amplitude must be between $\bar{v}^{\pm}$, $v^{\pm}$ and that e.g. $\bar{v}(p)\hat{p} = m\bar{v}(p)$ then everything becomes clear.

Now we are able to formulate a recipe for determining the asymptotics of any planar convergent graph with spinor lines. Let we have a sequence of independent (in the sense of definition of this section) t - subgraphs

$$V'_1 \dots V'_{r_1} \quad \text{with the index } k_1 \text{ (in the new sense)}$$
$$V^2_1 \dots V^2_{r_2} \quad \text{with the index } k_2$$
$$V^{\delta}_1 \dots V^{\delta}_{r_\delta} \quad \text{with the index } k_\delta$$

then the power of the main asymptotic term is determined by the minimal of the numbers $k_\kappa - h_\kappa$, where h_κ is the number of increasing pairs of spinor intercepts for a given sequence of subgraphs and the logarithm power is specified simply by the number of independent subgraphs entering the given set:

$$T \approx \frac{1}{S^{k-h}} (\ln S)^{r-1} .$$

4. R - Operation for Spinor Graphs

In considering divergent graphs we shall use the regularization method in the alpha representation developed in ref. /3/. As is known /6/ to regularize a divergent graph the following operation is employed

$$R = 1 + \sum_{2 \le m \le n-1} \hat{P}_{n_1} \dots \hat{P}_{n_m} + \hat{P}_n , \qquad (13)$$

where the summation is made over all possible divisions of the graph into blocks and the action of the operator $\hat{P}_{n'}$ on a generalized block (containing n' vertices and ℓ' external with respect to them lines) is given by the rules:

$$\hat{P}_{n'} = 1 \qquad \text{if} \quad n' = 1$$
$$\hat{P}_{n'} = 0 \qquad \text{if} \quad k' = \ell' - 2\mu' - r' > 0$$
$$\text{or if } \Gamma_{n'} \text{ is not strongly connected}$$

and

$$\hat{P}_{n'} = - \hat{M} \left(1 - \sum_{2 \le m' \le n'-1} \hat{P}_{n'_1} \dots \hat{P}_{n'_m}\right) \quad \text{if} \quad k' \le 0$$

where $\hat{M}$ is the operator which subtracts from the function of momenta the first $-2k$ of its expansion in the Maclaurin series.

We first consider graphs of the first class including all graphs two any divergent subgraphs of which either are contained into one another or have no common lines at all. For this case the formula (13) reads

$$R = (1-\hat{M}_1)\dots(1-M_a). \qquad (14)$$

It is necessary to note one pecularity of the spinor case. The fact is that the contribution from the given graph, as we know, is the sum of terms to which there corresponds a definite breaking of all the spinor lines into pairs and single lines (to these lines there correspond the functions $C_{\sigma\sigma'}$ or P_σ) But the number of essential spinor pairs defining the index may be different for different terms. For this reason operator $\hat{M}$ affects different terms in a different manner. However if we discuss the graph divergency in general then we bear in mind a minimal possible index which will be denoted by $-\omega$. But this singularity of the spinor case is automatically taken into account by the following procedure. Let us consider first a graph containing no internal divergent parts at all but divergent as a whole. If the degree of divergence of this graph is ω then the subtraction of the corresponding number of the first derivative, i.e. the residual sum of the Maclauren series is given by the formula /10/

$$T_R \approx \frac{1}{\omega!} \int \Pi d\alpha_\nu e^{iM(\alpha)} \int_0^1 d\rho (1-\rho)^\omega \frac{d^{\omega+1}}{d\rho^{\omega+1}} \frac{\rho^{2\mu+r}}{\Delta^2(\rho\alpha)} \prod_\sigma \left(\frac{1}{2\alpha_\sigma\rho} \epsilon_{\sigma j} \partial p_j + m_\sigma\right) \times$$

$$\times \exp[i d_{jk}(\alpha\rho) p_j p_k].$$

In fact this formula means the following. In a formal expression of (3) we replace each α_ν (except in the mass term $M(\alpha) = \Sigma m_\nu^2 \alpha_\nu$) by $\rho\alpha_\nu$ then multiply the whole expression by $\rho^{2\mu+r}$ and subtract from it ω first derivatives at $\rho = 0$ and then put $\rho = 1$. It is not difficult to see that this procedure is equivalent to the R-operator in the above sense. The same rule is also valid for graphs containing internal divergent blocks since they can be constructed from the above type graphs, additional integrations over the intermediate momenta not affecting the parameters $\rho\alpha$. So, for any graphs of the first class containing divergent subgraphs $\Gamma_1 \dots \Gamma_a$ we can immediately write the corresponding regularized contribution

$$T_R \approx \frac{1}{\omega_1! \dots \omega_a!} \int \Pi d\alpha_\nu e^{iM(\alpha)} \int_0^1 \prod_1^a \left[d\rho_b (1-\rho_b)^{\omega_b} \frac{d^{\omega_b+1}}{d\rho^{\omega_b+1}}\right] \times$$

$$\times \frac{\prod_1^a \rho_b^{2\mu_b+r_b}}{\Delta^2(\beta)} \prod_\sigma \left(\frac{1}{2i\beta_\sigma} \epsilon_{\sigma k} \partial p_k + m_\sigma\right) \exp[i d_{jk}(\beta) p_j p_k], \qquad (15)$$

where $\beta_\nu = \rho_0 \dots \rho_{0'} a_\nu$ if the line ν enters simultaneously divergent subgraphs $\Gamma_0 \dots \Gamma_{0'}$ and $\beta_\nu = a_\nu$ if it enters neither of the divergent subgraphs, the form $\Delta(\beta)$ and $d_{jk}(\beta)$ being constructed from parameters β according to usual rules. Show that the operator $(1-\hat{M})$ acts in just a manner as needed i.e. it subtracts from each term as many derivatives, as its degree of divergence is. Indeed, consider a term whose divergence is less than ω (maximal divergence), say ω'. This means that to the given term there corresponds not r but $r' < r$ essential pairs of spinor lines, i.e. the operator $(1-\hat{M})$ acts on a function which behaves as $\rho^{r-r'}$ if $\rho \to 0$. As a result the first $r-r'$ derivatives vanish and $\frac{1}{\omega!}\int_0^1 d\rho(1-\rho)^\omega \frac{d^{\omega+1}}{d\rho^{\omega+1}} F(\rho)\rho^{r-r'}$ transforms into $\frac{1}{\omega'!}\int_0^1 d\rho(1-\rho)^{\omega'}\frac{d^{\omega'+1}}{d\rho^{\omega'+1}}F(\rho)$ where $\omega' = \omega + r - r' = 2\mu - \ell + r'$.

5. Asymptotics of Divergent Graphs

Now we go over to consideration of the asymptotics of the scattering amplitude for which eq. (15) is rewritten in the form

$$T_R \approx \frac{1}{\omega_1! \dots \omega_a!}\int \Pi d a_\nu \int_0^1 \prod_1^a [d\rho_b (1-\rho_b)^{\omega_b} \frac{d^{\omega_b+1}}{d\rho^{\omega_b+1}}] \times$$

$$\times \frac{\prod_1^a \rho^{2\mu_b + r_b}}{\Delta^2(\beta)} f(\beta, \hat{\beta}, m_\sigma) \exp[i \frac{A(\beta)}{\Delta(\beta)} s + iB(\beta, t, m^2 a)].$$

As before, the two mechanisms operate here which increase the amplitude asymptotics. One of them is due to the presence of momenta in the preexponential factor and another to "minimal" t-subgraphs letting the coefficient $\frac{A}{\Delta}$ vanish. However now, in contrast to convergent graphs, this coefficient can be cancelled not only by vanishing some set of the parameters a but also by vanishing the parameters ρ corresponding to divergent graphs. The account of this effect on the asymptotics will be made by analogy with ref. /3/ removing the inaccuracies made there. A function $\Phi(\xi)$ analogous to (12) is now written in the form

$$\Phi(\xi) \approx \frac{i^\xi}{\omega_1! \dots \omega_a! \Gamma(\xi+1)}\int \Pi d a_\nu \int_0^1 \prod_1^a [d\rho_b (1-\rho_b)^{\omega_b} \frac{d^{\omega_b+1}}{d\rho^{\omega_b+1}}] \times \qquad (16)$$

$$\times \frac{\prod_1^a \rho_b^{2\mu_b + r_b}}{\Delta^2(\beta)} f(\beta, \hat{\beta}, m_\sigma)[\frac{A(\beta)}{\Delta(\beta)}]^\xi \exp[iB(\beta, t, m^2 a)].$$

Further on, for each term of eq. (16) it is possible to find a t-subgraph V with the index k which defines its power asymptotics. However now, in contrast to convergent graphs, if this V contains g allowed divergent subgraphs $\Gamma_1 \dots \Gamma_g$ (forbidden and allowed subgraphs are described below) it is possible to introduce $g+1$ set of parameters which lead to the appearance of the most right (for the given term) pole of the order $g+1$. These may be, e.g., the following sets

$$\{a_\nu \in V\}, \{a_\nu \in V - \Gamma_1, \rho_1\} \dots \{a_\nu \in V - \sum_1^g \Gamma_i, \rho_1 \dots \rho_g\}$$

each corresponds to onefold "covering" of any line of the subgraph V either by variables a but not simultaneously. The barycentric transformation corresponding to each set leads to an integral $\int d\lambda \lambda^{k+\xi-1}$ (since $\Delta \to \lambda^{\mu'} \Delta'$; $\frac{\partial}{\partial\rho} \to \frac{1}{\lambda}\frac{\partial}{\partial\rho}$ and $(\frac{A}{\Delta}) \to \lambda(\frac{A'}{\Delta'})$ when $\rho \to 0$) and to a new delta function of the type $\delta(1-\rho_1 - \Sigma a \in V - \Gamma_1)$. Such a set, as can be shown, is a maximal independent set related to the subgraph V. Thus, the integration over all the parameters λ corresponding to these sets leads to a singularity $(\xi + k)^{-g-1}$ being appeared.

However, in addition to the subgraph V, the graph may contain other independent subgraphs with the index k which include divergent blocks. Let $V_1 \dots V_r$ be a certain sequence of such independent t-subgraphs. After the barycentric transformation of just the same type as for the graph V and the integration over all λ we get at the point $\xi = -k$ a pole of the order $r + r'$ where

r' is the total number of allowed divergent subgraphs entering any of $V_1 \dots V_r$. Passing to the variable S by eq. (11) we obtain the following asymptotic behaviour corresponding to this singularity

$$T_R \approx S^{-k+h} \begin{cases} (\ln S)^{r+r'} & \text{for } k \le 0 \\ (\ln S)^{r+r'-1} & \text{for } k > 0 . \end{cases}$$

The exponent h is due to the influence on the asymptotics of the second mechanism, i.e. it is due to the appearance of the additional powers of S because of the momenta entering the preexponential factor. It must be calculated just as for convergent graphs. The number h is simply the number of increasing pairs of the spinor intercepts corresponding to the given sequence of t- subgraph. For each sequence the number of $-k+h$ will be different. So, to determine the main asymptotic term it is necessary to find a sequence with maximal number $-k+h$. Consider in more detail the concept of allowed divergent subgraphs, and the reason of their importance for the asymptotics. Let e.g. we have only one essential t- subgraph with the index k containing one divergent block. After the barycentric transformation and the integration over λ we find that the residue of the pole at $\xi = -k$ is determined in the scalar case by

$$\int d\rho\, (1-\rho)^{\omega} \frac{d^{\omega+1}}{d\rho^{\omega+1}} \frac{\rho^{2\mu}}{\Delta_v^2(\beta)} \left(\frac{A(\beta)}{\Delta(\beta)}\right)^{\xi}$$

since for $\lambda \to 0$ (see (3) or (11))

$$\Delta_G \to \Delta_V \Delta_{G'} \lambda^{\mu'+1}$$
$$A_G \to A_V A_{G''} \lambda^{\mu'+1} , \qquad (17)$$

where the graph G' is obtained from the graph G by contraction of the subgraph V, A_V is formed from V by means of S- section increasing by one the number of its components and the graph G'' is obtained from the graph G by contraction of the formed components of the subgraph V after breaking it by S - section of the graph G. However, if $\frac{\rho^{2\mu}}{\Delta_V(\beta)}$ and $\frac{A_V(\beta)}{\Delta_V(\beta)}$ are independent of ρ then this coefficient vanishes and the pole does not work. The latter is possible when and only when:

1. Any section of the graph Γ increases the number of components of V (in this case any of the trees of V can include only the trees of Γ and therefore $\Delta_V(\beta) = \rho^{\mu} \Delta_V(\alpha)$).

2. No one of the S- section of V increasing by one the number of its components affects Γ (in this case A_V may contain only trees of the subgraph and consequently $A_V(\beta) = \rho^{\mu} A_V(\alpha)$). The name "allowed" is attributed to those subgraphs for which on one of these conditions is fulfilled.

In spinor graphs terms such as $P_{\sigma_1} \dots P_{\sigma_j} C_{\sigma_{j+1}\sigma_{j+2}} \dots C_{\sigma_{m-1}\sigma_m}$ which are determined by (6), (8) and (9) with replacement of α by β enter under the sign of differentiation as well. But we notice that any $\frac{A(sp;qi)}{\beta_\sigma}$ is a sum over the product of chords of some selected trees of G. Indeed, $A(sp;qi)$ are 2-trees with vertices, s and q belonging to different components. If in any of these we reconstruct the line σ joining these vertices, what is equivalent in this case to the division by β_σ, then we obtain a tree. If after the replacement $\alpha_\nu \to \lambda\alpha_\nu$ for $\nu \in V$ and $\lambda \to 0$ we get

$$\frac{A(sp;qi)}{\beta_\sigma} \to \begin{cases} \lambda^{\mu'} \dfrac{\Delta_V A_{a'}(sp;qi)}{\beta_\sigma} & \text{for the line } \sigma \in V \\ \lambda^{\mu'}_{M} \dfrac{A'_V(s,q) A_{a''}}{\beta_\sigma} & \text{for the line } \sigma \in V \end{cases}$$

where the symbol A'_V denotes some 2-trees or V with vertices s and q belonging to different components. Taking into account the first of the relations (17) we become sure that the parameters ρ associated with the divergent subgraph Γ will affect only those P_σ and $C_{\sigma\sigma'}$ which have, at least, one line belonging to the subgraph. It is not difficult to see now that in fulfilling conditions (1) and (2) these additional terms in the spinor case are all the more independent of ρ. It is sufficient to note that $\frac{A'_V(s,q)}{\beta_\sigma}$ corresponds to some trees of the subgraph V and since, according to (1), any tree of V may contain only trees of Γ then $\frac{A'_V(s,q)}{\beta_\sigma \Delta_V}$ is independent of ρ. So, the concept of allowed divergent subgraphs remains in the spinor case the same.

Now we consider the graphs of the second class. The form of the R-operation for it can be obtained from the expression (14) by concelling all the terms containing the product $\hat{M}_b \hat{M}_o$ which corresponds to partially intersecting (in the sense of common lines) divergent subgraph Γ_b and Γ_o. This leads to a decrease of singularity of the integrant when the parameters ρ_b and ρ_o tend simultaneously to zero. As we already see this possibility corresponds to a "twofold covering" of some lines and therefore does not affect the main asymptotic term. For this reason all mentioned in this Section is extended to graphs of the second class with the only correction that in calculating the exponent r' only one of the intersecting divergent subgraphs are to be taken into account.

Finally we note one more important fact. If any essential t- subgraph contains e allowed divergent blocks $\Gamma_1 ... \Gamma_e$ satisfying however the condition (1) then the differentiations with respect to the corresponding ρ lead to the factor $\xi(\xi-1)...(\xi-N)$ being appeared in the nominator (where $N = \sum_1^e(\omega_i+1)$ and ω_i is the divergence of Γ_i). If furthermore $-N \leq k \leq 0$ then the effective power of the pole becomes smaller by one, the power of $\ln S$ will be smaller by one as well. Summarizing we formulate a final

6. Recipe

for finding the asymptotics of any planar graph with external momenta p_1, p_2, p_3, p_4 entering the vertices 1, 2, 3, 4; respectively.

$$S = (p_1+p_2)^2, \quad U = (p_1+p_2)^2, \quad t = (p_2+p_3)^2, \; S+U+t = \Sigma p_i^2$$

Definitions:

1. S- (or U) section is the cutting of the graph G into two connected components with separation of the vertices {1, 2 } from the vertices {3,4} (or {1,3 } from {2,4}).

2. The graph is called planar if it contains no U -section

3. The chain is the assembly of lines the end of any preceding line being the beginning of the subsequent one.

4. The subgraph is an arbitrary assembly of lines with the appropriate vertices.

5. The subgraph V' is called a t-subgraph with respect to V if any S-section increasing by one the number of components V and, at least, one of them increases the number of components by one.

6. The set of the subgraphs $V_1 ... V_r$ is called a sequence if each of them is the t- subgraph with respect to any of the foregoing and V_1 with respect to G.

7. The pair of spinor lines of the subgraph $V \subseteq G$ is called essential with respect to it if at least one of section of G into two components which separate the opposite ends of these lines increases the number of V's component by one.

8. The index of the subgraph V is the number $k = \ell - 2\mu - r$ where μ is the number of independent cycles of the subgraph, ℓ is the number of its lines and r is the number of essential pairs of spinor lines.

9. The subgraph Γ is divergent if its index $k \leq 0$.

10. The divergent subgraph $\Gamma \subset V \subseteq G$ is called allowed with respect to V if one of the following conditions is fulfilled: a) at least one of the sections of Γ does not increase the number of connected components of the subgraph V, b) at least one of the S- section increasing by one the number of components of V cut Γ.

11. The subgraph $V_1 ... V_r$ are independent if any of them cannot be completely constructed from the lines of the foregoing ones.

12. The pair of subsequent intercepts of the spinor chain is called increasing with respect to the given independent sequence of the t- subgraphs $V_1 ... V_r$ if a) each of the intercepts includes all the lines of the spinor chain lying on the one side of the set $V_1 ... V_r$ and entering the extreme of its subgraphs, b) each contains at least one of the lines not entering the number of essential pairs and c) no one of them is contiguous to the external spinor-line.

Rule

When $S \to \infty$ the contribution of the graph behaves like $S^{-k+h} (\ln S)^{r+r'-r''}$ where r is the number of subgraphs with the index k in the maximal sequence of the indepebdent set of t- subgraphs $V_1 ... V_r$ with h increasing pairs of spinor intercepts for which the number of $k-h$ is minimal; r' is the number of nonintersecting partially (in the sense of common lines) divergent subgraphs allowed with respect to the subgraphs of this sequence and begin not t- subgraphs, r'' when $k \leq 0$ is the number of subgraphs of sequence $V_1 ... V_r$ which contain e allowed divergent blocks not satisfying the condition a) in the definition 9 with indices satisfying unequality $\sum_1^e (q_i - 1) \leq k$ and $r'' = 1$ when $k > 0$.

7. Examples

Let us start from the simplest one

Example 1

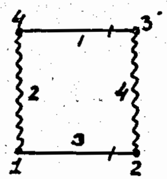


Fig. 3.

The essential subgraphs: V_1 = (the graph as a whole) $\ell - 2\mu = 2$, $r = 1$, $k = 1$ and two subgraphs which consist of one line: V_2=(4) and V_3=(2) with $\ell - 2\mu = 1$ and $r = 0$ consequently $T \approx \frac{1}{S} (\ln S)^2$.

Example 2 Let us show how the essential pair of spinor intercepts works

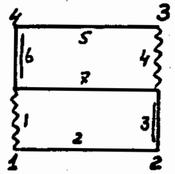
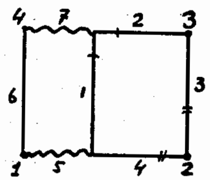


Fig.4.

The essential subgraphs: V_1 = (the graph as a whole) $\ell - 2\mu = 3$, $r = 1$ (e. g. the lines 2,7), $k = 2$; V_2 =(1,6); V_3 = (3,4) both with $k = 2$. The lines 6 and 3 form an increasing pair of intercepts, i.e. $h = 1$ as a result $T \approx \frac{1}{S} \ln^2 S$.

Two examples with divergent graphs.

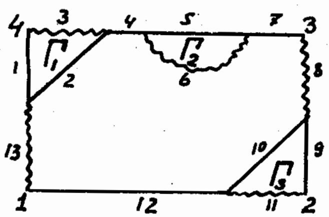
Example 3



$V = (1, 2, 3, 4)$, $\ell - 2\mu = 2$, $r = 2$

$T \approx \ln S$

Example 4



$V_1 = (1, 2, 3, 13)$ contains Γ_1 ; $\ell - 2\mu = 2$, $r = 1$, $k = 1$.

$V_2 = (9, 10, 11, 8)$ contains Γ_3 ; $\ell - 2\mu = 2$, $r = 1$, $k = 1$.

Γ_2 is forbidden and there is no essential pairs in intercepts, so that $T \approx \frac{1}{S} (\ln S)^2$.

8. Conclusion

So we have succeeded in formulating in a more of less compact form a rule for finding the asymptotics of one more wide class of graphs with a spinor lines. It may be said that these are quite "physical" graphs, as far as this class includes also electrodynamics though there can occur some pecularity related to the photon zero mass.

Till now however the wide class of non-planar graphs remains non-covered. An interesting attempt to study them has been made by Tictopulos /12/, however the problem of the connection between the asymptotics of a non-planar graph and its topology in the general sense remains still unsolved. However, it may be said that we "have bypassed" this problem. In fact, almost in each order of theory in which there is divergent "fourleg" (in electrodynamics, e.g. this is the photon-photon scattering) we may indicate the class of graphs which are the most important from the point of view of asymptotics and the non-planarity of which plays no role. These are graphs which contain divergent t- subgraphs. Indeed the pecularity of non-planar subgraphs is that the coefficient of it is not positive definite and therefore it can vanish somewhere in the middle of the integration region, not only at its boundary. However this vanishing has a character of the mutual cancelling out like $(a_j - a_j^0) \ldots (a_k - a_k^0)$ and apparently cannot lead to a singularity in the ξ -plane to the right of the point $\xi = -1$. (very likely it has a singularity at $\xi = -1$ namely) but the divergent subgraphs lead to the singularity at $\xi = 0$ i.e. it is they which define the asymptotics. For example, in the sixth order for the Compton effect the graph considered in the fig. 5 is the largest in the asymptotics.

Basing on this property, for various processes we may speak about the classes of the most important graphs for asymptotics. For meson-meson scattering, e.g. it may be states that the most important are graphs of the type

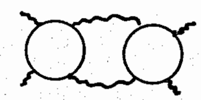


Fig. 7

what agrees with conclusions obtained on the basis of analiticity and crossing symmetry. The application of these methods for finding the class of asymptotically main graphs in many particle processes can give some interesting results as well.

One more domain of application of the developed methods is the graphs with two and three external lines. The point is that, in our sense, such graphs are always planar, i.e. the coefficient of p^2 is always positive. For the vertex part in the electrodynamics, e.g., it may be asserted that approximate equation of the type

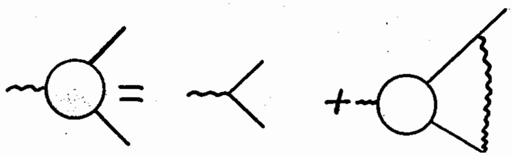


Fig. 8

will be incorrect since graphs such as

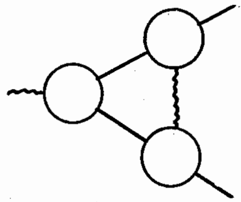


Fig. 9

give a singularity equal to that from the mentioned graphs.

Finally, one more application of these methods is a simple summation of graphs which are the main in each order. For this however, besides the asymptotics itself its coefficient is needed. In the present paper we are not dealing with the calculation of it but this can be certainly done. Simply we have to take into account that the leading asymptotic term of each graph can yield several sets of independent subgraphs and therefore it is necessary to sum up over all these possibilities.

In conclusion I would like to thank first of all O.I. Zavialov who gave me an apportunity to read his manuscript before publication and for interesting discussions. The author thanks also D.I. Blokhintsev, I.F. Ginsburg, A.T. Filippov for stimulating discussions.

APPENDIX

If any S - section cut the subgraph V with the index k at least into i' components then after barycentric transformation $a_\nu \to \lambda a_\nu$ for $\nu \in V$ and $\lambda \to 0$ $(\frac{A}{\Delta}) \to \lambda^{i'-i} (\frac{A'}{\Delta'})$ (i is the number of connected components of the subgraph V) what leads to a pole at the point $\xi = -\frac{k}{i'-i}$ if, of couse, V contains no forbidden divergent subgraphs. Let us show, however, that in this case we can single out from V such a subgraph V' that it will be the t-subgraph with respect to G its index being $k' \le \frac{k}{i'-i}$.

Indeed, let V consist of i connected components $V_1 \dots V_i$ the first r of which contain the t-path (i.e. a chain cut by any S-section, at least one of them cutting it into two parts) and each V_q be cut by any S-section at least into $i_q - 1$ components. We prove that if $i_q \ge 3$ then V_q consist of

$i_q - 1$ weakly connected t-subgraphs. We single out from V_q such connected subgraphs that for any S-section they should remain to one side (let us call them invariant). Let I, Q and J be three subsequent subgaphs such that any path from I to J belonging totally to V_q passes only through Q, Q and I, J lying to opposite sides of any S-section. It is clear that any path from I to Q as well as from Q to J must be a t-path, otherwise they would not be invariant. For definiteness we shall assume that Q can be joined by paths not cut by S-sections with vertices 3 and 4, and I and J with vertices 1 and 2. Now we show that any path from I to J passes through the same vertex belonging to Q i.e. V_q consists of two weakly connected in Q t-subgraphs In fact, let m_1 and m_2 be the ends of the t-paths joining Q with I and J respectively and let m_3 and m_4 be the ends of the paths joining Q with the vertices 3 and 4 then there exists no section separating the vertices $\{m_1, m_2\}$ from $\{m_3, m_4\}$ and dividing Q into two parts. Otherwise it would make a part of the S-section. And if this is so then taking into account that Q must be planar, i.e. there is no nonintersecting paths $m_1 \to m_3$ and $m_2 \to m_4$, all the paths from m_1 to m_2 intersect in the same vertex. Let another path from I to Q ends in the vertex m'_1, then all the paths from m'_1 to m_2 must go through the vertex m' but in virtue of the fact that Q is connected, the vertex m' concides with m otherwise there would exist a path from m_1 to m_2 not passing through m. Applying these considerations to any three subsequent invariant subgraphs it may be concluded that V_q consists of $i_q - 1$ weakly connected subgraphs $V'_q \dots V_q^{i_q - 1}$ Note that among the subgraphs $V_{r+1} \dots V_i$ these may not exist a divergent one otherwise it will be forbidden, therefore $k \ge \tilde{k}$ which is the index of the union $V_1 \dots V_r$. However, each of these V_q consists itself of weakly connected subgraphs, therefore.

$$\tilde{k} = \sum_{q=1}^{r} \sum_{p=1}^{i_q - 1} k_q^p$$

hence it follows that if V' entering the set $\{V_q^p\}$ has the minimal index then since

$$\sum_{q=1}^{r} (i - 1) = i' - i \,, \quad k \ge \tilde{k} \ge (i' - i)k'$$

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