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INFRARED ASYMPTOTICS OF THE GREEN FUNCTIONS

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Introduction

In meson theory without account of the electromagnetic interaction the Green function $G(p^2)$ of a particle of mass m in the infrared region $p^2 \approx m^2$ has the form ($x = p^2 m^{-2} - 1$)

$$G(p^2) \approx x^{-1} + \text{const}. \tag{1.1}$$

This formula follows from the Kallen-Lehmann representation /1,2/ and is valid in all orders in the coupling constant. If the electromagnetic interaction is taken into account and the particle is charged then the infrared asymptotics of its Green function becomes much more complicated. A large number of papers has been devoted to investigation of this asymptotics. Various methods, namely, method of renormalization group /3/, approximate solution of the Schwinger functional equations /4/, solution of the Dyson integral equations in the ladder approximation /5/, method of functional integration /6/ were used to show that the first term in the asymptotic expansion of the Green function in the infrared region is

$$G(p^2) \approx (-x)^{-1+\gamma} \tag{1.2}$$

where γ is, generally speaking, a series in the fine structure constant α, in which the first term of order α was obtained. With the aid of the method of renormalization group it has been shown that in γ there is no term of the order α^2 /7/.

The problem is to obtain the exponent γ in all orders in α and to find next terms in the expansion of the Green function in x. This problem was recently considered by Milekhin /8/ with the aid of the method of functional integration /9/, but next terms in the expansion were estimated by means of perturbation theory. Finally, in ref. /10/ it was shown that the infrared asymptotics of the Green function in all orders in the coupling constants is of the form

$$G(p^2) \approx (-x)^{-1+\gamma} + O(x^{\gamma}) + \text{const}, \tag{1.3}$$

where for the Feynman gauge in all orders in e

$$\gamma = -\frac{\alpha}{\pi}. \tag{1.4}$$

In the present paper we find an explicit form of the function $O(x^{\gamma})$ without any recourse to perturbation theory. We obtain the formula which like (1.1) explicitly contains all terms singular in the infrared region and thereby completely generalizes (1.1) to the case when the electromagnetic interaction is taken into account.

We consider the Green function for a particle of spin 0 and $\frac{1}{2}$ and employ the Kallen-Lehmann representation[1,2,10] and expansions of the matrix elements of field operators in the soft photon momenta obtained by the Low method[11] which for our purpose was generalized in ref.[12]. In ref.[12] it was indicated that the Low method generally speaking is not valid in higher orders in e for the matrix elements of real processes. The diagrams of such processes contain at least two external lines corresponding to real charged particles. The exchange of soft photons between these particles leads to infrared divergences and makes the Low method irrelevant. Here we shall consider the matrix elements of field operators the diagrams of which contain only one line corresponding to a charged real particle. These matrix elements contain no infrared divergence and the Low method for them is valid in all orders in e.

2. Particle of Spin Zero

Let us consider the Green function of a charged spinless particle which, for definiteness, we shall call meson. We first consider the matrix element[x)]

$$T_n = \langle 0 | \Phi | r, k_1, \ldots, k_n \rangle , \qquad (2.1)$$

where Φ is the Heisenberg operator of the meson field at the origin, r and m are the momentum and the mass of the meson, k_i is the momentum of a photon with polarization ϵ_i. This matrix element corresponds to the diagram of Fig. 1. In ref.[10] it has been shown that the expansion of T_n in the momentum $k_n = k$ is of the form

$$T_n \approx k^{-1} + O(1) \qquad (2.2)$$

In a similar way it can be shown that the next term in this expansion is $O(k \ln k)$. We shall find an explicit form of $O(1)$ in (2.2). For this purpose we use the following equality

$$T_n (\epsilon \to k) = e T_{n-1} , \qquad (2.3)$$

which is a generalization of the Ward identity. This equality can be derived from the relations obtained in ref.[13] It is not difficult to prove it directly. We notice

x) As in ref.[12], $\hbar = c = 1$; $ab = g^{mn} a_m b_n = a^0 b^0 - \vec{a}\,\vec{b}$; $\langle k | k' \rangle = (2\pi)^3\, 2k^0 \delta(\vec{k} - \vec{k}')$; $\tilde{d}\vec{k} = d\vec{k} / (2\pi)^3\, 2k^0$; $(F)_n = \prod_{i=1}^{n} F(k_i)$.

that if a photon line with momentum k and polarization ϵ is inserted into a line corresponding to a charged particle or into a simple meson-photon vertex and ϵ is replaced by k, then the result is obtained from the initial diagram by the substitution

$$F(q) \to e\,[\,F(q) - F(q+k)\,] . \qquad (2.4)$$

for the internal charged line and the vertex and $F(q) \to e F(q)$ for the external charged line, where $F(q)$ is the factor corresponding to the charged line or to the vertex and q stands for the momenta of the charged particle upon which the factor depends. Now we consider an arbitrary diagram of T_{n-1} corresponding to renormalizable interactions with all the counter-terms, insert into it a photon with momentum k and polarization ϵ in all possible ways and replace ϵ by k. Then we get the equality (2.3) for this diagram of T_{n-1} and for the corresponding class of diagrams of T_n. Summing over all the diagrams of T_{n-1} we get this equality for the renormalized matrix elements (2.1). Note that the equality (2.3) is valid for any charged particle interacting in a renormalizable way.

Next, following Low[11], we consider the class of diagrams $T_n^{(1)}$ of T_n in which the photon with momentum k and the incoming meson can be separated from the remaining part of the diagram if we cut one meson line (Fig.2.). The contribution of this class is

$$T_n^{(1)} = \Lambda_{n-1}(r+k)(2rk)^{-1} \epsilon I(r+k, r) . \qquad (2.5)$$

Let us consider arbitrary directions of ϵ, in particular such for which $\epsilon k \neq 0$. The vertex function I has the following general structure

$$I(r+k, r) = (2r+k) f + k g \qquad (2.6)$$

where f and g are invariant functions of $(r+k)^2$. From the equality (2.3) (for $n = 1$) it follows that

$$(2rk)^{-1} k I(r+k, r) = e \langle 0 | \Phi | r \rangle = e Z , \qquad (2.7)$$

where Z corresponds to the external meson line with all corrections. For the usual renormalization procedure these corrections vanish, and $Z = 1$. From eq. (2.7) we get

$$f = e Z . \qquad (2.8)$$

It is sufficient to take the function g into account only at $k = 0$, i.e. for the case of real meson external lines of the vertex I. But in this case $g = 0$. Thus, within the accuracy we are interested in

$$T_n^{(1)} = \Lambda_{n-1}(r+k)\frac{(2r+k)\epsilon}{2rk} e Z \,. \qquad (2.9)$$

In $\Lambda_{n-1}(r+k)$ it is sufficient to take into account two terms in the expansion in k

$$Z\Lambda_{n-1}(r+k) = (1+\sum_c ck\frac{\partial}{\partial rc})T_{n-1} + 2rk\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z \,, \qquad (2.10)$$

where $c = k_i$, ϵ_i, $i = 1, 2, \ldots n-1$.

Let us consider the contribution of the remaining diagrams $T_n^{(2)}$. It is sufficient to take it into account for $k = 0$. To find this contribution we use eq. (2.3). Inserting the expansion of $T_n^{(1)}$ (2.9, 10) and $T_n^{(2)}(k=0)$ into eq. (2.3) we get

$$e[(1+\sum_c ck\frac{\partial}{\partial rc})T_{n-1} + 2rk\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z] + T_n^{(2)}(k=0) = e\,T_{n-1} \,, \qquad (2.11)$$

from which it follows that

$$T_n^{(2)} = -e[\sum_c c\epsilon\frac{\partial}{\partial rc}T_{n-1} + 2r\epsilon\frac{\partial}{\partial r^2}\Lambda_{n-1}(r)Z] \,. \qquad (2.12)$$

Adding eqs. (2.9, 10) and (2.12) and remembering the order of the next term in the expansion we get the desirable expansion of T_n in k

$$T_n = (A + \sum_c B_c)T_{n-1} + O(k \ln k) \,, \qquad (2.13)$$

where

$$A = e\frac{(2r+k)\epsilon}{2rk} \,; \quad B_c = e\left(\frac{r\epsilon}{rk}ck - c\epsilon\right)\frac{\partial}{\partial rc} \qquad (2.14)$$

Notice that in this proof it was implied that a mass λ was introduced into the photon propagator. Otherwise the derivative $(\partial/\partial r^2)\Lambda_{n-1}(r)$ would not exist. However, this derivative does not enter the final result which remains valid for $\lambda = 0$. The next term in the expansion denoted in eq. (2.13) by O, for $\lambda \neq 0$ would be of the order k. For $\lambda = 0$ it is of the order $k \ln k$.

When deriving eq. (2.13) it was assumed that $k = k_n$ was much less than all the remaining momenta. Let now k_{n-1} be much smaller than all the other momenta but k_n and as before $k_n \ll k_{n-1}$. Then we can expand T_{n-1} in eq. (2.13) in k_{n-1}. We take into account two terms of this expansion in the terms of the order k_n^{-1} and only one in the terms of the order k_n^o. Then we get an expansion in k_n and k_{n-1} all terms of which are symmetrical with respect to their permutation, except for the term

$$B_{n,n-1} = (B_{k_{n-1}} + B_{\epsilon_{n-1}})A \,. \qquad (2.15)$$

However, this term is easily symmetrized/12/. Using the condition $k_n \ll k_{n-1}$ we replace the factor rk_{n-1} in the denominator of eq. (2.15) by $r(k_n + k_{n-1})$. Then we get the symmetrical expression

$$B_{ij} = \frac{e^2}{r(k_i+k_j)}\left(k_i\frac{r\epsilon_i}{rk_i} - \epsilon_i\right)\left(\epsilon_j - k_j\frac{r\epsilon_j}{rk_j}\right) \,, \qquad (2.16)$$

and the expansion in k_n, k_{n-1} becomes valid for any k_n, k_{n-1} much smaller than all the remaining momenta. Continuing the expansion in the remaining photon momenta we finally get

$$T_n = (A)_n[1 + \tfrac{1}{2}\sum_{i,j=1}^{n}(B_{ij}/A_i A_j) + \sum_{i=1}^{n} O(k_i^2 \ln k_i)]Z \qquad (2.17)$$

Now we can find the infrared asymptotics of the meson Green function $G(p^2)$ employing its spectral representation/1,2/ which we write in the form/10/

$$G(p^2) = \int_{m^2}^{a}\frac{g(r^2)\,dr^2}{p^2 - r^2 - i0} + v(p^2) \,, \qquad (2.18)$$

where a is arbitrary close to m^2 and the function $v(p^2)$ is continuous in the neighbourhood of m^2. The spectral function g is a tempered distribution belonging to the class S^*. It is given by the sum

$$g(p^2) = (2\pi)^3\sum_N \delta(p - p_N)\langle 0|\Phi|N\rangle\langle N|\Phi^+|0\rangle \,. \qquad (2.19)$$

and for p^2 sufficiently close to m^2 reduces to the sum over the states $|N\rangle = |r, k_1, \ldots k_n\rangle$ containing a meson and an arbitrary number of soft photons. In this case

$$(2\pi)^3\sum_N \delta(p - p_N) = (2\pi)^3\int \tilde{d}\vec{r}\sum_{n=0}^{\infty}\frac{1}{n!}(\Sigma\eta\int\tilde{d}\vec{k})_n\,\delta(p - r - \sum_{i=1}^{n}k_i) = \Sigma(\Sigma\eta)_n \qquad (2.20)$$

where $\Sigma\eta$ means the summation over all four polarizations of a photon[x])
(Feynman gauge) and

$$\Sigma = (2\pi)^3\int\tilde{d}\vec{r}\sum_n\frac{1}{n!}(\int\tilde{d}\vec{k})_n\,\delta(p - r - \sum_i k_i) = \frac{1}{2\pi}\int d^4x\int\tilde{d}\vec{r}\,e^{-i(p-r)x}\sum_n\frac{1}{n!}(\int\tilde{d}k\,e^{ikx})_n \quad (2.21)$$

In the last equality we have replaced the delta function of four-momentum conservation by its Fourier integral in order to factorize the contribution of each photon and to sum over n.

To avoid divergences in (2.19) in integrating over small momenta of intermediate photons we ascribe to them a fictitious small mass λ, i.e. we put $k^o = (k^2 + \lambda^2)^{1/2}$. We shall tend λ to zero earlier than p^2 to m^2.

Inserting eq. (2.17) into eq. (2.19) we have

$$g(p^2) = Z^2\Sigma(\Sigma\eta)_n(A^2)_n[1 + \sum_{i,j}(B_{ij}/A_i A_j) + \sum_i O(k_i^2 \ln k_i)] \,. \qquad (2.22)$$

Noting that

$$(A^2)_n = (a^2)_n[1 + \sum_i(d_i/a_i) + \sum_{i,j}O(k_i k_j)] \,, \qquad (2.23)$$

where

$$a = e\frac{r\epsilon}{rk} \,; \quad d = e\frac{k\epsilon}{rk} \,, \qquad (2.24)$$

we rewrite eq. (2.22) in the form

x) $\eta = -1$ for time and 1 for space polarizations, $\Sigma\eta\epsilon_m\epsilon_n = -g_{mn}$

$$g = Z^2 (g_1 + g_2); \qquad (2.25)$$

$$g_1 = \sum (\Sigma \eta)_n (a^2)_n \left[1 + \sum_{i,j} (B_{ij} / a_i a_j) + \sum_i O(k_i^2 \ln k_i)\right]; \qquad (2.26)$$

$$g_2 = \sum (\Sigma \eta)_n (a^2)_n \sum_i (d_i / a_i). \qquad (2.27)$$

Let us consider the function g_1. Summing over polarizations and changing the notations of the photon momenta we represent eq. (2.26) in the form

$$g_1 = \sum \left[(h)_n + (h)_{n-2}\, n(n-1) H_{n,n-1} + (h)_{n-1}\, n\, O(\ln k_n) \right], \qquad (2.28)$$

where

$$h = -\left(\frac{em}{rk}\right)^2; \qquad H_{ij} = \frac{e^4 m^2}{rk_i\; r(k_i + k_j)\; rk_j} \left(\frac{m^2 k_i k_j}{rk_i \; rk_j} - 1 \right). \qquad (2.29)$$

Inserting expression (2.21) into eq. (2.28) and summing over n we get

$$g_1 = \frac{1}{2\pi} \int d^4 x \int d\vec{\tilde r}\, e^{-i(p-r)x + F} \left[1 + \iint d\vec{\tilde k}_1\, d\vec{\tilde k}_2\, e^{i(k_1+k_2)x} H_{12} + \int d\vec{\tilde k}\, O(\ln k)\, e^{ikx} \right], \qquad (2.30)$$

where

$$F = \int d\vec{\tilde k}\, h e^{ikx} = \int^{y} d\vec{\tilde k}\, h + \int^{y} d\vec{\tilde k}\, h\, (e^{ikx} - 1) + \int_{y} d\vec{\tilde k}\, h\, e^{ikx}. \qquad (2.31)$$

Here the integral with upper (lower) limit y means the integral over the region $pk \leq y\sqrt{p^2}$ $(pk > y\sqrt{p^2})$ and

$$y = \sqrt{p^2} - m \qquad (2.32)$$

At $\lambda \to 0$

$$\int^{y} d\vec{\tilde k}\, h = \gamma \ln \frac{2y}{\lambda e} + B; \qquad B = \gamma \left[1 - \int_0^1 dz \left[1 - z^2 \left(1 - \left(\frac{mp}{pr}\right)^2 \right) \right]^{-1} \right] \qquad (2.33)$$

(γ is given by the formula (1.4)). In the remaining terms in eqs. (2.31, 30) we may put $\lambda = 0$. We denote

$$F = \gamma \ln \frac{2}{\lambda e} + B + D, \qquad (2.34)$$

$$D = \gamma \ln y + \int^{y} d\vec{\tilde k}\, h\, (e^{ikx} - 1) + \int_{y} d\vec{\tilde k}\, h\, e^{ikx}. \qquad (2.35)$$

It is convenient to make further calculations in the coordinate system where $\vec p = 0$. From the conservation of four-momentum in eq. (2.21) it follows that in this system, in fact, $\vec r^{\,2} < p^2 - m^2$. Therefore all the terms in eq. (2.30) (except for $\exp(-i\vec r \vec x)$) can be expanded in $\vec r$. After this the integartion over $\vec r$ leads to $\delta(\vec x)$ and its derivatives. Integrating over $\vec x$ we can easily estimate each term

of this expansion by replacing the variables $x^0 \to x^0/y$; $k \to ky$ [x)]. It is not difficult to see that expressions H_{12} (2.29) and B (2.33) give no contribution we are interested in. We have

$$g_1 = \frac{1}{2m} \left(\frac{2}{\lambda e}\right)^{\gamma} \frac{1}{2\pi} \int d^4 x\, e^{-iyx^0 + D} \left[1 - \frac{i x^0}{2m} \vec\nabla^2 \delta(\vec x) + i \vec\nabla \delta(\vec x) \frac{\partial D}{\partial \vec r} + \right. \qquad (2.36)$$

$$\left. + \delta(\vec x) \int d\vec{\tilde k}\, O(\ln k)\, e^{ikx} \right],$$

where all the quantities are taken at $\vec r = 0$. Integrating over $\vec x$ we get

$$g_1 = \frac{1}{2m} \left(\frac{2}{\lambda e}\right)^{\gamma} \left[R_1 + \frac{\gamma}{m} \left(\tfrac{1}{2} R_2 - 2 R_3 \right) + O(y^{1+\gamma} \ln y) \right], \qquad (2.37)$$

where

$$R_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, \exp\{ -ixy + \gamma S \}\, f_i(x); \qquad (2.38)$$

$$S = \ln y + \int_0^y \frac{dk}{k} (e^{ikx} - 1) + \int_y^{\infty} \frac{dk}{k} e^{ikx}; \qquad (2.39)$$

$$f_1(x) = 1; \quad f_2(x) = ix \int_0^{\infty} dk\, k\, e^{ikx}; \quad f_3(x) = \int_0^{\infty} dk\, e^{ikx}. \qquad (2.40)$$

It not difficult to show that

$$S = -C + i\frac{\pi}{2} - \ln(x + i0), \qquad (2.41)$$

where C is the Euler constant [14] and

$$-f_2(x) = f_3(x) = i(x + i0)^{-1}. \qquad (2.42)$$

We have therefore the integrals R_i which as it is shown by Gelfand and Shilov [15] unambiguously determine in the class S^* the Riesz distributions

$$R_1 = e^{-C\gamma + i\gamma\frac{\pi}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{-ixy} (x + i0)^{-\gamma} = e^{-C\gamma} \frac{y_+^{\gamma-1}}{\Gamma(\gamma)}; \qquad (2.43)$$

$$-R_2 = R_3 = e^{-C\gamma} \frac{y_+^{\gamma}}{\Gamma(1+\gamma)}. \qquad (2.44)$$

In the last expression the subscript $+$ may be omitted since it is integrable near $y = 0$ in the usual sense for γ given by eq. (1.4). Thus

$$g_1 = \frac{1}{2m} \left(\frac{2}{\lambda e}\right)^{\gamma} e^{-C\gamma} \frac{y_+^{\gamma-1}}{\Gamma(\gamma)} \left[1 - \frac{5}{2} \frac{y}{m} + O(y^2 \ln y) \right]. \qquad (2.45)$$

---

x) Note that the integral over x^0 in eq. (2.30) determines a function of y which is generally speaking a distribution near $y = 0$. Therefore this replacement is possible only for $y \neq 0$. The integrals after this replacement are defined as the values of this distribution at the regular point $y = 1$. This concerns the terms written down in eq. (2.36). The other terms are continuous at $y = 0$.

Introducing

$$x = p^2 m^{-2} - 1 \tag{2.46}$$

and noting that $y = \frac{1}{2} m x (1 - \frac{1}{4} x) + O(x^3)$ we finally get

$$g_1 = \frac{1}{m^2} \left(\frac{m}{\lambda e}\right)^{\gamma} e^{-C\gamma} \frac{x_+^{\gamma-1}}{\Gamma(\gamma)} [1 - x - \frac{\gamma}{4} x + O(x^2 \ln x)] . \tag{2.47}$$

Let us consider the function g_2 (2.27). Summing over polarizations we have

$$g_2 = \Sigma (h)_{n-1}\, n(-e^2/rk_n) . \tag{2.48}$$

Further calculations are performed in a similar way. We finally get

$$g_2 = \frac{1}{m^2} \left(\frac{m x}{\lambda e}\right)^{\gamma} e^{-C\gamma} \frac{1}{\Gamma(\gamma)} [1 + O(x)] . \tag{2.49}$$

Thus the spectral function of the meson Green function (2.25) is

$$g(p^2) = \frac{Z^2}{m^2} \left(\frac{m}{\lambda e}\right)^{\gamma} e^{-C\gamma} \frac{x_+^{\gamma-1}}{\Gamma(\gamma)} [1 - (\tfrac{1}{2} + \tfrac{1}{4}\gamma) x + O(x^2 \ln x)] . \tag{2.50}$$

Inserting it into eq. (2.18) and integrating as the Riecz distribution for $p^2 \neq m^2$ /15, 12/ we get the following asymptotic expansion of the meson Green function

$$G(p^2) = Z_1 \frac{1}{m^2} (-x)^{\gamma-1} [1 - (\tfrac{1}{2} + \tfrac{1}{4}\gamma) x] + \text{const} \tag{2.51}$$

$$Z_1 = Z^2 (m/\lambda e)^{\gamma} e^{-C\gamma} \Gamma(1-\gamma) . \tag{2.52}$$

Note that at the point $x = 0$ this function should be considered as the Gelfand-Shilov distribution $(-x + i0)^{\gamma-1}$ /15/. In the lowest order perturbation theory constant in (2.51) is equal to 1/2.

3. Particle of Spin One Half

In just the same way we shall find now the infrared asymptotics of the Green function of a charged particle of spin 1/2 (proton). We consider the matrix element (2.1) where Φ now stands for the proton field and r and m are the momentum and the mass of the proton.

We write it in the form

$$T_n = \mathcal{J}_n Z u , \tag{3.1}$$

where u is the proton spinor and the constant Z as earlier corresponds to the external proton line with all corrections.

Eq. (2.3) is valid in this case too. The contribution of the diagrams of Fig.2 now [x)] is

$$\mathcal{J}_n^{(1)} = \Lambda_{n-1} \frac{\hat{r} + \hat{k} + m}{2rk} \Gamma(k) \tag{3.2}$$

---

x) $\hat{a} = \gamma a$; $\{\gamma^m, \gamma^n\} = 2 g^{mn}$.

As it is shown in refs./11,16/ up to terms in eq. (3.1) independent of k the vertex function $\Gamma(k)$ is equal to

$$\Gamma(k) = e\hat{\epsilon} + \frac{\mu'}{2} [\hat{k}, \hat{\epsilon}] , \tag{3.3}$$

where μ' is the anomalous magnetic moment of the proton. Instead of the expression (3.2) we consider

$$\mathcal{J}_n^{(1)} = \Lambda_{n-1}(P) \frac{\hat{P} + M}{2rk} \Gamma(k) \tag{3.4}$$

where

$$P = r + k ; \quad M^2 = P^2 , \quad M = m + rk/m + O(k^2) \tag{3.5}$$

The difference between eqs. (3.2) and (3.4) does not contain k^{-1} and we include it into $\mathcal{J}_n^{(2)}$. Since $\hat{P}(\hat{P} + M) = M(\hat{P} + M)$ then $\Lambda_{n-1}(P)$ in eq. (3.4) has the same matrix structure as $\mathcal{J}_{n-1}$ and does not contain $\hat{P}$. Therefore the expansion of $\Lambda_{n-1}(P)$ in k is given by eq. (2.10) with the replacement T by $\mathcal{J}$ and Z by 1). Inserting now $T_n^{(1)}$ and $T_n^{(2)}(k=0)$ into eq.(2.3) we have

$$eZ\, [(1 + \sum_c ck \frac{\partial}{\partial r_c}) \mathcal{J}_{n-1} + 2rk \frac{\partial}{\partial r^2} \Lambda_{n-1}(r)] \frac{\hat{P} + M}{2rk} \hat{k} u + T_n^{(2)}(\epsilon \to k) = e\, T_{n-1} \tag{3.6}$$

Noting that

$$(2rk)^{-1} (\hat{P} + M) \hat{k} u = (1 + \hat{k}/2m) u \tag{3.7}$$

we get

$$T_n^{(2)} = -eZ \left(\mathcal{J}_{n-1} \frac{\hat{\epsilon}}{2m} + 2r\epsilon \frac{\partial}{\partial r^2} \Lambda_{n-1}(r) + \sum_c c\epsilon \frac{\partial}{\partial r_c} \mathcal{J}_{n-1}\right) u . \tag{3.8}$$

Adding this expressions to $T_n^{(1)}$ we obtain the following expansion for T_n

$$T_n = (a + \sum_c B_c) T_{n-1} + Z \mathcal{J}_{n-1} \delta u + O(k \ln k) , \tag{3.9}$$

where a and B_c are given by the formulas (2.24, 14) and

$$\delta = \frac{1}{2rk} \{ (\hat{r} + m) \frac{\mu'}{2} [\hat{k}, \hat{\epsilon}] + e \hat{k} \hat{\epsilon} \} . \tag{3.10}$$

Continuing the expansion in the remaining photon momenta we get

$$T_n = (a)_n [(1 + \sum_{i=1}^{n} \frac{\delta_i}{a_i} + \tfrac{1}{2} \sum_{i,j=1}^{n} \frac{B_{ij}}{a_i a_j}) Z u + \sum_{i=1}^{n} O(k_i^2 \ln k_i)] . \tag{3.11}$$

We consider now the proton Green function

$$G(p) = \hat{p}\, G_1(p^2) + m\, G_2(p^2) . \tag{3.12}$$

The spectral representation for $G_1(p^2)$ has the form of eq.(2.18)/1,2/

$$G_1(p^2) = \int_{m^2}^{a} \frac{s_1(r^2)\, dr^2}{p^2 - r^2 - i0} + v_1(p^2) , \tag{3.14}$$

where

$$\hat{p}\, s_1(p^2) + m\, s_2(p^2) = s(p) = (2\pi)^3 \sum_N \delta(p - p_N) \langle 0|\Phi|N\rangle \langle N|\bar{\Phi}|0\rangle . \tag{3.14}$$

Inserting the expansion (3.11) into eq. (3.14) and summing over the spin states of the intermediate proton with the aid of the formula $\Sigma u \bar{u}' = \hat{r} + m$ we obtain

$$s(p) = Z^2 \Sigma (\Sigma \eta)_n (a^2)_n \{ [1 + \sum_{i,j} \frac{B_{ij}}{a_i a_j} + \sum_i O(k_i^2 \ln k_i)](\hat{r} + m) + \sum_i [\delta_i (\hat{r} + m) + (\hat{r} + m) \gamma^0 \delta_i^+ \gamma^0] / a_i \} . \qquad (3.15)$$

The terms with μ' in this expression are cancelled and we have

$$s(p) = Z^2 \{ (\hat{p} + m) g_1 + \Sigma (\Sigma \eta)_n (a^2)_n \sum_i [-\hat{k}_i + (\hat{k}_i \hat{\epsilon}_i (\hat{r} + m) + (\hat{r} + m) \hat{\epsilon}_i \hat{k}_i) / 2 r \epsilon_i] \} \qquad (3.16)$$

where g_1 is given by the formula (2.26). Summing over the photon polarizations we get

$$s(p) = Z^2 [(\hat{p} + m) g_1 + m g_2] , \qquad (3.17)$$

where g_2 is given by the formula (2.48). Thus, the considered case reduces to the previous one:

$$s_1 = Z^2 g_1 ; \qquad s_2 = Z^2 (g_1 + g_2) . \qquad (3.18)$$

From (2.47, 49) we get the spectral functions of the proton Green function

$$s_1 (p^2) = \frac{Z^2}{m^2} \left(\frac{m}{\lambda e} \right)^{\gamma} e^{-C\gamma} \frac{x^{\gamma-1}}{\Gamma(\gamma)} [1 + L_1 x + O(x^2 \ln x)] ; \qquad (3.19)$$

$$L_1 = -1 - \tfrac{1}{4} \gamma ; \qquad L_2 = -\tfrac{1}{2} - \tfrac{1}{4} \gamma . \qquad (3.20)$$

The infrared asymptotics of this function is of the form

$$G_1 (p^2) = Z_1 \frac{1}{m^2} (-x)^{\gamma-1} (1 + L_1 x) + \text{const} \qquad (3.21)$$

where Z_1 is given by the formula (2.52). Instead of eqs. 3.12, 21) we can also write

$$G(p) = Z_1 \left(\frac{1}{m - \hat{p}} + \frac{2\hat{p} + m}{2 m^2} + \gamma \frac{\hat{p} + m}{4 m^2} \right) \left(\frac{m^2 - p^2}{m^2} \right)^{\gamma} + \text{const} . \qquad (3.22)$$

The first two terms in this formula coincide with the result obtained by Milekhin by functional integration /8/.

In conclusion we note that using method developed here we can find the infrared asymptotics of the vertex functions and the scattering matrix elements in all orders in the coupling constants /10,12/. In particular, it turns out that the elastic scattering of charged particles at small angles is described (if we neglect a phase factor) by the simplest one-photon-exchange diagram in all orders in e . This means that the scattering of charged particles at small angles obeys the Coulomb law at arbitrary high energies.

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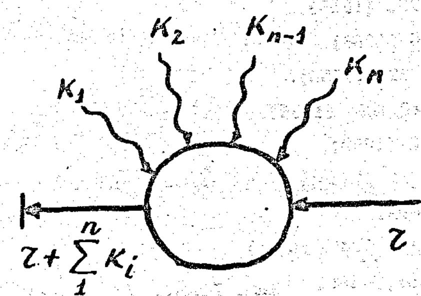

Fig. 1.

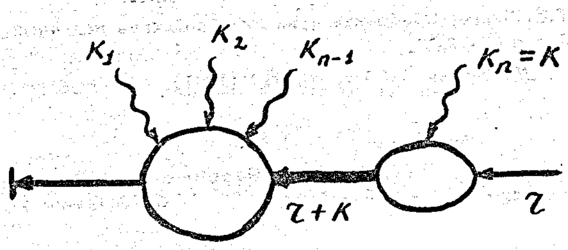

Fig. 2.