

с 347а

B-64

3/IV-65

✓

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E-2047



ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

J. Blank, I. Ulehla

SOLUTION OF THE BETHE-GOLDSTONE EQUATION
FOR THE REACTION MATRIX
IN FINITE NUCLEI

*Czech. J. Phys., 1965,
v 15, n 11, p 797-823.*

1965

J. Blank, I. Uehla

SOLUTION OF THE BETHE-GOLDSTONE EQUATION
FOR THE REACTION MATRIX
IN FINITE NUCLEI

30.58/3 49

Объединенный институт
ядерных исследований
БИБЛИОТЕКА

1. Introduction

The purpose of this paper is to develop a method for calculation of the binding energies and the r.m.s. radii of some nuclei with closed shells according to the Goldstone formula (ref.1.). This problem was solved by Eden et al. for O^{16} (ref.2.). They obtained a good result for the binding energy but the r.m.s. radius was found too small. In order to obtain a more correct value, Eden proposed to use instead of a harmonic oscillator single-particle potential another single-particle potential which would give the wave functions behaving asymptotically as e^{-x} . However for such a potential the separation in the CMS system cannot be practically performed and thus the problem becomes very complicated. On the other hand the discrepancy in the value of the r.m.s. radius can be due to the applied self-consistent method, which approximates the ground state wave function by a Slater determinant constructed from the self-consistent single-particle functions. It is possible that the influence of the pair correlations is essential.

Our idea is therefore to retain a harmonic oscillator potential with a free parameter and to calculate the first three terms of the Goldstone formula for the ground state energy and wave function imposing no self-consistency requirement. The free parameter is to be chosen to make the convergence as good as possible. Such calculation can also indicate whether the Goldstone series actually converges.

In order to perform this program we need a method which would give us the solution of the BG equation sufficiently exact to be sure that eventual negative results are not due to the used approximations. In this paper we develop such a method and prove numerically its accuracy for a mathematically simple one-dimensional case. Our starting assumptions are (1) a harmonic oscillator form for the single-particle potential is used, (2) the exclusion principle (EP) is taken exactly into account, (3) the internucleon potential $v(r)$ consists of a hard core (which is treated as the limit case of the rectangular repulsive potential with a variable height v_0) and of an attractive part of the Yukawa type. The calculations are performed in relative Cartesian coordinates and the dependence on the variables y and z is omitted. In this way the problem becomes one-dimensional. The reason for the use of Cartesian coordinates is the simple form of the transformation coefficients for the transition from the laboratory system to the CMS (ref. 3,4). Otherwise our one-dimensional equations are almost the same as the equations in spherical coordinates for an internucleon potential acting in the S -state only.

In our next paper we shall demonstrate that this method is applicable in the general three-dimensional case for different phenomenological internucleon potentials and derive the formulas for the more complicated matrix elements of the reaction matrix t occurring in the third term of the Goldstone formula for the ground state energy. The necessary matrix elements of t will be then calculated numerically.

II. The System of Integral Equations for the Wave Function of a Pair of Nucleons

The BG equation in operator form (ref.1)

$$t = v + v \frac{Q}{E_0 - H_0} t \quad (2.1)$$

determines the reaction matrix t of a pair of nucleons in nuclear medium if the internucleon potential v is known.

The Hamiltonian

$$H_0 = \sum_{j=1}^A (T_j + V_j)$$

contains a model single-particle potential V . E_0 is the energy of the lowest model eigenstate Φ_0 (we consider only such nuclei in which no degeneracy with respect to H_0 arises). The projection operator Q ensures that intermediate states of a pair of nucleons in (2.1) are both excited with respect to all occupied states in Φ_0 .

The single-particle potential V , normalized eigenfunctions and corresponding eigenvalues are of the form:

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{2} m \omega^2 r^2; & \phi_{\vec{r}}(\vec{r}) &= \phi_{n_x}(x) \phi_{n_y}(y) \phi_{n_z}(z); \\ \phi_n(x) &= [2^{-n} n! \sqrt{\pi}]^{-1/2} e^{-1/2(\alpha x)^2} H_n(\alpha x) \sqrt{\alpha}; & \alpha &= \sqrt{\frac{m\omega}{\hbar}}; \\ E_{n_x n_y n_z} &= \hbar \omega \left(n + \frac{3}{2} \right); & n &= n_x + n_y + n_z, \end{aligned} \quad (2.2)$$

where m is the mass of a nucleon, $H_n(x)$ is the Hermite polynomial as defined in ref.5.

Expressing (2.1) in the representation (2.2) we obtain

$$(\vec{n}_1 \vec{n}_2 | t | \vec{n}_1 \vec{n}_2) = (\vec{n}_1 \vec{n}_2 | v | \vec{n}_1 \vec{n}_2) + \sum'_{\vec{n}_1 \vec{n}_2'} \frac{(\vec{n}_1 \vec{n}_2 | v | \vec{n}_1 \vec{n}_2') (\vec{n}_1 \vec{n}_2' | t | \vec{n}_1 \vec{n}_2)}{E_{n_1} + E_{n_2} - E_{n_1'} - E_{n_2'} - \delta E} \quad (2.3)$$

Here δE is expressed by means of the single-particle energies E_n according to ref. 1 and Σ' means that the summation extends over excited states only.

Further we restrict ourselves to the study of the matrix elements of t corresponding to the transitions from occupied initial states, i.e. \vec{n}_1 and \vec{n}_2 denote occupied states and $\delta E = 0$. Multiplying (2.3) by $\frac{2}{\hbar \omega}$ and denoting $\bar{t} = \frac{2}{\hbar \omega} t$, $\bar{v} = \frac{2}{\hbar \omega} v$ we get

$$(\vec{n}_1, \vec{n}_2 | \bar{t} | \vec{n}_1, \vec{n}_2) = \int \phi_{\vec{n}_1}(\vec{r}_1) \phi_{\vec{n}_2}(\vec{r}_2) \bar{v}(\vec{r}_1 - \vec{r}_2) [\phi_{\vec{n}_1}(\vec{r}_1) \phi_{\vec{n}_2}(\vec{r}_2) + (2.3') \\ + \sum'_{\vec{n}'_1, \vec{n}'_2} \frac{\phi_{\vec{n}'_1}(\vec{r}_1) \phi_{\vec{n}'_2}(\vec{r}_2)}{2(n_1 + n_2 - n'_1 - n'_2)} \int \phi_{\vec{n}'_1}(\vec{r}'_1) \phi_{\vec{n}'_2}(\vec{r}'_2) \bar{v}(\vec{r}'_1 - \vec{r}'_2) \psi_{\vec{n}_1, \vec{n}_2}(\vec{r}'_1, \vec{r}'_2) d^3 r'_1 d^3 r'_2].$$

The solution of this equation retains all important features of the general case (2.3).

If we denote the square bracket in (2.3') as $\psi_{\vec{n}_1, \vec{n}_2}(\vec{r}'_1, \vec{r}'_2)$ we get

$$\psi_{\vec{n}_1, \vec{n}_2}(\vec{r}'_1, \vec{r}'_2) = \phi_{\vec{n}_1}(\vec{r}'_1) \phi_{\vec{n}_2}(\vec{r}'_2) + \\ + \sum'_{\vec{n}'_1, \vec{n}'_2} \frac{\phi_{\vec{n}'_1}(\vec{r}'_1) \phi_{\vec{n}'_2}(\vec{r}'_2)}{2(n_1 + n_2 - n'_1 - n'_2)} \int \phi_{\vec{n}'_1}(\vec{r}'_1) \phi_{\vec{n}'_2}(\vec{r}'_2) \bar{v}(\vec{r}'_1 - \vec{r}'_2) \psi_{\vec{n}_1, \vec{n}_2}(\vec{r}'_1, \vec{r}'_2) d^3 r'_1 d^3 r'_2. \quad (2.4)$$

Further we introduce CMS and relative coordinates

$$\vec{R} = \frac{\vec{r}'_1 + \vec{r}'_2}{\sqrt{2}}; \quad \vec{r} = \frac{\vec{r}'_1 - \vec{r}'_2}{\sqrt{2}}, \quad (2.5)$$

so that the form of the eigenfunctions remains the same as in (2.2). Denoting

$$\psi_{N, \vec{n}_1, \vec{n}_2}(\vec{r}) = \int \phi_N(\vec{R}) \psi_{\vec{n}_1, \vec{n}_2}(\frac{\vec{R} + \vec{r}}{\sqrt{2}}, \frac{\vec{R} - \vec{r}}{\sqrt{2}}) d^3 R \\ I_{N, \vec{n}_1, \vec{n}_2}(\vec{r}) = \int \phi_N(\vec{R}) \phi_{\vec{n}_1}(\frac{\vec{R} + \vec{r}}{\sqrt{2}}) \phi_{\vec{n}_2}(\frac{\vec{R} - \vec{r}}{\sqrt{2}}) d^3 R, \quad (2.6)$$

we have from (2.4)

$$\psi_{N, \vec{n}_1, \vec{n}_2}(\vec{r}) = I_{N, \vec{n}_1, \vec{n}_2}(\vec{r}) + \\ + \sum'_{\vec{n}'_1, \vec{n}'_2} \frac{I_{N, \vec{n}'_1, \vec{n}'_2}(\vec{r})}{2(n_1 + n_2 - n'_1 - n'_2)} \int I_{N, \vec{n}'_1, \vec{n}'_2}(\vec{r}') \bar{v}(\sqrt{2}\vec{r}') \psi_{N, \vec{n}_1, \vec{n}_2}(\vec{r}') d^3 r'. \quad (2.7)$$

This is an infinite system of integral equations equivalent with (2.4).

Now we shall drop the dependence on y and z . We can further express the functions $I_{N, n_1, n_2}(x)$ explicitly (see. ref. 3):

$$I_{N, n_1, n_2}(x) = T_{N, n_1 + n_2 - N}^{n_1, n_2} \phi_{n_1 + n_2 - N}(x) \quad (2.8) \\ T_{N, n_1}^{n_1, n_2} = \delta_{n_1 + n_2, N + n} \left[\frac{N! n!}{n_1! n_2! 2^{N+n}} \right] \sum_{j+k=n} (-1)^k \binom{n_1}{j} \binom{n_2}{k}.$$

Let the occupied states be $n_j = 0, 1, \dots, n_0 - 1$. Introducing a dimensionless variable $r = \alpha x$ and

$$\psi_{N, n_1, n_2}(r) = \frac{1}{\sqrt{\alpha}} \psi_{N, n_1, n_2}(x); \quad \phi_n(r) = (2^{-n} n! \sqrt{\pi})^{-1/2} e^{-1/2 r^2} H_n(r); \quad v(r) = \bar{v}(\sqrt{2}x)$$

we obtain from (2.7)

$$\psi_{N, n_1, n_2}(r) = T_{N, n_1+n_2}^{n_1, n_2} \phi_{n_1+n_2, N}(r) + \sum_{N'=0}^{\infty} \sum_{n_1'=n_0}^{\infty} \sum_{n_2'=n_0}^{\infty} \frac{\phi_{n_1'+n_2', N'}(r)}{2(n_1+n_2-n_1'-n_2')} T_{N, n_1'+n_2', N}^{n_1', n_2'} T_{N', n_1'+n_2', N'}^{n_1', n_2'} \int dr' \phi_{n_1'+n_2', N'}(r') v(r') \psi_{N', n_1', n_2'}(r') \quad (2.9)$$

The kernels in (2.9) can be transformed if we introduce new indices in the double sum and use the orthogonality relations and symmetry properties of the coefficients T

$$\nu = n_1' + n_2' \quad \nu \geq 2n_0; \quad \mu = n_2' \quad n_0 \leq \mu \leq \nu - n_0;$$

$$K_{NN'}(r, r') = \sum_{n_1'=n_0}^{\infty} \sum_{n_2'=n_0}^{\infty} \frac{\phi_{n_1'+n_2', N}(r) \phi_{n_1'+n_2', N'}(r')}{2(n_1+n_2-n_1'-n_2')} T_{N, n_1'+n_2', N}^{n_1', n_2'} T_{N', n_1'+n_2', N'}^{n_1', n_2'} = (2.10)$$

$$= \sum_{\nu=n_0}^{\infty} \frac{\phi_{\nu, N}(r) \phi_{\nu, N'}(r')}{2(n_1+n_2-\nu)} [\delta_{NN'} - \sigma_{\nu NN'}]$$

where

$$\sigma_{\nu NN'} = [1 + (-1)^{N+N'}] \sum_{\mu=0}^{n_0-1} T_{N, \nu-N}^{\nu-\mu, \mu} T_{N', \nu-N'}^{\nu-\mu, \mu}$$

Using the symmetry of the harmonic oscillator eigenfunctions we find from (2.4)

$$\psi_{n_1, n_2}(-r_1, -r_2) = (-1)^{n_1+n_2} \psi_{n_1, n_2}(r_1, r_2)$$

Hence

$$\psi_{N, n_1, n_2}(-r) = (-1)^{n_1+n_2+N} \psi_{N, n_1, n_2}(r) \quad (2.11)$$

Using this relation we can write (2.9) for $r \geq 0$ only:

$$\psi_{N, n_1, n_2}(r) = T_{N, n_1+n_2}^{n_1, n_2} \phi_{n_1+n_2, N}(r) + \sum_{N'=0}^{\infty} \int_0^{+\infty} K_{NN'}(r, r') v(r') \psi_{N', n_1', n_2'}(r') dr' \quad (2.12)$$

$$\bar{K}_{NN'}(r, r') = K_{NN'}(r, r') + (-1)^{n_1+n_2+N'} K_{NN'}(r, -r')$$

In addition, since the kernels $\bar{K}_{NN'}$ connect only N and N' of the same parity (see(2.10)), the system is splitted into two independent systems for even or odd N respectively.

We shall further specify the internucleon potential $v(r)$

$$v(r) = \begin{cases} v_0 & \dots\dots\dots 0 \leq r < a \\ -k \frac{e^{-\mu r}}{r} & \dots\dots\dots a < r < R \\ 0 & \dots\dots\dots R < r \end{cases}; \quad a = \frac{r_c}{\sqrt{2}}; \quad k > 0; \quad \mu > 0; \quad v_0 > k \frac{e^{-\mu a}}{a}$$

where r_c is the hard core radius, equal approximately 0.4 f and R is arbitrarily large, $R \gg a$. Hence it is sufficient to solve (2.12) for $0 \leq r \leq R$.

III. Reduction of the Infinite System of Integral Equations

In order to reduce the infinite system (2.12) a method suggested in ref. 6 will be applied. Let us define for $0 \leq r \leq +\infty$ the following functions

$$\left. \begin{aligned} \psi_{n_1 n_2}(r) &= \psi_{N n_1 n_2}(r - NR) \\ F_{n_1 n_2}(r) &= T_{N, n_1 + n_2}^{n_1 n_2} \psi_{n_1 + n_2, N}(r - NR) \\ V(r) &= v(r - NR) \end{aligned} \right\} r \in \langle NR, R + NR \rangle$$

$$K(r, r') = \bar{K}_{NN'}(r - NR, r' - N'R) \quad \left\{ \begin{array}{l} r \in \langle NR, R + NR \rangle \\ r' \in \langle N'R, R + N'R \rangle \end{array} \right.$$

Then the system (2.12) can be written as one equation

$$\psi_{n_1 n_2}(r) = F_{n_1 n_2}(r) + \int_0^{+\infty} K(r, r') V(r') \psi_{n_1 n_2}(r') dr' \quad (3.1)$$

We decompose the kernel $K(r, r')$ into two parts

$$K(r, r') = K_0^{(M)}(r, r') + K_1^{(M)}(r, r'),$$

where

$$K_{(j)}^{(M)}(r, r') = \bar{K}_{(j)NN'}^{(M)}(r - NR, r' - N'R) \quad \left\{ \begin{array}{l} r \in \langle NR, R + NR \rangle \\ r' \in \langle N'R, R + N'R \rangle \\ j = 0, 1 \end{array} \right.$$

and

$$\begin{aligned} \bar{K}_{(0)NN'}^{(M)}(r, r') &= \sum_{\nu = \max(2n_0, N)}^{\infty} \frac{\phi_{\nu-N}(r) \phi_{\nu-N}(r')}{2(n_1 + n_2 - \nu)} [1 + (-1)^{\nu + n_1 + n_2}] \delta_{NN'} \\ &- \sum_{\nu = \max(2n_0, N, N')}^M \frac{\phi_{\nu-N}(r) \phi_{\nu-N'}(r')}{2(n_1 + n_2 - \nu)} [1 + (-1)^{\nu + n_1 + n_2}] \sigma_{\nu NN'} \end{aligned} \quad (3.2)$$

$$\bar{K}_{(1)NN'}^{(M)}(r, r') = - \sum_{\nu = \max(M+1, N, N')}^{\infty} \frac{\phi_{\nu-N}(r) \phi_{\nu-N'}(r')}{2(n_1 + n_2 - \nu)} [1 + (-1)^{\nu + n_1 + n_2}] \sigma_{\nu NN'}$$

M positive integer

(cf. (2.10), (2.12)).

Substituting (3.2) in (3.1) and denoting

$$f_{n_1 n_2}(r) = F_{n_1 n_2}(r) + \int_0^{+\infty} K_0^{(M)}(r, r') V(r') \psi_{n_1 n_2}(r') dr', \quad (3.3)$$

we get

$$\psi_{n_1 n_2}(r) = f_{n_1 n_2}(r) + \int_0^{+\infty} K_1^{(M)}(r, r') V(r') \psi_{n_1 n_2}(r') dr'.$$

Using the well-known inequality

$$\int_0^{+\infty} f(r) g(r) dr \leq \|f\| \|g\|,$$

where

$$\|f\|^2 = \int_0^{+\infty} |f(r)|^2 dr$$

for any quadratic integrable functions f and g , one finds:

$$\|\psi_{n_1 n_2} - f_{n_1 n_2}\| \leq \nu_0 B_M \|\psi_{n_1 n_2}\|; \quad B_M = \|K_1^{(M)}\|. \quad (3.4)$$

Only quadratic integrable solutions are of interest i.e. we require

$$\|\psi_{n_1 n_2}\| < +\infty.$$

Direct evaluation of B_M gives

$$B_M^2 = \sum_{N, N'=0}^{\infty} \int_0^R \int_0^R [K_{(1)NN'}^{(M)}(r, r')]^2 dr' dr < \sum_{N, N'=0}^{\infty} \int_0^{+\infty} \int_0^{+\infty} [K_{(1)NN'}^{(M)}(r, r')]^2 dr dr' = \\ = \sum_{N, N'=0}^{\infty} \sum_{\nu=\max(M+1, N, N')}^{\infty} \frac{1 + (-1)^{\nu+n_1+n_2}}{2} \frac{\sigma_{\nu NN'}^2}{4(n_1 + n_2 - \nu)^2}.$$

Interchanging the order of summations we obtain finally

$$B_M^2 < \frac{1}{8} \sum_{\nu=M+1}^{\infty} \frac{1 + (-1)^{\nu+n_1+n_2}}{(n_1 + n_2 - \nu)^2} \sum_{N=0}^{\nu} \sum_{N'=0}^{\nu} \sigma_{\nu NN'}^2.$$

The double sum can be evaluated by means of the orthogonality relations of the coefficients T , the result being equal to $2n_0$. Hence

$$B_M^2 < \frac{n_0}{4} \sum_{\nu=M+1}^{\infty} \frac{1 + (-1)^{\nu+n_1+n_2}}{(n_1 + n_2 - \nu)^2} \quad (3.5)$$

and the quantity B_M^2 decreases as $1/M$. Consequently the approximation

$$\psi_{n_1 n_2}(r) = f_{n_1 n_2}(r) \quad (3.6)$$

can be made arbitrarily exact, i.e. the kernel $K(r, r')$ can be replaced by $K_0^{(M)}(r, r')$ if M is sufficiently large.

Using this approximation we get from (2.12) and (3.2)

$$\psi_{N n_1 n_2}(r) = T_{N, n_1+n_2-N}^{n_1 n_2} \phi_{n_1+n_2-N}(r) + \\ + \sum_{\nu=\max(2n_0, N)}^{\infty} \frac{\phi_{\nu-N}(r)}{2(n_1 + n_2 - \nu)} [1 + (-1)^{n_1+n_2+\nu}] \int_0^R \phi_{\nu-N}(r') v(r') \psi_{N n_1 n_2}(r') dr' - \\ - \sum_{\nu=\max(2n_0, N)}^M \frac{\phi_{\nu-N}(r)}{2(n_1 + n_2 - \nu)} [1 + (-1)^{\nu+n_1+n_2}] \sum_{N'=0}^{\nu} \sigma_{\nu NN'} \int_0^R \phi_{\nu-N}(r') v(r') \psi_{N n_1 n_2}(r') dr'. \quad (3.7)$$

In the system (3.7) only the functions ψ_{N, n_1, n_2} ($N=0; 1; \dots; M$) are coupled which implies the possibility to solve first this finite system. If $N \geq M+1$ the equations for ψ_{N, n_1, n_2} are decoupled since the third term in (3.7) is non-zero for $N \leq M$ only. Further $\Gamma_{M, n_1+n_2-M}^{n_1, n_2} = 0$ $M > n_1 + n_2$ and hence

$$\psi_{N, n_1, n_2}(r) = 0 \quad N \geq M > n_1 + n_2. \quad (3.8)$$

The approximation (3.6) is clearly an approximation of the EP which includes for $M < 2n_0$ the "triangular approximation" used in ref.2. From (3.7) one further sees, that the "triangular approximation" is the only case when no coupling arises. The approximation (3.6) represents a modified form of the EP which is demonstrated in fig.1.

In this way the reduction of the infinite system (2.12) has been performed for a finite value of v_0 . It should be noted that the estimation (3.4) is not applicable for the limit case $v_0 = +\infty$, but in this case all previous formulation of the problem is meaningless from the mathematical point of view. The only way how to treat this case exactly is to solve the problem with a finite value of v_0 , then find the limit form of the solution and investigate its dependence on M . This will be performed in the next sections.

IV. The Solution of the Reduced System

In this section the solution of the system (3.7) will be found. Denoting

$$\alpha_{\nu N}^{(n_1, n_2)} = \int_0^a \phi_{\nu-N}(r) v_0 \psi_{N, n_1, n_2}(r) dr$$

$$\bar{\alpha}_{\nu N}^{(n_1, n_2)} = \int_a^R \phi_{\nu-N}(r) v(r) \psi_{N, n_1, n_2}(r) dr$$

$$Y_{\nu N}^{(n_1, n_2)} = \begin{cases} \alpha_{\nu N}^{(n_1, n_2)} & \dots \dots 0 \leq \nu \leq 2n_0 - 1 \\ \nu \\ \sum_{N=0}^{\nu} \sigma_{\nu NN} \alpha_{\nu N}^{(n_1, n_2)} & \dots \dots \nu \geq 2n_0 \end{cases} ;$$

$$Y_{\nu N}^{-(n_1, n_2)} = \begin{cases} \bar{\alpha}_{\nu N}^{(n_1, n_2)} & \dots \dots 0 \leq \nu \leq 2n_0 - 1 \\ \nu \\ \sum_{N=0}^{\nu} \sigma_{\nu NN} \bar{\alpha}_{\nu N}^{(n_1, n_2)} & \dots \dots \nu \geq 2n_0 \end{cases} \quad (4.1)$$

$$\Phi_{N, n_1, n_2}(r) = \int_a^R [G^{(n_1+n_2-N)}(r, r') + (-1)^{n_1+n_2+N} G^{(n_1+n_2-N)}(r, r')] v(r') \psi_{N, n_1, n_2}(r') dr'$$

$$G^{(n)}(r, r') = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\phi_k(r) \phi_k(r')}{2(n-k)},$$

we obtain from (3.7)

$$\psi_{N, n_1, n_2}(r) = T_{N, n_1 + n_2 - N}^{n_1, n_2} \phi_{n_1 + n_2 - N}(r) + \Phi_{N, n_1, n_2}(r) + \int_0^a [G^{(n_1, n_2)}(r, r') + (-1)^{n_1 + n_2 + N} G^{(n_1, n_2)}(r, -r')] v_0 \phi_{N, n_1, n_2}(r') dr' - \sum_{\substack{\nu=N \\ \nu \neq n_1 + n_2}}^M \frac{\phi_{\nu, N}(r)}{2(n_1 + n_2 - \nu)} [1 + (-1)^{n_1 + n_2 + \nu} \{ \gamma_{\nu, N}^{(n_1, n_2)} + \gamma_{\nu, N}^{-(n_1, n_2)} \}]. \quad (4.2)$$

In the definition of G weak convergence is meant i.e. for any quadratic integrable function f any fixed r it shall hold

$$\int_{-\infty}^{+\infty} G^{(n)}(r, r') f(r') dr' = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\phi_k(r)}{2(n-k)} \int_{-\infty}^{+\infty} \phi_k(r') f(r') dr' \quad (4.3)$$

while the usual convergence of the series (4.1) is not guaranteed (see ref. 7).

The kernel of the system (4.2) contains a very large (or infinite) quantity v_0 for $0 \leq r \leq a$ and hence it is impossible to solve it by means of an iteration method. We shall transform (4.2) to an usual algebraic linear system and linear system of integral equations for $a \leq r \leq R$. The kernel of the integral system contains only the attractive part of the internucleon potential and allows to find the solution of the system by a very rapidly convergent iteration method. This transformation can be applied without any change to the case $v_0 = +\infty$ (see sec. V).

To perform the transformation let us suppose first of all that the functions $\psi_{N, n_1, n_2}(r)$ for $a \leq r \leq R$, $N = 0, 1, \dots, \infty$ are given i.e. all quantities $\bar{\alpha}_{\nu, N}^{(n_1, n_2)}$ and the functions $\Phi_{N, n_1, n_2}(r)$ for all r and N can be immediately calculated. Further let all $\alpha_{\nu, N}^{(n_1, n_2)}$ occurring in (4.2) be any fixed real numbers. Then the system (4.2) is formally decoupled and by means of the properties of the function $G^{(n)}(r, r')$ (see appendix A, formula (?)) we convert for $0 \leq r \leq a$ each equation into a differential one:

$$\left[\frac{d^2}{dr^2} - r^2 + 2(n_1 + n_2 - N) + 1 - v_0 \right] \psi_{N, n_1, n_2}(r) = - \sum_{\substack{\nu=N \\ \nu \neq n_1 + n_2}}^M \phi_{\nu, N}(r) [1 + (-1)^{\nu + n_1 + n_2} \{ \gamma_{\nu, N}^{(n_1, n_2)} + \gamma_{\nu, N}^{-(n_1, n_2)} \}] - \left\{ \begin{array}{l} 2\phi_{n_1 + n_2 - N}(r) \left[\alpha_{n_1 + n_2, N}^{(n_1, n_2)} + \alpha_{n_1 + n_2, N}^{-(n_1, n_2)} \right] \dots \dots \dots N \leq n_1 + n_2 \\ 0 \dots \dots \dots N > n_1 + n_2 \end{array} \right. \quad (4.4)$$

The properly symmetrized (see (2.11)) general solution of (4.4) is:

$$\psi_{N, n_1, n_2}(r) = \frac{1 + (-1)^{n_1 + n_2 + N}}{2} A_N e^{-\frac{r^2}{2}} {}_1F_1\left(\frac{v_0}{4} + \frac{N - n_1 - n_2}{2}, \frac{1}{2}, r^2\right) + \frac{1 - (-1)^{n_1 + n_2 + N}}{2} B_N r e^{-\frac{r^2}{2}} {}_1F_1\left(\frac{v_0}{4} + \frac{N + 1 - n_1 - n_2}{2}, \frac{3}{2}, r^2\right) + \sum_{\substack{\nu=N \\ \nu \neq n_1 + n_2}}^M \frac{1 + (-1)^{\nu + n_1 + n_2}}{v_0 + 2(\nu - n_1 - n_2)} \phi_{\nu, N}(r) \left[\gamma_{\nu, N}^{(n_1, n_2)} + \gamma_{\nu, N}^{-(n_1, n_2)} \right] + \left\{ \begin{array}{l} \frac{2}{v_0} \left[\alpha_{n_1 + n_2, N}^{(n_1, n_2)} + \alpha_{n_1 + n_2, N}^{-(n_1, n_2)} \right] \phi_{n_1 + n_2 - N}(r) \dots \dots \dots N \leq n_1 + n_2 \\ 0 \dots \dots \dots N > n_1 + n_2 \end{array} \right. \quad (4.5)$$

where ${}_1F_1(a, b, r)$ is the confluent hypergeometric function (ref. 5).

Further we shall put $n_0 = 2$ which corresponds to 0^{16} and restrict ourselves to the case $n_1 = n_2 = 0$ omitting the indices n_1, n_2 . This restriction represents no simplification of the problem and the solution for other n_1 and n_2 can be found in the same way.

With this restriction the only non-zero coefficients γ is $T_{00}^{\infty} = 1$. Hence the system for $\psi_N(r)$ with N odd is homogeneous i.e. $\psi_N(r) = 0$ for N odd, and only $\psi_N(r)$ with N even are non-zero. We determine now the constants A_N comparing at $r = a$ (4.2.) and (4.5) (M is even) and using the explicit form of $G^{(n)}(r, r')$ given in appendix A:

$$A_0 = \frac{1}{e^{-\frac{a^2}{4}} {}_1F_1\left(\frac{\nu_0}{4}, \frac{1}{2}, a^2\right)} \left\{ \phi_0(a) + \Phi_0(-) + \sum_{\nu=1}^M [1 + (-1)^\nu] \{ \gamma_{\nu 0} + \bar{\gamma}_{\nu 0} \} \left[\frac{1}{2\nu} - \frac{1}{\nu_0 + 2\nu} \right] \phi_{\nu}(a) - \frac{2}{\nu_0} \bar{a}_{00} \phi_0(a) - a_{00} \left[\frac{2}{\nu_0} + \ln 2 + C + h_0(-a) \right] \phi_0(a) - \frac{\phi_0(a)}{2\pi^{1/4}} \beta_0 \right\}$$

$$\beta_0 = \int_0^a e^{-\frac{r^2}{2}} [h_0(r) + h_0(-r)] \nu_0 \psi_0(r) dr \quad (4.6)$$

$$A_N = \frac{1}{e^{-\frac{a^2}{4}} {}_1F_1\left(\frac{\nu_0 + N}{4}, \frac{1}{2}, a^2\right)} \left\{ \Phi_N(a) + \sum_{\nu=N}^M (1 + (-1)^\nu) \{ \gamma_{\nu N} + \bar{\gamma}_{\nu N} \} \phi_{\nu-N}(a) \left[\frac{1}{2\nu} - \frac{1}{\nu_0 + 2\nu} \right] - \frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(a\sqrt{2}) \beta_N \right\}$$

$$\beta_N = \int_0^a [D_{-N}(r\sqrt{2}) + D_{-N}(-r\sqrt{2})] \nu_0 \psi_N(r) dr \quad N = 2, 4, \dots, M.$$

The quantities a_{00} and β_N ($N = 0, \dots, M$) can be excluded if we substitute in their definition formulae from (4.5) and (4.6) and carry out the integrations:

$$a_{00} = \nu_0 A_0 J_{00} + 2j_{00}(a_{00} + \bar{a}_{00}) + \sum_{\nu=1}^M \frac{1 + (-1)^\nu}{\nu_0 + 2\nu} \nu_0 (\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}) j_{0\nu}$$

$$\beta_0 = \nu_0 A_0 L_0 + 2K_{00}(a_{00} + \bar{a}_{00}) + \sum_{\nu=1}^M \frac{1 + (-1)^\nu}{\nu_0 + 2\nu} \nu_0 (\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}) K_{0\nu} \quad (4.7)$$

$$\beta_N = \nu_0 A_N P_N + \sum_{\nu=N}^M \frac{1 + (-1)^\nu}{\nu_0 + 2\nu} \nu_0 (\gamma_{\nu N} + \bar{\gamma}_{\nu N}) Q_{N, \nu-N} \quad N = 2, 4, \dots, M.$$

(The notation used in (4.7) is explained in appendix B). Excluding $a_{00}, \beta_0, \dots, \beta_M$ from (4.7) and (4.6) we obtain new expressions for A_N :

$$A_N = C_N \frac{\exp\left(-\frac{a^2}{2}\right)}{{}_0 {}_1F_1\left(\frac{\nu_0 + N}{4} + 1, \frac{3}{2}, a^2\right)} = C_N \frac{Z_1^{(N)}}{\nu_0 J_{N0}} \quad (4.8)$$

$$C_0 = \frac{1}{Z_2^{(0)} + Z_1^{(0)} \left[Z_3^{(0)} \frac{\phi_0(a)}{2} \frac{L_0}{J_{00}} \pi^{-1/4} \right]} \{ \phi_0(a) + \Phi_0(a) + \sum_{\nu=1}^M [1+(-1)^\nu] [\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}] \left[\left(\frac{1}{2\nu} - \frac{1}{\nu_0 + 2\nu} \right) \phi_\nu(a) - Z_3^{(0)} \frac{\nu_0 j_{0\nu}}{\nu_0 + 2\nu} - \frac{\phi_0(a)}{2} \frac{\pi^{-1/4} \nu_0 K_{0\nu}}{\nu_0 + 2\nu} \right] - \bar{a}_{00} \left[\frac{2}{\nu_0} \phi_0(a) + 2j_{00} Z_3^{(0)} + \phi_0(a) \pi^{-1/4} K_{00} \right] \} \quad (4.8)$$

$$C_N = \frac{1}{Z_2^{(N)} + Z_1^{(N)} Z_3^{(N)} \frac{P_N}{J_{N0}}} \{ \Phi_N(a) + \sum_{\nu=N}^M [1+(-1)^\nu] [\gamma_{\nu N} + \bar{\gamma}_{\nu N}] \left[\left(\frac{1}{2\nu} - \frac{1}{\nu_0 + 2\nu} \right) \phi_{\nu-N}(a) - Z_3^{(N)} \frac{\nu_0}{\nu_0 + 2\nu} Q_{N, \nu-N} \right] \}$$

$$Z_1^{(N)} = \frac{\exp\left(\frac{a^2}{2}\right)}{a} \frac{J_{N0}}{F_1\left(\frac{\nu_0}{4} + \frac{N}{2} + 1, -\frac{3}{2}, a^2\right)}; \quad Z_2^{(N)} = \frac{F_1\left(\frac{\nu_0}{4} + \frac{N}{2} + 1, \frac{1}{2}, a^2\right)}{a \nu_0 F_1\left(\frac{\nu_0}{4} + \frac{N}{2} + 1, -\frac{3}{2}, a^2\right)}$$

$$Z_3^{(0)} = \frac{\phi_0(a)}{1-2j_{00}} \left[\frac{2}{\nu_0} + \ln 2 + C + b_0(-a) + K_{00} \pi^{-1/4} \right]; \quad Z_3^{(N)} = \frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(a\sqrt{2})$$

$$N = 2, 4, \dots, M.$$

In this way the functions $\psi_N(r) (0 \leq r \leq a)$ have been found for arbitrary values of $a_{\nu N}$ and arbitrary functions $\psi_N(r) (a \leq r \leq R)$.

Now the quantities $a_{\nu N}$ can be determined from their original definition (4.1):

$$a_{\nu N} = \int_0^a \phi_{\nu-N}(r) \nu_0 \psi_N(r) dr,$$

i.e.,

$$a_{\nu N} = C_N \frac{Z_1^{(N)}}{J_{N0}} \left[\sum_{\mu=N}^M \frac{1+(-1)^\mu}{\nu_0 + 2\mu} \nu_0 [\gamma_{\mu N} + \bar{\gamma}_{\mu N}] j_{\mu-N, \nu-N} + \begin{cases} 2(a_{00} + \bar{a}_{00}) j_{00} & \dots & N=0 \\ 0 & \dots & N>0 \end{cases} \right] \quad (4.9)$$

These equations together with the definitions (4.1) for $\gamma_{\nu N}$ and (4.8) for C_N represent a linear algebraic system of $\frac{1}{2} \left(\frac{M}{2} + 1 \right) \left(\frac{M}{2} + 2 \right)$ equations. The solution depends linearly on $\bar{a}_{\nu N}$ and $\Phi_N(a)$ i.e. on integrals of the hitherto arbitrary functions $\psi_N(r)$ over the interval $a \leq r \leq R$. These functions can be now determined if we substitute the solution of (4.9) and (4.5) in our original system (4.2). We get for $a \leq r \leq R$ a system of linear integral equations where only $\psi_N(r)$ for $a \leq r \leq R$ are unknown. This system can be solved by iteration. If the functions of the n -th approximation $\Psi_{(n)N}(r)$ are known we obtain $\Psi_{(n+1)N}(r)$ in the following way:

1) We calculate

$$\left. \begin{aligned} {}_{(n)}\bar{\alpha}_{\nu N} &= \int_a^R \phi_{\nu-N}(r) v(r) \psi_N(r) dr \\ {}_{(n)}\bar{\gamma}_{\nu N} &= \begin{cases} {}_{(n)}\bar{\alpha}_{\nu N} & \dots\dots\dots 0 \leq \nu \leq 3 \\ \sum_{N'=0}^{\nu} \sigma_{\nu NN'} {}_{(n)}\bar{\alpha}_{\nu N'} & \nu \geq 4 \end{cases} \end{aligned} \right\} \begin{aligned} N &= 0, 2, \dots M \\ \nu &= N; N+2, \dots M \end{aligned}$$

$${}_{(n)}\Phi_N(r) = \int_a^R [G^{(-N)}(r, r') + G^{(-N)}(r, -r')] v(r') \psi_N(r') dr'$$

$a \leq r \leq R$

2) We substitute ${}_{(n)}\bar{\alpha}_{\nu N}$ and ${}_{(n)}\Phi_N(a)$ into (4.8) and find the solution of the algebraic system:

$$\begin{aligned} {}_{(n)}\alpha_{\nu N} = {}_{(n)}C_N \frac{Z_1^{(N)}}{J_{N0}} J_{N, \nu-N} + \sum_{\substack{\mu=N \\ \mu \neq 0}}^M \frac{1+(-1)^\mu}{v_0 + 2\mu} v_0 [{}_{(n)}\gamma_{\mu N} + {}_{(n)}\bar{\gamma}_{\mu N}] j_{\mu-N, \nu-N} + \\ + \begin{cases} [{}_{(n)}\alpha_{00} + {}_{(n)}\bar{\alpha}_{00}] 2j_{00} & \dots N=0 \\ 0 & \dots N > 0 \end{cases} \end{aligned}$$

3) Putting ${}_{(n)}\alpha_{\nu N}$ into (4.5) we get for $0 \leq r \leq a$; $N=0, 2, \dots M$:

$$\begin{aligned} {}_{(n)}\psi_N(r) = {}_{(n)}C_N \frac{Z_1^{(N)}}{v_0 J_{N0}} e^{-\frac{r^2}{2}} {}_1F_1\left(\frac{v_0}{4} + \frac{N}{2}, \frac{1}{2}, r^2\right) + \\ + \sum_{\substack{\nu=N \\ \nu \neq 0}}^M \frac{1+(-1)^\nu}{v_0 + 2\nu} [{}_{(n)}\gamma_{\nu N} + {}_{(n)}\bar{\gamma}_{\nu N}] \phi_{\nu-N}(r) + \begin{cases} \frac{2}{v_0} \phi_0(r) [{}_{(n)}\alpha_{00} + {}_{(n)}\bar{\alpha}_{00}] & \dots N=0 \\ \dots & \dots N > 0. \end{cases} \end{aligned}$$

Then we define for $a \leq r \leq R$; $N=0, 2, \dots M$:

$$\begin{aligned} {}_{(n+1)}\psi_N(r) = \delta_{N0} \phi_0(r) + {}_{(n)}\Phi_N(r) + \int_a^R [G^{(-N)}(r, r') + G^{(-N)}(r, -r')] v_0 {}_{(n)}\psi_N(r') dr' + \\ + \sum_{\substack{\nu=N \\ \nu \neq 0}}^M \frac{\phi_{\nu-N}(r)}{2^\nu} [1+(-1)^\nu] [{}_{(n)}\gamma_{\nu N} + {}_{(n)}\bar{\gamma}_{\nu N}]. \end{aligned}$$

From the properties of $G^{(-N)}(r, r')$ we find that for each approximation it holds:

$$\lim_{r \rightarrow a^-} {}_{(n)}\psi_N(r) = \lim_{r \rightarrow a^+} {}_{(n+1)}\psi_N(r); \quad \lim_{r \rightarrow a^-} {}_{(n)}\psi'_N(r) = \lim_{r \rightarrow a^+} {}_{(n+1)}\psi'_N(r). \quad (4.10)$$

Since the system is linear, the convergence of the iteration method does not depend on the zeroth approximation. We put

$${}_{(0)}\psi_N(r) \equiv 0 \quad a \leq r \leq R \quad N = 0, 2, \dots M.$$

Then the quantities ${}_{(0)}a_{\nu N}$, ${}_{(0)}\psi_N(r)$ $0 \leq r \leq a$ and ${}_{(1)}\psi_N(r)$ $a \leq r \leq R$ represent the exact solution of the system (4.2) for the internucleon potential containing no attractive part.

We have not tried to prove generally that this iteration method converges since there exist only sufficient conditions for convergence and it would be very difficult to verify them in our case. The convergence has been tested only numerically (see sec. VI).

As to the practical calculations according to this method, the only difficult part is to solve the algebraic system (4.9) for different M , since its dimension increases very rapidly with M . Fortunately the solution can be found by a simple iteration method. If we know the m -th approximation ${}_{(n)}^{(m)}a_{\nu N}$ (the order of approximation is now denoted by the upper index on the left) we calculate

$${}_{(n)}^{(m)}\gamma_{\nu N} = \begin{cases} {}_{(n)}^{(m)}a_{\nu N} & 0 \leq \nu \leq 3 & N = 0, 2, \dots, M \\ \sum_{N'=0}^{\nu} \sigma_{\nu N N'} {}_{(n)}^{(m)}a_{\nu' N'} & \nu = N, N+2, \dots, M \end{cases}$$

and with these values of ${}_{(n)}^{(m)}\gamma_{\nu N}$ the quantities ${}_{(n)}^{(m)}C_N$ according to (4.8).

If we substitute all this into the right-hand side of (4.9) we get by definition on the left the quantities ${}_{(n)}^{(m+1)}a_{\nu N}$. The iteration sequence converges sufficiently rapidly (see sec. VI) for all values of v_0 and also for $v_0 = +\infty$.

The possibility of solving the algebraic system by iteration can seem rather surprising since, in fact, it is equivalent with the system (4.2) for $0 \leq r \leq a$ and this system cannot be solved by iteration for $v_0 \rightarrow +\infty$. The explanation is in the following argument. First of all the system is linear and hence the convergence of the iteration sequence does not depend on the inhomogeneous term. We can therefore choose $e.g.$ $\Phi_N(a) = 0$ and $\bar{\gamma}_{\nu N} = 0$ for all ν and N . Further for simplicity we take $M=4$ i.e. the lowest value of M for which the equations are coupled. Substituting now from (4.8) into (4.5) we get a system of linear integral equations on the interval $0 \leq r \leq a$ with a degenerate kernel. The sufficient condition for the convergence of the iteration series of the system is (ref.6):

$$\int_0^a \int_0^a |K(x, s)|^2 dx ds < 1.$$

It has been found that this condition is fulfilled for $v_0 \leq 1000$ if $a = 0,1$ i.e. the system can be solved by iteration for rather large values v_0 . If we now pass to the equivalent algebraic system (4.10) we find that all its coefficients have finite limit values (see sec. V) and that the difference between these limit

values and the values for $v_0 \approx 10000$ is very small. Since the applicability of an iteration method for solving such a linear algebraic system depends only on the matrix of its coefficients we can expect that the solution of (4.10) can be found by iteration for all values v_0 . This result has been confirmed by numerical calculations. (see VI).

In conclusion of this section we want to give reason why we do not use the generalized perturbation method of ref.2. The main feature of our method is that we calculate corrections of both $a_{\nu N}$ and $\bar{a}_{\nu N}$ and obtain therefore in each approximation continuous functions $\psi_N(r)$ for $0 < r \leq R$. In the method of ref.2 $a_{\nu N}$ are taken in the zeroth approximation and only corrections of $\bar{a}_{\nu N}$ are calculated. Consequently the functions $\psi_N(r)$ are discontinuous at $r = a$. In order to correct the influence of this discontinuity the model functions $\phi_{n_1 + n_2, N}(r)$ in the expressions for $\bar{a}_{n_1 + n_2, N}$ are replaced by the first approximation ${}^{(1)}\psi_{N, n_1, n_2}(r)$ which represents the exact solution of the problem without the attractive potential and is continuous at $r = a$. i.e.

$$\lim_{r \rightarrow a^+} {}^{(1)}\psi_{N, n_1, n_2}(r) = \lim_{r \rightarrow a^-} {}^{(0)}\psi_{N, n_1, n_2}(r)$$

Since the functions $\psi_{N, n_1, n_2}(r)$ are known only for n_1 and n_2 representing occupied states, in the Goldstone formula only the first term which contains the matrix elements of t between occupied states can be calculated.

However, if we want to calculate higher terms we need the other matrix elements of t as well. The general matrix element of t with the occupied initial states (denoted n_j) and arbitrary final states (denoted m_j) can be expressed by means of $a_{\nu N}$ and $\bar{a}_{\nu N}$ from (2.3'), (2.10), (3.8) and (4.1):

$$\begin{aligned} (m_1, m_2 | t | n_1, n_2) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_{m_1}(x_1) \phi_{m_2}(x_2) \bar{v}(x_1 - x_2) \psi_{n_1, n_2}(x_1, x_2) dx_1 dx_2 = \\ &= \sum_{N=0}^{\min(m_1 + m_2, M)} T_{N, m_1 + m_2, N}^{m_1, m_2} [1 + (-1)^{n_1 + n_2 + m_1 + m_2}] [a_{m_1 + m_2, N}^{(n_1, n_2)} + a_{m_1 + m_2, N}^{-(n_1, n_2)}]. \end{aligned} \quad (4.11)$$

Hence we can calculate by means of (4.12) the first and the second term of the Goldstone formula. For the third term the matrix elements $(m_1, n_2 | t | n_1, m_2)$ and also the matrix elements with changed energy denominators $\delta E \neq 0$ are necessary. They can be calculated similarly, which will be described in detail in our next paper.

V. The Limit Form of the Solution for a Hard Core Potential

The reformulation of our problem given in sec. IV is fully convenient for consideration of the limit case $v_0 = +\infty$. We can simply find the limit values of all

coefficients of the algebraic system (4.9) and then its solution gives us directly the matrix elements of t for $v_0 = +\infty$, provided the existence of the finite limit values of $a_{\nu N}$ is guaranteed. It is practically impossible to obtain the analytic dependence of $a_{\nu N}$ on v_0 for large M . This dependence has been found for $M < 4$ i.e. for the triangular approximation and the limit values have been evaluated directly, the relation

$$\lim_{v_0 \rightarrow +\infty} \frac{{}_1F_1\left(\frac{v_0}{4} + \frac{1}{2}, \frac{a^2}{2}\right)}{v_0 {}_1F_1\left(\frac{v_0}{2} + 1, \frac{3}{2}a^2\right)} = 0 \quad (5.1)$$

which follows from the asymptotic behaviour of the confluent hypergeometric function, being used. It turns out that there exist the finite limit values of $a_{\nu N}$ and the functions $a_{\nu N}(v_0)$ are continuous and very slowly increasing for large v_0 . Since the form of (4.9) is analogous for all values of M we can assume the existence of the finite limit values of $a_{\nu N}$ for all M . Using (5.1) and appendix B we get these limit values:

$$Z_1^{(N)} \rightarrow \phi_0^{(N)}(a); \quad Z_2^{(N)} \rightarrow 0; \quad \frac{J_{N\nu}}{J_{N0}} \rightarrow \frac{\phi_\nu(a)}{\phi_0(a)}; \quad \frac{L_0}{J_{00}} \rightarrow \pi^{1/4} [h_0(a) + h_0(-a)];$$

$$\frac{P_N}{J_{N0}} \rightarrow \frac{D_{-N}(a\sqrt{2}) + D_{-N}(-a\sqrt{2})}{\phi_0(a)}; \quad Z_3^{(0)} \rightarrow \frac{\phi_0(a)}{1 - 2j_{00}} [\ln 2 + C + h_0(-a) + \pi^{-1/4} K_{00}] = \phi_0(a) Z(a).$$

Denoting the limit values of C_N as \bar{C}_N we obtain from (4.8):

$$\bar{C}_0 = \frac{1}{\phi_0^2(a) [Z(a) + \frac{1}{2}(h_0(a) + h_0(-a))]} \{ \phi_0(a) + \Phi_0(a) - \bar{a}_{00} \phi_0(a) [2Z(a)j_{00} + \pi^{-1/4} K_{00}] + \sum_{\nu=1}^M [1 + (-1)^\nu] [\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}] \left[\frac{\phi_\nu(a)}{2\nu} - \phi_0(a) (Z(a)j_{0\nu} + \pi^{-1/4} \frac{K_{0\nu}}{2}) \right] \}$$

$$\bar{C}_N = \frac{\Phi_N(a) + \sum_{\nu=N}^M [1 + (-1)^\nu] [\gamma_{\nu N} + \bar{\gamma}_{\nu N}] \left[\frac{\phi_{\nu-N}(a)}{2\nu} - \frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(a\sqrt{2}) Q_{N,\nu-N} \right]}{\frac{(N-1)!}{\sqrt{\pi}} [D_{-N}(a\sqrt{2}) + D_{-N}(-a\sqrt{2})] \cdot D_{-N}(a\sqrt{2})} \quad N = 2, 4, \dots, M.$$

The limit form of the algebraic system (4.10) is then

$$a_{\nu N} = \bar{C}_N \phi_{\nu-N}(a) + \sum_{\substack{\mu=N \\ \mu \neq 0}}^M [1 + (-1)^\mu] [\gamma_{\mu N} + \bar{\gamma}_{\mu N}] j_{\mu-N, \nu-N} \begin{cases} 2j_{00}(a_{00} + \bar{a}_{00}) \dots & N=0 \\ 0 \dots & N>0. \end{cases} \quad (5.3)$$

Further from (4.5) we find for $0 \leq r \leq a$

$$\lim_{v_0 \rightarrow +\infty} |\psi_N(r)| \leq \frac{|\bar{C}_N|}{a \exp(-\frac{a^2}{2})} \lim_{v_0 \rightarrow +\infty} \frac{{}_1F_1\left(\frac{v_0}{4} + \frac{N}{2}, \frac{1}{2}a^2\right)}{v_0 {}_1F_1\left(\frac{v_0}{2} + N + 1, \frac{3}{2}a^2\right)} = 0 \quad (5.4)$$

$N = 2, 4, \dots, M.$

Hence the functions $\psi_N(r)$ converge uniformly to zero for $0 \leq r \leq a$. In addition, differentiating (4.5) and taking the limit value of the result at $r = a$ we have

$$\lim_{\nu_0 \rightarrow +\infty} \left[\frac{d}{dr} \psi_N(r) \Big|_{r=a^-} \right] = \bar{C}_N; \quad \frac{d}{dr} \left[\lim_{\nu_0 \rightarrow +\infty} \psi_N(r) \Big|_{r=a^-} \right] = 0. \quad (5.5)$$

On the other hand one finds easily from (4.2) and (4.7):

$$\frac{d}{dr} \left[\lim_{\nu_0 \rightarrow +\infty} \psi_N(r) \Big|_{r=a^+} \right] = \lim_{\nu_0 \rightarrow +\infty} \left[\frac{d}{dr} \psi_N(r) \Big|_{r=a^+} \right] = \lim_{\nu_0 \rightarrow +\infty} \left[\frac{d}{dr} \psi_N(r) \Big|_{r=a^-} \right] = \bar{C}_N. \quad (5.6)$$

Comparing (5.5) and (5.6) we see that the derivative of the limit function

$\lim_{\nu_0 \rightarrow +\infty} \psi_N(r)$ is discontinuous at $r = a$.

Using the derived properties (5.4) - (5.6) of the solution for the internucleon potential containing a hard core we can simplify the method described in sec. IV. We can namely avoid the solving of the differential equations (4.4) and finding the limit value of the solution.

Rewriting (4.4) and excluding a_{00} according to (4.7) we get for any function $f(r)$ with the integrable second derivative for $0 \leq r \leq a$:

$$\begin{aligned} \lim_{\nu_0 \rightarrow +\infty} \int_0^a f(r) \nu_0 \psi_N(r) dr &= \lim_{\nu_0 \rightarrow +\infty} \int_0^a f(r) \frac{d^2}{dr^2} \psi_N(r) dr + \sum_{\substack{\nu=N \\ \nu \neq 0}}^M [1 + (-1)^\nu] [\gamma_{\nu N} + \\ &+ \bar{\gamma}_{\nu N}] \int_0^a f(r) \phi_{\nu-N}(r) dr + \quad (5.7) \\ &+ \left\{ \frac{2}{1-2j_{00}} \left[\bar{C}_0 \phi_0(a) + \bar{a}_{00} + \sum_{\nu=1}^M [1 + (-1)^\nu] [\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}] j_{\nu 0} \right] \int_0^a f(r) \phi_0(r) dr, \right. \\ &\quad \dots\dots N=0 \\ &\quad \dots\dots N>0 \end{aligned}$$

where the uniform convergence of $\psi_N(r)$ to zero has been used. Since $\psi_N(r)$ are symmetrical with respect to $r=0$ it follows that $\psi'_N(0) = 0$ and hence:

$$\lim_{\nu_0 \rightarrow +\infty} \int_0^a f(r) \frac{d^2}{dr^2} \psi_N(r) dr = \bar{C}_N f(a).$$

Now (5.7) can be rewritten formally as:

$$\begin{aligned} \lim_{\nu_0 \rightarrow +\infty} \nu_0 \psi_N(r) &= \bar{C}_N \delta(r-a) + \sum_{\substack{\nu=N \\ \nu \neq 0}}^M [1 + (-1)^\nu] [\gamma_{\nu N} + \bar{\gamma}_{\nu N}] \phi_{\nu-N}(r) + \quad (5.8) \\ &+ \left\{ \frac{2}{1-2j_{00}} \left[\bar{C}_0 \phi_0(a) + \bar{a}_{00} + \sum_{\nu=1}^M [1 + (-1)^\nu] [\gamma_{\nu 0} + \bar{\gamma}_{\nu 0}] j_{\nu 0} \right] \phi_0(r) \dots\dots N=0 \right. \\ &\quad \dots\dots N > 0, \end{aligned}$$

which is an analogon of the Bethe-Goldstone formula derived in ref. 8 for nuclear matter.

Using (5.7) we can exclude from the system (4.2) the terms containing the products $\lim_{v_0 \rightarrow +\infty} v_0 \psi_N$ and investigate this system from the beginning only for $a \leq r \leq R$. Consequently the method of sec. IV is simplified as follows. The first step does not change. In the second step we use (5.2) instead of (4.8) and solve the system (5.3) by iteration. As soon as its solution is found we can determine the $(n+1)$ -th approximation for $a \leq r \leq R$:

$$\begin{aligned} \psi_{(n+1)0}(r) = & \phi_0(r) + \Phi_0(r) - \phi_0(r) \phi_0(a) \bar{C}_{0(n)} \left[Z(r) + \frac{h_0(a) + h_0(-a)}{2} \right] + \\ & + \sum_{\nu=1}^M [1 + (-1)^\nu] \left\{ \gamma_{\nu 0(n)} \bar{\gamma}_{\nu 0(n)} \left[\frac{\phi_\nu(r)}{2\nu} - \phi_0(r) (Z(r) j_{0\nu} + \pi^{-1/4} \frac{K_{0\nu}}{2}) \right] - \right. \\ & \left. - \bar{\alpha}_{00(n)} \phi_0(r) [2j_{00} Z(r) + \pi^{-1/4} K_{00}] \right\}. \end{aligned} \quad (5.9)$$

$$\begin{aligned} \psi_N(r) = & \Phi_N(r) - \frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(\sqrt{2}r) \bar{C}_{N(n)} \left[D_{-N}(a\sqrt{2}) + D_{-N}(-a\sqrt{2}) \right] + \\ & + \sum_{\nu=N}^M [1 + (-1)^\nu] \left\{ \gamma_{\nu N(n)} \bar{\gamma}_{\nu N(n)} \left[\frac{\phi_{\nu-N}(r)}{2\nu} - \frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(r\sqrt{2}) Q_{N,\nu-N} \right] \right\} \\ & N = 2, 4, \dots, M. \end{aligned}$$

In conclusion of this section we complete the consideration performed in sec. III for the case $v_0 \rightarrow +\infty$. From (3.1) and (3.3) we get

$$\begin{aligned} \left\| \psi_{n_1 n_2} - F_{n_1 n_2} - \int_0^{+\infty} K_0^{(M)}(r, r') V(r') \psi_{n_1 n_2}(r') dr' \right\|^2 = \\ = \int_0^{+\infty} \left[\int_0^{+\infty} K_1^{(M)}(r, r') V(r') \psi_{n_1 n_2}(r') dr' \right]^2 dr. \end{aligned} \quad (5.10)$$

We put $n_1 = n_2 = 0$, $n_0 = 2$ and omit the indices n_1, n_2 . Let $\psi^{(M)}(r)$ be the solution of the equation

$$\psi(r) = F(r) + \int_0^{+\infty} K_0^{(M)}(r, r') V(r') \psi(r') dr'.$$

This solution has been found for any M and, in addition, we have obtained for $0 \leq r \leq R$

$$\lim_{v_0 \rightarrow +\infty} v(r) \psi_N^{(M)}(r) = \bar{C}_N^{(M)} \delta(r-a) + \chi_N^{(M)}(r) \quad (5.8')$$

where $\chi_N^{(M)}(r)$ are quadratic integrable. The function $\psi^{(M)}(r)$ is, of course, not the solution of (3.1) since the r.h.s. of (5.10) is not zero for $\psi(r) = \psi^{(M)}(r)$. However, let the constants $\bar{C}_N^{(M)}$ be bounded for all M and N and let for any $\epsilon > 0$ M exist so that for any $M > M_0, M' > M_0$

$$\left\| \chi^{(M)} - \chi^{(M')} \right\| < \epsilon. \quad (5.11)$$

Hence two positive constants B_1 and B_2 exist for which

$$\left| \bar{C}_N^{(M)} \right| \leq B_1 ; \quad \left\| \chi^{(M)} \right\| \leq B_2 . \quad (5.12)$$

It is practically impossible to verify generally whether (5.11) is fulfilled since the function $\psi^{(M)}(r)$ can be determined only numerically. However, numerical results given in sec. VI suggest that this assumption holds.

From (5.11) it follows that there exists a quadratic integrable limit function $\psi(r) = \lim_{M \rightarrow +\infty} \psi^{(M)}(r)$ since

$$\left\| \chi^{(M)} - \chi^{(M')} \right\|^2 = \sum_{N=0}^{m \max(M, M')} \left\{ \int_0^a \left[\chi_N^{(M)}(r) - \chi_N^{(M')}(r) \right]^2 dr + \int_0^b v^2(r) \left[\psi_b^{(M)}(r) - \psi_b^{(M')}(r) \right]^2 dr \right\} \geq \text{const.} \cdot \left\| \psi - \psi' \right\|^2 .$$

With the help of (5.12) we further get (see appendix C) :

$$\int_0^{+\infty} \left[\int_0^{+\infty} K_i^{(M)}(r, r') V(r') \psi^{(M)}(r') dr' \right]^2 dr = O\left(\frac{1}{\sqrt{M}}\right) . \quad (5.13)$$

Consequently $\psi(r)$ is the solution of (3.1) for $v_0 \rightarrow +\infty$. In this way our method yields for a sufficiently large M an approximate solution of (3.1) with arbitrary accuracy.

VI. Numerical Results and Conclusions

In this section we check numerically some statements which we have not been able to prove generally, and demonstrate the practical applicability of our method. The calculations have been performed for the hard core radius $r_c = 0,4 f$ and the corresponding value $a = 0,1$, which gives $\alpha = 0,35f^{-1}$ and $hw = 5,17 \text{ MeV}$.

First of all we must check the validity of the assumption (5.11) which proves for the hard core potential that the solutions of the finite systems converge with increasing M to the solution of the infinite system. Since the validity of this assumption and the conclusions based upon it do not depend on the attractive part of the potential, we have calculated only the case when the potential contains no attractive part. To prove (5.11) it is now sufficient to show that the quantities $a_{\nu N}$ converge with increasing M to certain finite values. The numerical calculations have been carried out for $M = 2,4 \dots 22$. In the last case the algebraic system for $a_{\nu N}$ consists of 78 equations. The number n of iterations which gives the solution with the accuracy

$$\left| {}^{(n+1)}a_{\nu N} - {}^{(n)}a_{\nu N} \right| < 10^{-4}$$

is about 25. This represents on the average for each value of $a_{\nu M}$ approximately one minute on a computer having 20000 operations per. sec.

From the definition of $\alpha_{\nu N}$ it follows, that all these quantities can be divided into different groups, each group consisting of the quantities with the common second index, i.e. with the common function $v_0 \psi_N^{(M)}$. Then all members of the given group depend on M in the same way and therefore we give from each group only $\alpha_{NN}^{(M)}$. The results are given in table 1 where we have denoted

$$R_N^{(1)}(M, M') = 100 \left| \frac{\alpha_{NN}^{(M)} - \alpha_{NN}^{(M')}}{\alpha_{NN}^{(M)}} \right|$$

$$R_N^{(2)}(M) = 100 \left| \frac{\alpha_{NN}^{(M)}}{\alpha_{00}^{(M)}} \right|$$

$$\alpha_{NN}^{(M)} = \lim_{v_0 \rightarrow +\infty} \int_0^{\infty} \phi_0(r) v_0 \psi_N^{(M)}(r) dr$$

and defined the accuracy $\epsilon_{M'M}$ of the solution for given M and any

$$\epsilon_{M'M} = \sqrt{\sum_{N=0}^M [\alpha_{NN}^{(M)} - \alpha_{NN}^{(M')}]^2} = 10 \left| \alpha_{00}^{(M')} \right| \sqrt{\sum_{N=0}^{M'} [R_N^{(1)}(M', M) R_N^{(2)}(M')]^2}$$

From the table it is obvious that $\alpha_{\nu N}$ converge to finite values. Further one sees that for $N \geq 12$ the functions $\lim_{v_0 \rightarrow +\infty} v_0 \psi_N$ are very small, while those for $N = 0, 2, \dots, 10$ are important. Consequently the coupling in the system cannot be neglected and the triangular approximation which gives $v_0 \psi_N = 0$ for $N \neq 0$ is very rough. For comparison the only non-zero quantity given by this approximation is $\alpha_{00}^{(2)} = 3,406$, while $\alpha_{00}^{(22)} = 5,089$. The importance of the coupling is demonstrated in the last column where the solutions for $M = 10$ and $M = 22$ are compared. Although the functions $\lim_{v_0 \rightarrow +\infty} v_0 \psi_N$ for $N \geq 12$ are very small, the complete neglect of them (i.e. the case $M = 10$) makes the accuracy about ten times worse than for $M = 20$.

Further we have to prove numerically the convergence of the iteration method involving the attractive part of the potential. The calculations have been done for $M = 10$ for three potentials

$$v(x) = \begin{cases} +\infty & \dots \dots 0 \leq x \leq r_0 \\ -V_0 \frac{e^{-\mu x}}{\mu x} & \dots \dots x > x_0 \end{cases} \quad r_0 = 0,4 f \quad (6.1)$$

- (1) $V_0 = 450$ MeV; $\mu = 1,5 f^{-1}$
- (2) $V_0 = 250$ MeV; $\mu = 1,25 f^{-1}$
- (3) $V_0 = 100$ MeV; $\mu = 1 f^{-1}$.

For all cases the rate of convergence is approximately the same. The results for the first potential are given in table 2, and the functions $\psi_N(r)$ are plotted in fig. 2 and 3. Four iterations are sufficient to determine the values of $a_{\nu N}$ and $\bar{a}_{\nu N}$ with the error about 1%. For each iteration approximately 20 minutes of computer time are necessary. The figures show that the attractive part shifts the functions nearer to the core edge and enlarges essentially the derivative \bar{C}_0 . Since the most important part of $a_{\nu N}$ is the term containing \bar{C}_N (cf. (5.3)) we can expect that the quantities $a_{\nu N}$ which represent the hard core contribution to the matrix elements of t will be influenced rather strongly by the attractive part. This is demonstrated quantitatively in the first and last columns in table 2. On the other hand the general form of the functions $\psi_N(r)$ is influenced rather weakly. It can be thus expected that small variations of the attractive potential will not change these functions at all. We have investigated the effect of addition of two weak long range potential (1) the repulsive Coulomb potential describing the $p-p$ electrostatic interaction, (2) the harmonic oscillator potential which is connected with the motion of the centre of mass of the whole nucleus (ref. 9). The influence of both these potentials upon the wave functions is negligible, which implies the possibility of solving the problem without these potentials and taking them into account only in the calculation of the quantities $\bar{a}_{\nu N}$.

Finally we have calculated for the case (1) in (6.1) the matrix elements of t with $n_1 = n_2 = 0$, $n_0 = 2$, which contribute to the first and second terms of the expression for energy. These contributions are $(00|t|00)$ in the first term, and $\sum_{n=2}^{\infty} \frac{1}{E_0 - E_n} (00|t|0n)(0n|t|00)$ in the second term. From (2.3'), (2.4) and (4.11) we find

$$(00|t|00) = h\omega [a_{00} + \bar{a}_{00}] = 6,55 \text{ MeV};$$

$$t_n = (0n|t|00) = (n0|t|00) = \frac{1 + (-1)^n}{2} h\omega 2^{\frac{n}{2}} \sum_{N=0}^{n \text{ min } (n,10)} [a_{nN} + \bar{a}_{nN}] (-1)^N \sqrt{\binom{n}{N}}$$

$t_2 = -4,65 \text{ MeV}$, $t_4 = 2,83 \text{ MeV}$, $t_6 = -1,33 \text{ MeV}$, $t_8 = 0,324 \text{ MeV}$, $t_{10} = 0,258 \text{ MeV}$. Since t is hermitian we get $(00|t|0n) = \overline{(0n|t|00)}$ and if we restrict ourselves to $n \leq 10$ (the contribution of the term with $n=10$ is less than 0,05%) we find

$$\sum_{n=2}^{10} \frac{(00|t|0n)(0n|t|00)}{E_0 - E_n} = - \sum_{n=2}^{10} \frac{|(00|t|0n)|^2}{h\omega n} = -2,56 \text{ MeV}$$

It should be noted that these quantities do not describe any real physical situation since we have considered only a one-dimensional case in Cartesian coordinates.

On the basis of calculations given in this section we can conclude that the proposed method for solving the BG equation is practically applicable. It gives the solution with arbitrary accuracy, the accuracy being determined only by the number N of coupled equations and by the number of iterations. In addition a sufficiently exact solution can be obtained using a middle-speed computer.

At present calculations are being performed in the three-dimensional case using the general internucleon potential of Hamada and Johnston and of the Yale group.

The authors are indebted to the Computing Centre of the Joint Institute for Nuclear Research and especially to Li Da-tu who carried out all the numerical calculations described in the paper. They also wish to acknowledge the help of the Numerical Computer Centre of Charles University and the Mathematical Laboratory of Czech Technical University in Prague in performing preliminary calculations.

APPENDIX A. Kernels for the Integral Equations

We want to find an explicit formula for the function $G^{(n)}(r, r')$ defined by the relation (4.3). The function $G(r, r', z)$ given in ref. 7 for any complex $z \neq 0, 1, 2, \dots$

$$G(r, r', z) = \frac{1}{\omega(z)} \Phi_0(s, z) \Phi_0(-t, z) \quad \begin{matrix} s = \{ \begin{matrix} r & r \geq r' \\ r' & r < r' \end{matrix} \\ t = \{ \begin{matrix} r & r \leq r' \\ r' & r > r' \end{matrix} \end{matrix} \quad (A.1)$$

$$\Phi_0(r, z) = \exp\left(-\frac{r^2}{2}\right) \int_{+\infty}^{(0+)} e^{-rt - \frac{1}{2}t^2} t^{-z-1} dt$$

$$\omega(z) = -[1 - \exp(-2\pi i z)]^2 2^{1-z} \sqrt{\pi} \Gamma(-z)$$

satisfies for any quadratic integrable function f

$$\int_{-\infty}^{+\infty} G(r, r', z) f(r) dr = \sum_{j=0}^{\infty} \frac{\phi_j(r)}{2(z-j)} \int_{-\infty}^{+\infty} \phi_j(r') f(r') dr' \quad (A.2)$$

i.e. (see (4.3).)

$$G(r, r', z) = G^{(z)}(r, r') \quad z \neq 0, 1, 2, \dots$$

For $z = -N$, $N = +1, +2, \dots$ the contour integrations in the expression for $\Phi_0(r, z)$ can be carried out and we get

$$G^{(-N)}(r, r') = -\frac{(N-1)!}{2\sqrt{\pi}} D_{-N}(\sqrt{2}s) D_{-N}(-\sqrt{2}t) \quad (A.3)$$

where $D_{-n}(r)$ are the functions of parabolic cylinder, as defined in ref.5. Now let n be a non-negative integer and z any complex number near to n , $z \neq n$. From (A.2) it follows

$$\int_{-\infty}^{+\infty} [G(r, r', z) - \frac{\phi_n(r)\phi_n(r')}{2(z-n)}] f(r') dr' = \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{\phi_k(r)}{2(z-k)} \int_{-\infty}^{+\infty} \phi_k(r') f(r') dr'. \quad (A.2')$$

On the right-hand side there is a function analytic in some neighbourhood of n . Hence

$$\begin{aligned} G^{(n)}(r, r') &= \lim_{z \rightarrow n} [G(r, r', z) - \frac{\phi_n(r)\phi_n(r')}{2(z-n)}] \\ &= \lim_{z \rightarrow n} \frac{\partial}{\partial z} [(z-n)G(r, r', z)]. \end{aligned} \quad (A.4)$$

Using (A.1) we find

$$\begin{aligned} G^{(n)}(r, r') &= -\frac{1}{2} [\Gamma(n+1) C - \sum_{j=1}^n \frac{1}{j}], \phi_n(r) \phi_n(r') \\ &\quad - \frac{(-1)^n}{2} \frac{\sqrt{n!}}{2^n \sqrt{\pi}} [\phi_n(-s) h_n(t) e^{-\frac{t^2}{2}} + \phi_n(t) h_n(-s) e^{-\frac{s^2}{2}}] \end{aligned} \quad (A.5)$$

where the functions h_n are defined by a recurrence formula

$$h_n(r) = \frac{1}{r} [\frac{H_n(r)}{n!} + 2r h_{n-1}(r) - 2h_{n-2}(r)] \quad n = 1, 2, \dots$$

$$h_{-1}(-r) = \frac{\sqrt{\pi}}{2} e^{r^2} [1 + \Phi(r)] \quad \Phi(r) = \frac{2}{\sqrt{\pi}} \int_0^r e^{-u^2} du \quad (A.6)$$

$$h_0(r) = \sqrt{\pi} \int_0^r e^{-u^2} [1 + \Phi(u)] du - \frac{1}{2} C$$

$$C = 0,577215.. \text{ (Euler constant)}$$

and satisfy

$$[\frac{d}{dr^2} - r^2 + 2n + 1] \cdot \frac{1}{2} \frac{\sqrt{n!}}{2^n \sqrt{\pi}} e^{-\frac{r^2}{2}} h_n(r) = \phi_n(r). \quad (A.6')$$

From (A.2) and (A.2') we have also

$$\begin{aligned} [\frac{d}{dr^2} - r^2 + 2n + 1] \int_{-\infty}^{+\infty} G^{(n)}(r, r') f(r') dr' &= f(r) - \\ - [\phi_n(r) \int_{-\infty}^{+\infty} \phi_n(r') f(r') dr' &\dots \quad n = 0, 1, \dots \\ 0 &\dots \dots \dots \text{other } n. \end{aligned} \quad (A.7)$$

APPENDIX B. Evaluation of Integrals

Let $f_\alpha(r)$ and $g_\beta(r)$ satisfy

$$\left(\frac{d}{dr^2} - r^2 + \alpha \right) f_\alpha(r) = 0 \quad \left(\frac{d}{dr^2} - r^2 + \beta \right) g_\beta(r) = 0$$

Then

$$f''_{\alpha} g_{\beta} - f_{\alpha} g''_{\beta} = (\beta - \alpha) f_{\alpha} g_{\beta} \quad (B1)$$

$$\int_0^a f_{\alpha}(r) g_{\beta}(r) dr = \frac{1}{\beta - \alpha} [f'_{\alpha}(r) g_{\beta}(r) - f_{\alpha}(r) g'_{\beta}(r)]_0^a$$

provided $\alpha \neq \beta$.

In this way the integrals

$$J_{N\nu} = \int_0^a \exp(-\frac{r^2}{2}) {}_1F_1(\frac{\nu_0}{4} + \frac{N}{2}, \frac{1}{2}, r^2) \phi_{\nu}(r) dr; \quad Q_{N\nu} = \int_0^a \phi_{\nu}(r) [D_{-N}(\sqrt{2}) + D_{-N}(-\sqrt{2})] dr$$

$$P_N = \int_0^a \exp(-\frac{r^2}{2}) {}_1F_1(\frac{\nu_0}{4} + \frac{N}{2}, \frac{1}{2}, r^2) [D_{-N}(\sqrt{2}) + D_{-N}(-\sqrt{2})] dr; \quad j_{\mu\nu} = \int_0^a \phi_{\mu}(r) \phi_{\nu}(r) dr \quad \mu \neq \nu$$

can be evaluated. To evaluate the last integral for $\mu = \nu$ we use simple properties of the Hermite polynomials and get a recurrence formula

$$j_{\nu\nu} = -\frac{1}{\sqrt{2\nu}} \phi_{\nu}(a) \phi_{\nu-1}(a) + j_{\nu-1, \nu-1} \quad \nu = 1, 2, \dots$$

or explicitly

$$j_{\nu\nu} = \frac{1}{\sqrt{\pi}} \int_0^a e^{-t^2} dt - \sum_{n=1}^{\nu} \frac{1}{\sqrt{2n}} \phi_n(a) \phi_{n-1}(a). \quad \nu = 0, 1, \dots \quad (B2)$$

Further using (A.6') we can evaluate integrals containing h_n in the same way as in (B.1)

$$L_0 = \int_0^a e^{-r^2} {}_1F_1(\frac{\nu_0}{4}, \frac{1}{2}, r^2) [h_0(r) + h_0(-r)] dr \quad (B.3)$$

$$K_{0\nu} = \int_0^a \exp(-\frac{r^2}{2}) [h_0(r) + h_0(-r)] \phi_{\nu}(r) dr \quad \nu \neq 0.$$

Only for $\nu = 0$ the integration must be carried out numerically which is, however, very simple since, a is smaller than unity.

APPENDIX C, Estimation of the Kernel $K_1^{(M)}$

We derive (5.13) assuming that (5.12) is valid. From (3.2) it follows

$$\int_0^{\infty} \left[\int_0^{\infty} K_1^{(M)}(r, r') v(r') \psi_N^{(M)}(r') dr' \right]^2 dr = \sum_{N=0}^{\infty} \int_0^R \left[\sum_{N'=0}^{\infty} \int_0^R K_{(1)NN'}^{(M)}(r, r') v(r') \psi_N^{(M)}(r') dr' \right]^2 dr$$

$$\leq \sum_{N=0}^{\infty} \int_0^{\infty} \left[\sum_{N'=0}^{\infty} \int_0^R K_{(1)NN'}^{(M)}(r, r') v(r') \psi_N^{(M)}(r') dr' \right]^2 dr.$$

Using (5.8') we get

$$\int_0^R K_{(1)NN}^{(M)}(r, r') v(r') \psi_N^{(M)}(r') dr' = \bar{C}_{N'}^{(M)} K_{(1)NN}^{(M)}(r, a) + \int_0^R K_{(1)NN'}^{(M)}(r, r') \chi_N^{(M)}(r') dr'.$$

Then

$$\int_0^{+\infty} \left[\int_0^{+\infty} K_1^{(M)}(r, r') V(r') \psi^{(M)}(r') dr' \right]^2 dr \leq \sum_{N=0}^{+\infty} \int_0^{+\infty} \left\{ \left[\sum_{N'=0}^{\infty} C_{N'}^{(M)} K_{(1)NN'}^{(M)}(r, a) \right]^2 + \right. \\ \left. + \left[\sum_{N'=0}^{\infty} \int_0^R K_{(1)NN'}^{(M)}(r, r') \chi_{N'}^{(M)}(r') dr' \right]^2 + 2 \sum_{N'=0}^{\infty} C_{N'}^{(M)} K_{(1)NN'}^{(M)}(r, a) \sum_{N''=0}^{\infty} \int_0^R K_{(1)NN''}^{(M)}(r, r') \chi_{N''}^{(M)}(r') dr' \right\} dr. \quad (C.1)$$

We shall estimate successively the three terms in the square bracket.

1) The functions $\phi_n(r)$ are orthonormal on the interval $(-\infty, +\infty)$ i.e.,

$$\int_0^{+\infty} \phi_n(r) \phi_m(r) dr = \frac{1}{2} \delta_{nm}$$

if n and m are of the same parity. Hence by (3.2)

$$\int_0^{+\infty} K_{(1)NN'}^{(M)}(r, a) K_{(1)NN''}^{(M)}(r, a) dr = \sum_{\nu = \max(N, N', N'', M+1)}^{\nu} [1 + (-1)^\nu] \frac{\sigma_{\nu NN'} \sigma_{\nu NN''}}{4\nu^2} \phi_{\nu-N}^-(a) \phi_{\nu-N''}^-(a).$$

Interchanging the order of summations we get

$$\sum_{N=0}^{\infty} \int_0^{+\infty} \left[\sum_{N'=0}^{\infty} C_{N'}^{(M)} K_{(1)NN'}^{(M)}(r, a) \right]^2 dr = \\ = \sum_{\nu=M+1}^{\infty} \frac{1+(-1)^\nu}{4\nu^2} \sum_{N'=0}^{\nu} \sum_{N''=0}^{\nu} \phi_{\nu-N'}^-(a) \phi_{\nu-N''}^-(a) C_{N'}^{(M)} C_{N''}^{(M)} \sum_{N=0}^{\nu} \sigma_{\nu NN'} \sigma_{\nu NN''} = \\ = \sum_{\nu=M+1}^{\infty} \frac{1+(-1)^\nu}{4\nu^2} \frac{1}{2\nu} \left\{ \left[\sum_{N=0}^{\nu} \sqrt{\binom{\nu}{N}} C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 + \left[\sum_{N=0}^{\nu} (-1)^N \sqrt{\binom{\nu}{N}} C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 + \right. \\ \left. + \frac{1}{\nu} \left[\sum_{N=0}^{\nu} \sqrt{\binom{\nu}{N}} (2N-\nu) C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 + \frac{1}{\nu} \left[\sum_{N=0}^{\nu} (-1)^N \sqrt{\binom{\nu}{N}} (2N-\nu) C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 \right\}. \quad (C.2)$$

Here the orthogonality properties of the coefficients T , which give

$$\sum_{N=0}^{\nu} \sigma_{\nu NN'} \sigma_{\nu NN''} = \sigma_{\nu N'N''} \quad \nu \geq 2n_0 - 2$$

and the explicit form of $\sigma_{\nu NN'}$ for $n_0 = 2$, have been used. Further we get

$$\left[\sum_{N=0}^{\nu} (\pm 1)^N \sqrt{\binom{\nu}{N}} C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 \leq \sum_{N=0}^{\nu} \binom{\nu}{N} (C_N^{(M)})^2 \sum_{N=0}^{\nu} \phi_{\nu-N}^2(a) \quad (C.3)$$

and with the help of the Christoffel-Darboux formula for the Hermite polynomials

and the asymptotic behaviour of $\phi_n(r)$ for large n (ref. 10) we find

$$\sum_{N=0}^{\nu} \phi_{\nu-N}^2(a) < 0(\sqrt{\nu}).$$

Finally by (5.12) the inequality (C.3) becomes

$$\left[\sum_{N=0}^{\nu} (\pm 1)^N \sqrt{\binom{\nu}{N}} C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 \leq 2^\nu B_1^2 0(\sqrt{\nu}). \quad (C.4)$$

Similarly we obtain

$$\left[\sum_{N=0}^{\nu} (\pm 1)^N \sqrt{\binom{\nu}{N}} (2N-\nu) C_N^{(M)} \phi_{\nu-N}^-(a) \right]^2 < B_1^2 0 \left((\sqrt{\nu}) \sum_{N=0}^{\nu} \binom{\nu}{N} (2N-\nu)^2 \right) = \\ = B_1^2 2^\nu \nu 0(\sqrt{\nu}). \quad (C.5)$$

Then it follows from (C₂) that the first term in (C₁) behaves asymptotically as $O(\frac{1}{\sqrt{M}})$.

2) For the second term in (C₁) we find

$$\sum_{N=0}^{\infty} \int_0^{+\infty} \left[\sum_{N'=0}^{\infty} \int_0^R K_{(1)NN'}^{(M)}(r, r') \chi_N^{(M)}(r') \right]^2 dr \leq \sum_{N=0}^{\infty} \int_0^{+\infty} \left[\sum_{N'=0}^{\infty} \|\chi_N^{(M)}\| \sqrt{\int_0^R \frac{K_{(1)NN'}^{(M)}(r, r')^2}{(1)NN'} dr'} \right]^2 dr$$

$$< \sum_{N=0}^{\infty} \sum_{N'=0}^{\infty} \|\chi_N^{(M)}\|^2 \int_0^{+\infty} \int_0^R \left[K_{(1)NN'}^{(M)}(r, r') \right]^2 dr' dr \leq B_2^2 \sum_{N=0}^{\infty} \int_0^{+\infty} \int_0^R \left[K_{(1)NN'}^{(M)}(r, r') \right]^2 dr' dr.$$

The double sum can be estimated as in (3.5) and hence the second term in (C₁) behaves as $O(\frac{1}{M})$.

3) From (3.2) it is obvious that the third term can be obtained from the first replacing $\bar{C}_N^{(M)} \phi_{\nu-N}(a)$ by

$$k_{\nu-N}^{(M)} = \int_0^R \phi_{\nu-N}(r) \chi_N^{(M)}(r) dr.$$

Since

$$\sum_{N=0}^{\nu} (k_{\nu-N}^{(M)})^2 \leq \sum_{N=0}^{\nu} \int_0^R \phi_{\nu-N}^2(r) dr \int_0^R (\chi_N^{(M)}(r))^2 dr \leq B_2^2$$

we find

$$\left| \sum_{N=0}^{\nu} (\pm 1)^N k_{\nu-N}^{(M)} \sqrt{\binom{\nu}{N}} \right| \leq \sqrt{2^{\nu} \sum_{N=0}^{\nu} (k_{\nu-N}^{(M)})^2} < 2^{\frac{\nu}{2}} B_2$$

$$\left| \sum_{N=0}^{\nu} (\pm 1)^N k_{\nu-N}^{(M)} \sqrt{\binom{\nu}{N}} (2N - \nu) \right| \leq 2^{\frac{\nu}{2}} \sqrt{\nu} B_2.$$

Hence

$$\left| \sum_{N=0}^{\infty} \int_0^{+\infty} \sum_{N'=0}^{\infty} \bar{C}_{N'}^{(M)} K_{(1)NN'}^{(M)}(r, a) \sum_{N=0}^{\infty} \int_0^R K_{(1)NN'}^{(M)}(r, r') \chi_N^{(M)}(r') dr' dr \right| <$$

$$\leq B_1 B_2 \sum_{\nu=M+1}^{\infty} \frac{1+(-1)^{\nu}}{\nu^2} O(\nu^{-1/2}) = O(M^{-3/4}).$$

Consequently from (C₁) we get

$$\int_0^{+\infty} \left[\int_0^R K_1^{(M)}(r, r') \mathbf{V}(r') \psi^{(M)}(r') dr' \right]^2 dr = O\left(\frac{1}{\sqrt{M}}\right).$$

References

1. J. Goldstone. Proc. Roy. Soc., A 237, 267 (1957).
2. R. J. Eden, V. J. Emery. Proc. Roy. Soc., A 238, 266 (1958).
3. Yu. F. Smirnov. Nuclear Physics, 39, 346 (1962).
4. M. Moshinsky. Nuclear Physics, 13, 104 (1959).
5. И. М. Рыжик, И. С. Градштейн. Таблицы интегралов, сумм и произведений. Москва, Физматгиз, 1963.
6. С. Г. Михлин. Лекции по линейным интегральным уравнениям. Москва, Физматгиз, 1959.
7. E. C. Titchmarsh. Eigenfunction Expansions I. (1950) Oxford.
8. H. A. Bethe, J. Goldstone. Proc. Roy. Soc., A 238, 551 (1957).
9. K. Kumar. Perturbation Theory and the Nuclear Many Body Problem. North Holland Publishing Co. Amsterdam (1962).
10. G. Szegő. Orthogonal Polynomials. Am. Math. Soc. (1959).

Received by Publishing Department
on March 8, 1965.

Table 1

N	$R_N^{(1)}(12,10)$	$R_N^{(2)}(12)$	$R_N^{(1)}(14,12)$	$R_N^{(2)}(14)$	$R_N^{(1)}(16,14)$	$R_N^{(2)}(16)$
0	0,37	100	0,22	100	0,20	100
2	2,02	26,4	0,87	26,5	1,34	25,75
4	1,25	16,37	4,22	17,10	4,49	17,88
6	7,58	13,54	6,18	14,41	5,85	15,28
8	22,3	6,15	11,74	7,02	9,05	7,65
10	51,6	1,69	23,5	2,21	12,31	2,52
12	100	0,22	47,0	0,41	6,22	0,43
14	—	—	100	0,03	683,0	0,005
16	—	—	—	—	100	0,01

$$\mathcal{E}_{12,10} = 0,103$$

$$\mathcal{E}_{14,12} = 0,079$$

$$\mathcal{E}_{16,14} = 0,075$$

N	$R_N^{(1)}(18,16)$	$R_N^{(2)}(18)$	$R_N^{(1)}(20,18)$	$R_N^{(2)}(20)$	$R_N^{(1)}(22,20)$	$R_N^{(2)}(22)$	$a_{NN}^{(22)}$	$R_N^{(1)}(22,10)$
0	0,19	100	0,15	100	0,05	100	5,089	1,07
2	0,92	25,5	0,47	25,3	0,29	25,2	-1,283	1,87
4	3,65	18,5	2,44	18,9	1,28	19,2	-0,978	16,3
6	4,94	16,1	3,68	16,5	2,33	17,0	0,867	27,0
8	8,09	8,32	6,79	8,91	4,98	9,36	-0,604	49,2
10	10,87	2,82	11,3	3,18	10,3	3,54	0,180	76,9
12	12,08	0,39	1,31	0,39	20,8	0,49	-0,025	100
14	96,6	0,15	56,7	0,34	29,8	0,49	-0,025	100
16	87,2	0,09	65,0	0,25	45,1	0,45	0,023	100
18	100	0,01	81,3	0,07	63,0	0,20	-0,010	100
20	—	—	100	0,009	80,5	0,05	0,0023	100
22	—	—	—	—	100	0,005	-0,0002	100

$$\mathcal{E}_{18,16} = 0,067$$

$$\mathcal{E}_{20,18} = 0,055$$

$$\mathcal{E}_{22,20} = 0,041$$

$$\mathcal{E}_{22,10} = 0,399$$

N	(0) ^a NN	(0) ^b NN	(1) ^a NN	(1) ^b NN	(2) ^a NN	(2) ^b NN	(3) ^a NN	(3) ^b NN	(4) ^a NN	(4) ^b NN
0	5,035	0	5,399	-3,695	5,656	-4,293	5,739	-3,453	5,766	-4,499
2	-1,307	0	-1,097	0,637	-1,043	0,599	-1,016	0,562	-1,003	0,544
4	-0,819	0	-0,737	0,543	-0,691	0,534	-0,562	0,496	-0,547	0,473
6	0,633	0	0,550	-0,339	0,520	-0,330	0,506	-0,309	0,500	-0,299
8	-0,242	0	-0,217	0,128	-0,209	0,126	-0,205	0,121	-0,203	0,118
10	0,0414	0	0,0387	-0,0239	0,0377	-0,0236	0,371	-0,0226	0,0369	-0,0224

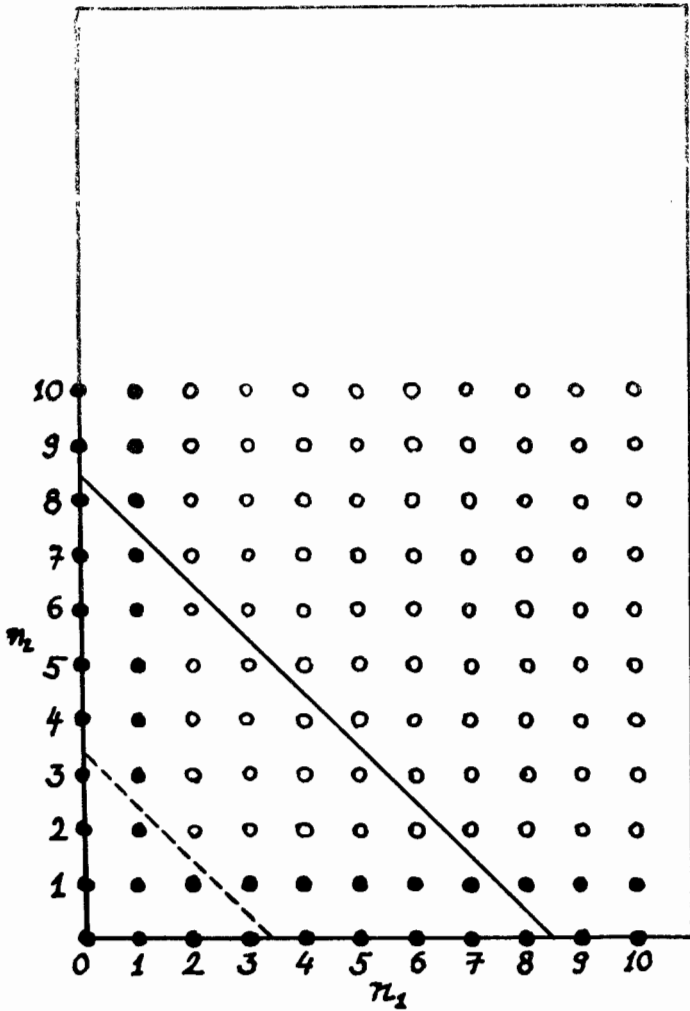


Fig. 1. The correct form of the EP for $n_0 = 2$ excludes the points marked with a black circle and allows those marked with a light circle. The modified form used in (3.7) allows all states above the solid line and excludes only the black circles below it (the case $M = 8$ is demonstrated as an example). The dashed line illustrates the triangular approximation.

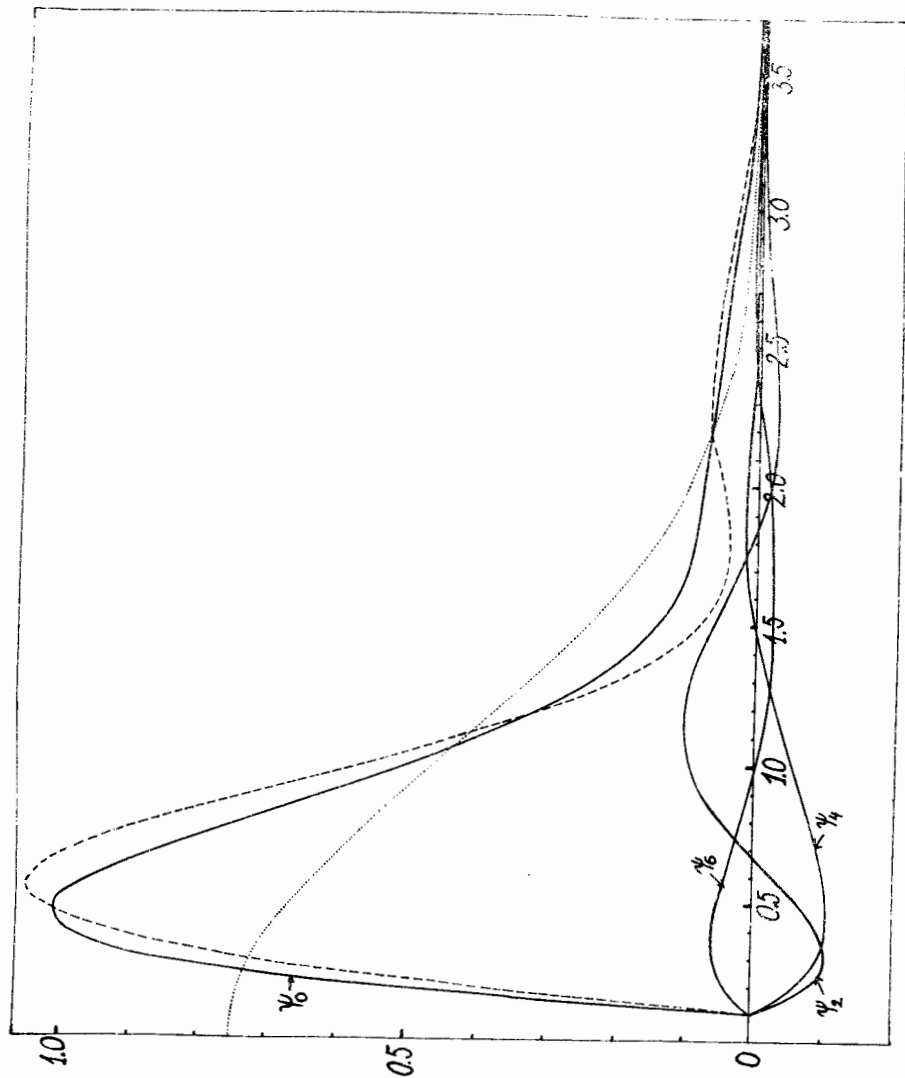


Fig.2. The solid curves represent the solution for the first potential in (6.1). For comparison the "unperturbed" function $\phi_0(r)$ (dotted line) and the function $\psi_0(r)$ for the potential without attraction (dashed line) are given. The functions ψ_8 and ψ_{10} are not plotted since they are very small if compared with ψ_0 .

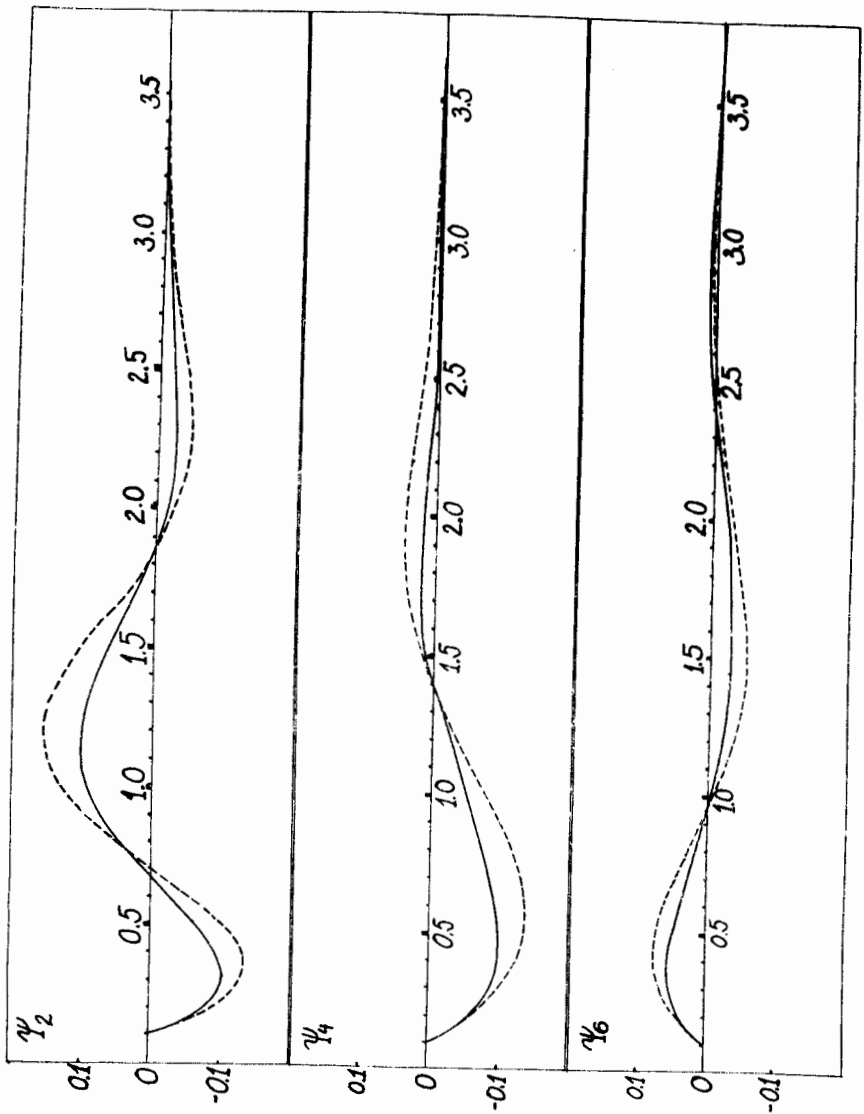


Fig.3. The solutions for the first potential in (6.1) (solid lines) and for the potential without attraction (dashed lines) are compared.