ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E-2-2932

Dao Vong Duc, Nguyen Van Hieu

ANALYTICITY, LOCALITY AND SYMMETRY WITH INFINITE MULTIPLETS

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1966

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Submitted to Jad.Phys.

1. Introduction

It was suggested /1-7/ that for a relativistic generalization of SU(6) symmetry one can use a group $\mathcal{G}$ which is the semi-direct product of the Poincare group $\mathcal{P}$ and a non--compact internal symmetry group S containing some subgroup SL(2,C). In such a theory elementary particles are classified on the basis of infinite-dimensional unitary representations of the internal symmetry group S.

In a series of paper /7-10/ the structure of S-matrix in such a symmetry was studied. It was shown that in such a relativistic scheme there exists no contradiction between symmetry and unitary of S-matrix /9,10/. The possibility of formulating symmetry with infinite multiplets within the framework of quantum field theory was discussed in refs. /10,11/. According to the method suggested in these papers elementary particles belonging to each infinite multiplet are described by means of an infinite number of spinor (or tensor) quantized fields which are transformed as finite-dimensional non-unitary representations of homogeneous Lorentz group. In such a scheme there exist the usual commutation relations with the normal connection between spin and statistic.

In the present paper we study the analyticity properties of the scattering amplitudes and the vertex functions in the theory of symmetry with non-compact group. The similar problem was also treated in a recent paper by Fronsdal /11/. For simplicity we shall consider the case S = SL(2,C). Our conclusions hold also for the general case.

2. Construction of physical basis for irreducible representations of internal symmetry group SL(2,C)

Before to study the vertex functions and the scattering amplitudes we must construct in an explicit form the basis of unitary representations of the group SL(2,C) according to which elementary particles are classified. The group SL(2,C) contains the group SU(2) as a maximal compact subgroup. In our previous paper /12/ each irreducible representation of the SL(2,C) group is realized in the form of an infinite set of SU(2) spinors - generalized spinors. This basis will be called canonical basis. However, particles with given spins in each multiplet are described by irreducible representations of the little group $SU(2)_p$ rather than by irreducible representations of the SU(2) subgroup. Therefore, to describe the particle states we must construct each irreducible representation of the SL(2,C) group in the from of an infinite set of $SU(2)_p$ spinors. We shall show that under the Lorentz transformation these spinors transform according to spin non-unitary representations of the homogeneous Lorentz group. We remind that each unitary representation of

SL(2,C) group (from the principal series) is characterized by a real number ρ and an integer or half-integer ν . Moreover a representation with given ν contains states with $j=|\nu|+n$ (n=0,1,2, ...). As illustration we first consider the simple case with $\nu=0$. This irreducible representation is realized in the Hilbert space of homogeneous functions $f(z_1,z_2)$ on two complex variables with degree of homogeneity $\left(\frac{i\rho}{2}-1,\frac{i\rho}{2}-1\right)$ The transformation law for these functions is of the form:

$$T_g f(z)=f(z')$$
$$z'_a=z_b g_{ba}\,,\quad \det g=1.\quad (a,b)=1,2$$

From the commutation relations between the generators of group S and Lorentz group it follows that the variables z_a transform according to the spinor representation of homogeneous Lorentz group

$$z_a \xrightarrow{\lambda} z_b A_{ba}(\lambda) \tag{1}$$

where $A(\lambda)$ is the unimodular 2x2 matrix corresponding to the Lorentz transformation λ. The variables z^*_a (complex conjugate of z_a) are transformed according to the conjugate representation and will be denoted by $z^{*\dot a}$:

$$z^{*\dot a} \xrightarrow{\lambda} z^{*\dot b} A^*_{\dot b\dot a}(\lambda) \tag{2}$$

Let us introduce new variables x_μ :

$$x_\mu = z_b(\sigma_\mu)^{b}_{\dot a}\, z^{*\dot a}, \tag{3}$$

$$(\sigma_4)^{b}_{\dot a}=\delta_{ba}\,,\qquad (\sigma_k)^{b}_{\dot a}=-i(\sigma_k)_{ba}\,, \tag{4}$$

where $(\sigma_k)_{ba}$ are the elements of Pauli matrices σ_k . It is easy to show that $x^2\equiv x_\mu x_\mu=0$. Under Lorentz transformation (1) and (2) the new variables x_μ transform as the components of a four-dimensional vector.

The homogeneous functions $f(z_1,z_2)$ with degree of homogeneity $\left(\frac{i\rho}{2}-1,\frac{i\rho}{2}-1\right)$ are also homogeneous functions on x_μ with the same degree of homogeneity. Hence a given unitary representation of the SL(2,C) group can be also realized in the Hilbert space of homogeneous functions $f(x)$ on the cone with degree of homogeneity $\frac{i\rho}{2}-1$. In our previous paper /12/ the canonical basis corresponding to the reduction SL(2,c) $\supset$ SU(2) has been constructed. The basis vector corresponding to the state with different spins are of the form [1]:

$$\text{spin } 0:\quad \phi \sim (z_c z^{*c})^{\frac{i\rho}{2}-1}$$
$$\text{spin } 1:\quad \phi_a^b \sim (z_c z^{*c})^{\frac{i\rho}{2}-2}\left\{z_a z^{*b}-\tfrac{1}{2}\delta_a^b z_c z^{*c}\right\} \tag{5}$$
$$\cdots\cdots\cdots$$
$$\text{spin } j:\quad \phi^{b_1b_2\dots b_j}_{a_1a_2\dots a_j} \sim (z_c z^{*c})^{\frac{i\rho}{2}-1-j}\sum_{s=0}^{j}(-1)^s\frac{(j!)^2(2j-s)!}{s![(j-s)!]^2(2j)!}(z_c z^{*c})^s\cdot \mathcal{S}\,\delta_{a_1}^{b_1}\cdots\delta_{a_s}^{b_s} z_{a_{s+1}}\cdots z_{a_j} z^{*b_{s+1}}\cdots z^{*b_j}$$

In terms of variables x_μ we have respectively:

$$\phi \sim x_0^{\frac{i\rho}{2}-1}$$
$$\phi_k \sim x_0^{\frac{i\rho}{2}-2}x_k \tag{6}$$
$$\cdots\cdots\cdots$$
$$\phi_{k_1k_2\dots k_j} \sim x_0^{\frac{i\rho}{2}-1-j}\sum_{s=0,2,\dots(j,j-1)}(-1)^{\frac{s}{2}}\frac{(2j-s-1)!!\,j!}{s!!(2j-1)!!(j-s)!}(\vec{x}^{\,2})^{\frac{s}{2}}\cdot \mathcal{S}\,\delta_{k_1k_2}\cdots\delta_{k_{s-1}k_s}x_{k_{s+1}}x_{k_{s+2}}\cdots x_{k_j}$$

These formulae admit a simple physical interpretation: x_0 is a scalar of the SU(2) group (but not of Lorentz group), x_k is the three-dimensional vector, and basis vectors $\phi_{k_1k_2\dots}$ are three-dimensional symmetrical traceless tensors constructed from the products of x_0 and x_k .

Now in terms of x_μ it is easy to construct the basis corresponding to the reduction SL(2,c) $\supset$ SU(2 $)_p$, this basis will be referred to as physical basis. As is well known in non-relativistic theory a spin 1 particle is described by a vector x_k but in relativistic theory this particle is described by a four-vector y_μ satisfying the condition

$$p_\mu y_\mu = 0\,, \tag{7}$$

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[1] Since this basis is not relativistic invariant there is no sense to consider its transformation properties under the Lorentz transformation and therefore we need not introduce dotted indices.

where p_μ is four-momentum of the particle. Therefore, instead of non-relativistic division of four-dimensional vector x_μ into space part x_k and time part x_0 we put

$$x_\mu = \left(x_\mu + \frac{p_\mu}{m^2} p_\nu x_\nu\right) - \frac{p_\mu}{m^2} p_\nu x_\nu = y_\mu - \frac{i p_\mu}{m} W, \tag{8}$$

where

$$y_\mu \equiv x_\mu + \frac{p_\mu}{m^2} p_\nu x_\nu \tag{9}$$

satisfies condition (7) and describes the spin 1 particle, and

$$W \equiv -i \frac{p_\mu x_\mu}{m} = z_a \left(-\frac{i\hat{p}}{m}\right)^a_{\dot{b}} z^{*\dot{b}} \tag{10}$$

is a scalar under the Lorentz group and describes the spin 0 particle. The physical basis corresponding to the reduction SL(2,c) $\supset$ SU(2)$_p$ can be constructed by analogy with formula (6), where instead of x_0 we must use W and instead of $x_k - y_\mu$. For example, the particles with momentum p and spin $j = 0,1,2$ will be described respectively by the following tensors:

$$\Phi \sim W^{\frac{i\varrho}{2}-1}$$
$$\Phi_\mu \sim W^{\frac{i\varrho}{2}-2} \cdot y_\mu$$
$$\Phi_{\mu_1\mu_2} \sim W^{\frac{i\varrho}{2}-3} \cdot \left(y_{\mu_1} y_{\mu_2} - \frac{1}{3} y_\nu y_\nu \cdot \Delta_{\mu_1\mu_2}\right),$$

where

$$\Delta_{\mu\nu} = \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}$$

For particle states with arbitrary spin j we have:

$$\Phi_{\mu_1\mu_2\cdots\mu_j} \sim W^{\frac{i\varrho}{2}-1-j} \, Y_{\mu_1\mu_2\cdots\mu_j},$$

$$Y_{\mu_1\mu_2\cdots\mu_j} = \sum_{s=0,2,\dots(j,j-1)} \alpha(j,s) \left(y_\mu y_\mu\right)^{\frac{s}{2}} S \, \Delta_{\mu_1\mu_2} \cdots \Delta_{\mu_{s-1}\mu_s} y_{\mu_{s+1}} \cdots y_{\mu_j}$$

$$\alpha(j,s) = (-1)^{\frac{s}{2}} \frac{(2j-1-s)!!\, j!}{s!!\,(2j-1)!!\,(j-s)!}$$

Now let us return to the original Hilbert space of homogeneous functions $f(z_1,z_2)$ of two complex variables z_1 and z_2. It follows from (11) that the states with

spins $j = 0,1,2,$ and momentum p are described by the following SU(2)$_p$ spinors:

$$\Phi \sim \left(z_c \left(-\frac{i\hat{p}}{m}\right)^c_{\dot{d}} z^{*\dot{d}}\right)^{\frac{i\varrho}{2}-1} \equiv \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^{\frac{i\varrho}{2}-1}$$

$$\Phi_a^{\dot{b}} \sim \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^{\frac{i\varrho}{2}-2} \cdot \left\{ z_a z^{*\dot{b}} - \frac{1}{2}\left(-\frac{i\hat{p}}{m}\right)_a^{\dot{b}} \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right) \right\}$$

$$\begin{aligned}\Phi_{a_1a_2}^{\dot{b}_1\dot{b}_2} \sim \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^{\frac{i\varrho}{2}-3} \cdot \Big\{ & z_{a_1} z_{a_2} z^{*\dot{b}_1} z^{*\dot{b}_2} - \\ & -\frac{1}{4}\left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)\Big[\left(-\frac{i\hat{p}}{m}\right)_{a_1}^{\dot{b}_1} z_{a_2} z^{*\dot{b}_2} + \left(-\frac{i\hat{p}}{m}\right)_{a_2}^{\dot{b}_1} z_{a_1} z^{*\dot{b}_2} + \left(-\frac{i\hat{p}}{m}\right)_{a_1}^{\dot{b}_2} z_{a_2} z^{*\dot{b}_1} + \left(-\frac{i\hat{p}}{m}\right)_{a_2}^{\dot{b}_2} z_{a_1} z^{*\dot{b}_1}\Big] + \\ & + \frac{1}{12}\left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^2 \Big[\left(-\frac{i\hat{p}}{m}\right)_{a_1}^{\dot{b}_1}\left(-\frac{i\hat{p}}{m}\right)_{a_2}^{\dot{b}_2} + \left(-\frac{i\hat{p}}{m}\right)_{a_2}^{\dot{b}_1}\left(-\frac{i\hat{p}}{m}\right)_{a_1}^{\dot{b}_2}\Big]\Big\}\end{aligned}$$

and for arbitrary spin we have:

$$\Phi_{a_1a_2\cdots a_j}^{\dot{b}_1\dot{b}_2\cdots\dot{b}_j} \sim \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^{\frac{i\varrho}{2}-1-j} \sum_{s=0}^{j} (-1)^s \frac{(j!)^2(2j-s)!}{s!\,[(j-s)!]^2(2j)!} \left(z\left(-\frac{i\hat{p}}{m}\right)z^*\right)^s \cdot S\left(-\frac{i\hat{p}}{m}\right)_{a_1}^{\dot{b}_1} \cdots \left(-\frac{i\hat{p}}{m}\right)_{a_s}^{\dot{b}_s} z_{a_{s+1}} \cdots z_{a_j} z^{*\dot{b}_{s+1}} \cdots z^{*\dot{b}_j} \tag{12}$$

These spinors of the little group SU(2)$_p$ are transformed according to corresponding spinor representations of homogeneous Lorentz group and satisfy the condition:

$$\left(\frac{i\hat{p}}{m}\right)^{a_l}_{\dot{b}_k} \Phi_{a_1\cdots a_l\cdots a_j}^{\dot{b}_1\cdots\dot{b}_k\cdots\dot{b}_j} = 0 \tag{13}$$

They differ from the SU(2) spinors by the fact that instead of the relativistic non-invariant summation $z_c z^{*c}$ we use the invariant one $z_c\left(-\frac{i\hat{p}}{m}\right)^c_{\dot{d}} z^{*\dot{d}}$. Otherwise speaking, the SU(2)$_p$ spinors are obtained from the SU(2) spinors defined by formula (6) by the substitution $\delta^a_b \longrightarrow \left(-\frac{i\hat{p}}{m}\right)^a_{\dot{b}}$.

Consider now the general case of unitary representation with arbitrary ν and ϱ. In our previous paper[12] the canonical basis constructed from SU(2) spinors was found:

$$\phi^{b_1 b_2\cdots b_{j-\nu}}_{a_1 a_2\cdots a_{j+\nu}} \sim (z_c z^{*c})^{\frac{i\varrho}{2}-1-j} \sum_{s=0}^{j-\nu} (-1)^s \frac{(j-\nu)!(j+\nu)!(2j-s)!}{s!(j-\nu-s)!(j+\nu-s)!(2j)!} (z_c z^{*c})^s \qquad (14)$$

$$\int \delta^{b_1}_{a_1}\cdots\delta^{b_s}_{a_s} z_{a_{s+1}}\cdots z_{a_{j+\nu}} z^{*b_{s+1}}\cdots z^{*b_{j-\nu}}$$

As in the case ν =0 in order to get the physical basis constructed from $SU(2)_p$ spinors it is sufficient to start from the spinors (14) making the substitution $z_c z^{*c} \longrightarrow z_c(-\frac{i\hat{p}}{m})^c_{\dot d} z^{*\dot d}$ and $\delta^b_a \longrightarrow (-\frac{i\hat{p}}{m})^b_a$. We have:

$$\phi^{b_1 b_2\cdots b_{j-\nu}}_{a_1 a_2\cdots a_{j+\nu}}(p) \sim \left(z(-\frac{i\hat{p}}{m})z^*\right)^{\frac{i\varrho}{2}-1-j} \sum_{s=0}^{j-\nu} (-1)^s \frac{(j-\nu)!(j+\nu)!(2j-s)!}{s!(j-\nu-s)!(j+\nu-s)!(2j)!} \left(z(-\frac{i\hat{p}}{m})z^*\right)^s \qquad (15)$$

$$\int \left(-\frac{i\hat{p}}{m}\right)^{b_1}_{a_1}\cdots\left(-\frac{i\hat{p}}{m}\right)^{b_s}_{a_s} z_{a_{s+1}}\cdots z_{a_{j+\nu}} z^{*b_{s+1}}\cdots z^{*b_{j-\nu}}$$

These $SU(2)_p$ spinors are transformed according to corresponding spinor representations of homogeneous Lorentz group and satisfy the condition:

$$\left(\frac{i\hat{p}}{m}\right)^{a_\ell}_{b_k} \phi^{b_1\cdots b_k\cdots b_{j-\nu}}_{a_1\cdots a_\ell\cdots a_{j+\nu}}(p)=0, \qquad (16)$$

which means that they describe states with definite spins.

Now let χ be any element from Hilbert space realizing a given unitary representation of the SL(2,c) group. This generalized spinor can be represented in the form:

$$\chi_p(z)=\sum_j \chi^{a_1\cdots a_{j+\nu}}_{b_1\cdots b_{j-\nu}}(p)\,\phi^{b_1\cdots b_{j-\nu}}_{a_1\cdots a_{j+\nu}}(z) \qquad (17)$$

The components $\chi^{a_1 a_2\cdots}_{b_1 b_2\cdots}(p)$ can always be chosen in such a way that they are symmetrical in indices of each kind and satisfy the condition:

$$\left(\frac{i\hat{p}}{m}\right)^{b_k}_{a_\ell}\chi^{a_1\cdots a_\ell\cdots a_{j+\nu}}_{b_1\cdots b_k\cdots b_{j-\nu}}(p)=0 \qquad (18)$$

where

$$(\sigma_4)^b_a=\delta_{ba}\quad , (\sigma_k)^b_a=i(\sigma_k)_{ba} \qquad (19)$$

Further on we shall express the vertex functions and matrix elements of scattering process in terms of these components $\chi^{a_1 a_2\cdots}_{b_1 b_2\cdots}$.

3. <u>Vertex functions and scattering amplitudes</u>

Now we study the structure of the vertex functions and scattering amplitudes in the theory of symmetry with infinite multiplets. First of all, for simplicity let us consider trilinear interaction between some infinite multiplet and a singlet, and in addition for the infinite multiplet we put $\varrho=0$. In this case the invariant vertex is ofthe form

$$\Gamma(p_2,p_1)=f(p_1^2,p_2^2,k^2)\,\varphi(k)\int\chi^*_{p_2}(z)\chi_{p_1}(z)\,d\omega_z\ ,\ k=p_2-p_1 \qquad (20)$$

where $f(p_1^2,p_2^2,k^2)$ is an arbitrary form-factor, $\varphi(k)$ is the wave function of the singlet. Using formula (17) for $\chi_{p_1}(z)$ and $\chi_{p_2}(z)$ we have:

$$\Gamma(p_2,p_1)=f(p_1^2,p_2^2,k^2)\varphi(k)\sum_{j,j'}\chi^{*b_1\cdots b_{j'-\nu}}_{\dot a_1\cdots\dot a_{j'+\nu}}(p_2)\,M^{\dot a_1\cdots\dot a_{j'+\nu},\,d_1\cdots d_{j-\nu}}_{b_1\cdots b_{j'-\nu},\,c_1\cdots c_{j+\nu}}(p_1,p_2)\,\chi^{c_1\cdots c_{j+\nu}}_{d_1\cdots d_{j-\nu}} \qquad (21)$$

where the matrices

$$M^{\dot a_1\cdots\dot a_{j'+\nu},\,d_1\cdots d_{j-\nu}}_{b_1\cdots b_{j'-\nu},\,c_1\cdots c_{j+\nu}}(p_1,p_2)$$

are kinematic factors and fully determined by the following integral:

$$M^{\dot a_1\cdots\dot a_{j'+\nu},\,d_1\cdots d_{j-\nu}}_{b_1\cdots b_{j'-\nu},\,c_1\cdots c_{j+\nu}}(p_1,p_2)=\int\phi^{*\dot a_1\cdots\dot a_{j'+\nu}}_{b_1\cdots b_{j'-\nu}}(p_2)\,\phi^{d_1\cdots d_{j-\nu}}_{c_1\cdots c_{j+\nu}}(p_1)\,d\omega_z\,. \qquad (22)$$

In order to find out the properties of these factors let us consider a special case when ν =0. Then the first factor contained in the vertex with three Lorentz scalar particles equals:

$$M(p_1,p_2)=\frac{1}{\pi}\int\left(z(-\frac{i\hat{p}_2}{m})z^*\right)^{-1}\left(z(-\frac{i\hat{p}}{m})z^*\right)^{-1}d\omega_z=\frac{1}{2\sqrt{\alpha^2-1}}\ln\frac{\alpha+\sqrt{\alpha^2-1}}{\alpha-\sqrt{\alpha^2-1}}\ ,\ \alpha\equiv-\frac{(p_1}{m}$$

This means that the part of vertex corresponding to the interaction of scalar particles is equal to

$$\langle 0|\Gamma|0\rangle = f(p_1^2,p_2^2,k^2)\,M(p_1,p_2)\,\varphi(k)\,\chi^*(p_2)\,\chi(p_1) \tag{23}$$

Here we note that the kinematical factors $M(p_1,p_2)$ are fully determined only in the physical region of corresponding processes. Thus, for example, this factor is equal to:

$$M(p_2,p_1)=\Gamma(t)=\frac{m^2}{\sqrt{t(t-4m^2)}}\ln\frac{2m^2-t+\sqrt{t(t-4m^2)}}{2m^2-t-\sqrt{t(t-4m^2)}}, \tag{24}$$

for the scattering channel $t=-(p_1-p_2)^2<0$

and

$$M(-p_2,p_1)=\Gamma(s)=\frac{m^2}{\sqrt{s(s-4m^2)}}\ln\frac{2m^2-s+\sqrt{s(s-4m^2)}}{2m^2-s-\sqrt{s(s-4m^2)}} \tag{25}$$

for annihilation channel $s=-(p_1+p_2)^2>4m^2$. An analogous result has been also obtained in an earlier paper of Fronsdal /11/ by another method.

Note that on the basis of formula (23) it is impossible to compute kinematic factor $\Gamma(t)$ for complex t . Otherwise speaking, there exists no theoretical basis for its analytical continuation.

In an analogous way from the formula (21) we get the following expression for the matrix element corresponding to the transition $j=1\to j'=0$ in the scattering channel:

$$\langle 0|\Gamma(p_2,p_1)|1\rangle = f(p_1^2,p_2^2,k^2)\frac{4im^3}{t(t-4m^2)}\left[1-\frac{2m^2-t}{2\sqrt{t(t-4m^2)}}\ln\frac{2m^2-t+\sqrt{t(t-4m^2)}}{2m^2-t-\sqrt{t(t-4m^2)}}\right]\cdot\varphi(k)\,\chi^*(p_2)\,V_\mu(p_1)\,p_{2\mu}, \tag{26}$$

where $V_\mu(p_1)$ is the relativistic wave function of meson with spin 1 in the initial state:

$$V_\mu(p_1)=\frac{1}{2}(\sigma_\mu)_b^{\dot a}\,\chi^b_{\dot a}(p_1) \tag{27}$$

It is clear that here the $L-S$ coupling is automatically obtained.

Consider now the case $\nu=\frac{1}{2}$. All the well-known spin 1/2 baryons must belong to the multiplets of this type. The part of the vertex corresponding to the interaction of spinors with spin 1/2 equals:

$$\langle\tfrac12|\Gamma(p_2,p_1)|\tfrac12\rangle = f(p_1^2,p_2^2,k^2)\cdot\frac{1}{2(1+\alpha)\sqrt{\alpha^2-1}}\left[\sqrt{\alpha+\sqrt{\alpha^2-1}}-\sqrt{\alpha-\sqrt{\alpha^2-1}}\right]\chi^*_{\dot a}(p_2)\left[-\frac{i(\hat p_1+\hat p_2)^{\dot a}_b}{m}\right]\chi^b(p_1) \tag{28}$$

For convenience let us introduce the Dirac spinor /13,14/ instead of two-component spinor χ^c:

$$\psi=(\psi^A)=\begin{pmatrix}\chi^a\\ \chi^{\dot a}\end{pmatrix} \tag{29}$$

$$\chi^{\dot a}=\left(-\frac{i\hat p}{m}\right)^{\dot a}_b\chi^b, \tag{30}$$

and put

$$\bar\psi=\psi^+\gamma_4\,,\quad \gamma_4=\begin{pmatrix}0&I\\ I&0\end{pmatrix}.$$

Then the last multiplier in the right-hand side of (28) can be rewritten in the form:

$$\chi^*_{\dot a}(p_2)\left[-\frac{i(\hat p_1+\hat p_2)^{\dot a}_b}{m}\right]\chi^b(p_1)=\chi^*_{\dot a}\chi^{\dot a}+\chi^*_a\chi^a=\bar\psi(p_2)\psi(p_1). \tag{31}$$

For the states with other spins we can obtain the expressions for vertex functions in an analogous way.

We see that together with arbitrary form-factors depending on the dynamics of the process the vertices contain also kinematical form-factors which are fully determined by the symmetry properties. We shall show below that the dynamical form-factors $f(p_1^2,p_2^2,k^2)$ have usual analyticity properties and are crossing symmetrical. As to kinematical factors they satisfy the usual Low's substitution law (passing from the scattering channel to the annihilation channel it is sufficient to substitute s for t), but they cannot be analytically continued into oomplex plane t.

Finally, consider the elastic scattering of a singlet on a particle from the multiplet with $\nu=1/2$. For the process

$$0+\tfrac12\longrightarrow\tfrac12+0$$

we have the following matrix element

$$M(q_1,p_1;p_2,q_2)=A(s,t)\cdot\frac{\sqrt{2}\,m^3}{(4m^2-t)\sqrt{t(t-4m^2)}}\left\{\sqrt{2m^2-t+\sqrt{t(t-4m^2)}}-\sqrt{2m^2-t-\sqrt{t(t-4m^2)}}\right\}\bar\psi(p_2)\psi(p_1), \tag{32}$$

where A(s,t) is some invariant amplitude which is determined by the dynamics of the.

process and has usual analyticity properties. Here it is interesting to note that the physical amplitudes are not analytical on t in the Lehman ellipse /15/.

Formula (32) shows that if π- meson is a singlet then in the scattering process of π-mesons on nucleons the polarization would be equal to zero in contradiction with experimental data. Therefore, for the classification of π- mesons it is necessary to use also an infinite multiplet.

4. Local interaction and analyticity of the scattering amplitudes

Now we study the connection between the results obtained above and the possibility of constructing local interaction Lagrangian. For simplicity we consider the trilinear interaction between particles from the multiplet with ν =0 and some singlet.

In the «p»-representation the interaction Lagrangian invariant under the SL(2,C) group is obtained from formula (21) if the dynamical form-factor is assumed to be constant. Then passing to the «x»-representation we get immediately the interaction Lagrangian invariant under the given group. The part of Lagrangian corresponding to the interaction of three particles with spin zero is:

$$\mathcal{L}_{int}(x) = f \cdot \varphi(x)\left\{\chi^*(x)\Gamma(\Box)\chi(x)\right\}, \tag{33}$$

where

$$\Gamma(\Box) = \sum_{k=0}^{\infty} C_k \Box^k$$

$$\Box \equiv \left(\frac{\overrightarrow{\partial}}{\partial x_\mu} + \frac{\overleftarrow{\partial}}{\partial x_\mu}\right)^2$$

$$C_k = \sum_{n=0}^{k} \sum_{\substack{a=0,\, b=0 \\ c=\frac{k}{2} \text{ or } \frac{k+1}{2}}}^{\infty} (-1)^{b+k}\binom{2c}{n}\frac{(2c-1)!!}{(2a+1)\cdot 2^{k+c} m^{2k}\, b!\, c!\, (k-n)!}\cdot\frac{\Gamma(a+\frac{3}{2})\Gamma(1-2b)}{\Gamma(a+\frac{3}{2}-b)\Gamma(1-2b-k+n)}$$

where $\frac{\overrightarrow{\partial}}{\partial x_\mu}$ acts on $\chi(x)$ and $\frac{\overleftarrow{\partial}}{\partial x_\mu}$ on $\chi^*(x)$. The analogous results can be also obtained for other cases.

The interaction Lagrangian (33) contains an infinite number of derivatives which appear namely because of the requirement of symmetry. The reason of their appearance is the following. The elementary particles contained in each infinite SL(2,C) multiplet are classified according to irreducible representations of the little group $SU(2)_p$. To describe these particles within the framework of quantum field theory we must introduce an infinite number of Bargman-Wigner-spinors

$$\psi^{A_1A_2\cdots A_m}_{B_1B_2\cdots B_n}(x) = \frac{i}{(2\pi)^{3/2}}\int\left\{\psi^{(+)A_1A_2\cdots A_m}_{B_1B_2\cdots B_n}(p)e^{ipx} + \psi^{(-)A_1A_2\cdots A_m}_{B_1B_2\cdots B_n}(p)e^{-ipx}\right\}\delta(p^2+m^2)\theta(p^0)d^4p \tag{35}$$

Let X be some element of internal symmetry group S which does not depend on momentum p. Since particles with definite spins form canonical basis corresponding to the reduction (depending on p) SL(2,C) $\supset$ SU$(2)_p$ the matrix element for transformation of Fourier-components of the field operators $\psi^{(\pm)A_1A_2\cdots}_{B_1B_2\cdots}(p)$ under X depends on p:

$$X\,\psi^{(\pm)A_1\cdots A_m}_{B_1\cdots B_n}(p)X^{-1} = \sum_{k,l} X^{A_1\cdots A_m;\,D_1\cdots D_l}_{B_1\cdots B_n;\,C_1\cdots C_k}(\pm p)\,\psi^{(\pm)C_1\cdots C_k}_{D_1\cdots D_l} \tag{36}$$

(for details see /10/.

Then it follows that under X the field operator $\psi^{A_1A_2\cdots}_{B_1B_2\cdots}(x)$ are transformed according to non-local transformation concerning derivatives of all orders:

$$X\psi^{A_1\cdots A_m}_{B_1\cdots B_n}(x)X^{-1} = \sum_{k,l} X^{A_1\cdots A_m;\,D_1\cdots D_l}_{B_1\cdots B_n;\,C_1\cdots C_k}\left(-i\frac{\partial}{\partial x}\right)\psi^{C_1\cdots C_k}_{D_1\cdots D_l}(x) \tag{37}$$

The interaction Lagrangian can be invariant under transformation of this type only in the case when it contains an infinite number of derivatives. Otherwise speaking, the appearance of infinite number of derivatives in the Lagrangian in due to the fact that the non-compact symmetry group SL(2,C) is the group of non-local transformations of quantum fields describing infinite multiplets of the given group.

Further, studying the structure of S-matrix many authors introduced also unphysical basis corresponding to the reduction (independent of p) SL(2,C) $\supset$ SU(2) together with a physical basis corresponding to the reduction (depending on p) SL(2,C) $\supset$ SU$(2)_p$. Each basic vector of this unphysical basis is a linear combination of an infinite number of particle wave functions:

$$|pj\mu\rangle_{unph.} = \sum_{j'\mu'} F_{j\mu,j'\mu'}(p)\,|pj'\mu'\rangle_{ph.} \tag{38}$$

where $|pj\mu\rangle_{ph.}$ and $|pj\mu\rangle_{unph.}$ are physical and unphysical states. In the physical state j and μ are spin and spin projection of the particle. Under X these states are transformed into:

$$X|pj\mu\rangle_{ph.} = \sum_{j'\mu'} X_{j\mu;j'\mu'}(p)|pj'\mu'\rangle_{ph.} \qquad (39)$$

$$X|pj\mu\rangle_{unph.} = \sum_{j'\mu'} \mathcal{X}_{j\mu;j'\mu'} |pj'\mu'\rangle_{unph.} \qquad (40)$$

The explicit expressions for $X_{j\mu;j'\mu'}(p)$ can be found by using the method given in refs. /9,10,12,16/ and $\mathcal{X}_{j\mu;j'\mu'}$ does not depend on p.

From the Fourier-components $\psi^{(\pm)A_1A_2\dots}_{B_1B_2\dots}(p)$ we first pick out the creation and annihilation operators:

$$\alpha_{j\mu}(p) = \bar{u}^{B_1\dots B_{n_j}}_{A_1\dots A_{m_j}}(pj\mu)\,\psi^{(+)A_1\dots A_{m_j}}_{B_1\dots B_{n_j}}(p) \qquad (41)$$

$$\beta^+_{j\mu}(p) = \bar{v}^{B_1\dots B_{n_j}}_{A_1\dots A_{m_j}}(pj\mu)\,\psi^{(-)A_1\dots A_{m_j}}_{B_1\dots B_{n_j}}(p) \qquad (42)$$

where $u^{A_1A_2\dots}_{B_1B_2\dots}(pj\mu)$ and $v^{A_1A_2\dots}_{B_1B_2\dots}(pj\mu)$ are respectively positive frequency and negative frequency wave functions of particles with spin j, spin projection μ, momentum p, and then we form the linear combinations of the type (38):

$$\tilde{\alpha}_{j\mu}(p) = \sum_{j'\mu'} F^*_{j\mu,j'\mu'}\,\alpha_{j'\mu'}(p) \qquad (43)$$

$$\tilde{\beta}^+_{j\mu}(p) = \sum_{j'\mu'} F^*_{j\mu,j'\mu'}\,\beta^+_{j'\mu'}(p) \qquad (44)$$

Under X these new operators are transformed independently of p:

$$X\tilde{\alpha}_{j\mu}(p) = \sum_{j'\mu'} \mathcal{X}^*_{j\mu;j'\mu'}\,\tilde{\alpha}_{j'\mu'}(p) \qquad (45)$$

$$X\tilde{\beta}^+_{j\mu}(p) = \sum_{j'\mu'} \mathcal{X}^*_{j\mu;j'\mu'}\,\tilde{\beta}^+_{j'\mu'}(p) \qquad (46)$$

Now we go from these operators to "x" representation:

$$\tilde{\psi}_{j\mu}(x) = \frac{1}{(2\pi)^{3/2}}\int\left[\tilde{\alpha}_{j\mu}(p)e^{ipx} + \tilde{\beta}^+_{j\mu}(p)e^{-ipx}\right]\delta(p^2+m^2)\theta(p^0)d^4p \qquad (47)$$

Operators $\tilde{\psi}_{j\mu}(x)$ form canonical basis corresponding to the reduction $SL(2,C) \supset SU(2)$. From them the polylinear invariant combinations can be formed immediately, the Clebsh-Gordan coefficients contained in these combinations being the usual numerical coefficients (independent both of p and x). This means that the symmetry is compatible with the locality with respect to unphysical fields $\tilde{\psi}_{j\mu}(x)$.

It is easy to show that these unphysical fields are related to the physical fields $\psi^{A_1A_2\dots}_{B_1B_2\dots}(x)$ describing particles with definite spins by means of the following integral (non-local) transformation:

$$\tilde{\psi}_{j\mu}(x) = \sum_{j'\mu'}\int \left\{F_{j\mu;j'\mu'}(x,y)\right\}^{B_1B_2\dots B_{n_{j'}}}_{A_1A_2\dots A_{m_{j'}}}\,\psi^{A_1A_2\dots A_{m_{j'}}}_{B_1B_2\dots B_{n_{j'}}}(y)\,dy \qquad (48)$$

$$\left\{F_{j\mu;j'\mu'}(x,y)\right\}^{B_1\dots B_{n_{j'}}}_{A_1\dots A_{m_{j'}}} = \frac{1}{(2\pi)^4}\int d^4p\, F^*_{j\mu;j'\mu'}(p)\left[e^{ip(x-y)}\bar{u}^{B_1\dots B_{n_{j'}}}_{A_1\dots A_{m_{j'}}}(pj'\mu') + e^{-ip(x-y)}\bar{v}^{B_1\dots B_{n_{j'}}}_{A_1\dots A_{m_{j'}}}(pj'\mu')\right] \qquad (49)$$

By substituting these expressions into the local interaction Lagrangian containing $\tilde{\psi}_{j\mu}(x)$ explicitly and satisfying the requirements of symmetry we obtain a non-local interaction Lagrangian. Analogously, if we start from the intitial and final unphysical states $|pj\mu\rangle_{unph.}$ defined by formula (38) then the symmetry does not contradict the analyticity of corresponding unphysical amplitudes. The form-factor $f(p_1^2,p_2^2,k^2)$ in formula (28) is an example of amplitude of this type. However, when we return to the real states $|pj\mu\rangle_{ph.}$ because of the presence of the kinematical factors $F_{j\mu,j'\mu'}(p)$ in formula (38) there appear the kinematical singularities in the physical scattering amplitudes,

Therefore, the higher symmetry with infinite multiplets is incompatible with the usual local properties of quantum fields and also with the usual analyticity properties of the scattering amplitudes. The symmetry group represents a group of non-local transformations and the invariance under this group requires a peculiar non-locality of interaction.

We are indebted to N.N.Bogoliubov, Ya.A.Smorodinsky and A.N.Tavkhelidze for their interest in this work. Our thanks are due also to C.Fronsdal for sending us his preprint on the same problem.

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Received by Publishing Department
on September 16,1966.