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# SPONTANEOUS BREAKDOWN OF SYMMETRIES IN QUANTUM FIELD THEORY

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# SPONTANEOUS BREAKDOWN OF SYMMETRIES IN QUANTUM FIELD THEORY

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## I. Introduction

The spontaneous breakdown of symmetries of strong interactions has received some attention in recent time. Guided by the analogy with the theory of superconductivity and superfluidity, one assumes that the Lagrangean of strong interactions is exactly symmetric under some group (e.g. SU (3) ) however, the physical solution does not share the complete symmetry of the Lagrangean (and commutation relations). In this way the mass differences of hadrons could be explained - one hopes- without introducing explicit symmetry breaking terms into the Lagrangean (See e.g.[1]). Another interesting proposition of the theory of spontaneous symmetry breaking is the Goldstone theorem[2] which - in analogy with the Hughenholz-Pines-theorem[3] in the many-body-problem- predicts a singularity of the Green function at $q^2 = 0$ (The latter is usually interpreted as a zero-mass particle arising in consequence of the breakdown of the original symmetry).

In a local relativistic quantum field theory, however, the situation is much more complicated because of ultraviolet divergences. All the existing proofs of the Goldstone theorem (e.g.[2,4]) implicitly assume that the quantities involved are finite - which is certainly not true without renormalization or the introduction of an artificial cut-off[x]. The same criticism applies to the investigations of the breakdown of SU (3).

The aim of the present note is a formal investigation of the questions raised above, without assuming any specific dynamical model. We define certain functionals, which are essentially the generators of the Green functions and irreducible vertices respectively. By expanding the symmetry breaking solution around the symmetric ("normal") one, we show that the irreducible vertices are renormalization invariant, provided the normal solution is (Sect. 2). The next sections (Sect. 3-5) are devoted to the study of the orthogonal symmetry group and SU (3) , respectively. Assuming the renormalizability of the theory in the sense defined above, we give a simple proof of the Goldstone theorem[xx].

[x] Prof. N. Bogolubov even raised the following question: is it not possible that what we claim as a singularity of the Green function at $q^2 = 0$ , (when the external sources tend to zero) comes from the infinity of the wave function renormalization constant ? Thus the Green function possibly is not singular at $q^2 = 0$ but identically infinite.

[xx] It appears that recently no paper can be written about degenerate vacuum without giving an alternative proof of the Goldstone theorem. We just follow this habit.

What concerns SU(3) the usual self-consistency condition gives that the preferred direction in unitary spin space can be the third axis only. The renormalization procedure developed in sect. 2 is illustrated on the example of a $\lambda\phi^4$ interaction. The generator of the irreducible vertices is calculated approximately by means of a generalized diagram technique. (The same technique can be easily generalized to any other field theory). The approximate expressions we find satisfy Goldstone's theorem (sect. 6). We conclude the paper by discussing several problems of the anomalous solutions in quantum field theory (sect. 7).

## 2. Definition of the Generating Functionals and Renormalization

We define

$$W = e^{iZ} = \langle T \exp i \int j_i(x)\, \Phi_i(x)\, dx \rangle_0 \tag{2.1}$$

the generator of time ordered products for a Heisenberg field $\Phi_i(x)$ $(i = 1, 2, \ldots, n)$ with external source $j_i(x)$. The index $i$ stands for some internal symmetry index to be specified in each special case. Then, of course, $Z$ is the generator of the connected parts of the Green's functions. A further functional which will prove useful, is obtained by means of a contact transformation from $Z$:

$$F = Z - \int dx\, j_i(x)\, \phi_i(x), \tag{2.2}$$

considered as the functional of

$$\phi_i(x) \equiv \langle T\, \Phi_i(x) \exp i \int j_k(y)\, \Phi_k(y)\, dy \rangle_0 = \frac{\delta Z}{\delta j_i(x)} .$$

We can proceed further by defining:

$$W' = e^{iZ'} = \langle T \exp i \Big[ \int j_i(x)\, \Phi_i(x)\, dx + \int h_{ik}(x,y)\, (\Phi_i(x) - \phi_i(x))(\Phi_k(y) - \phi_k(y))\, dx\, dy \Big] \rangle_0 \tag{2.3}$$

and

$$S = Z' - \int j_i(x)\, \phi_i(x)\, dx - \int h_{ik}(x,y)\, G_{ik}(x,y)\, dx\, dy , \tag{2.4}$$



considered as a functional of $\phi_i$ and $G_{ik}$, where $G_{ik}$ is the connected part of the one-particle Green function, and $h_{ik}$ is some other external source (the functionals we just defined, are analoga of well-known thermodynamic functions, e.g. $F$ the free energy, $S$ the entropy etc.[5]. Their usefulness in field theory has been pointed out by Jona-Lasinio[4]).

For the sake of completeness we list some relations (they are partly given in[4]):

$$\begin{aligned}
\frac{\delta Z}{\delta j_i(x)} &= \phi_i(x) \\
\frac{\delta^2 Z}{\delta j_i(x)\, \delta j_k(y)} &= G_{ik}(x,y) \\
\frac{dF}{\delta \phi_i(x)} &= -j_i(x) \\
\frac{\delta^2 F}{\delta \phi_i(x)\, \delta \phi_k(y)} &= -G^{-1}_{ik}(x,y) \\
\frac{\delta S}{\delta \phi_i(x)} &= -j_i(x) \\
\frac{\delta S}{\delta G_{ik}(x,y)} &= -h_{ik}(x,y)
\end{aligned} \tag{2.5}$$

The useful property of $F$ and $S$ is that in the physical limit ($j_i \to 0$, $h_{ik} \to 0$) they are stationary with respect to variations of $\phi_i$ and/or $G_{ik}$. In what follows this limit (unless otherwise stated) is always understood.

We shall speak of a spontaneous breakdown of the symmetry of the first and second kind, if in the limit $j_i \to 0$ , $h_{ik} \to 0$ , $\phi_i \neq 0$ and $\phi_i = 0$

but $G_{ik}(x,y) \neq \delta_{ik} G(x,y)$

respectively.

After this preparation, we can solve the renormalization problem as follows. In the case of a breakdown of the first kind, we expand $F(\phi_i)$ around $\phi_i = 0$ (the "normal" solution) in a formal Volterra series:

$$F(\phi_i) = F(0) + \int dx\, \phi(x) \left[\frac{\delta F}{\delta \phi_i(x)}\right]_{\phi_i=0} + \frac{1}{2!} \iint dx\, dy\, \phi_i(x)\, \phi_k(y) \left[\frac{\delta^2 F}{\delta\phi_i(x)\, \delta\phi_k(y)}\right]_{\phi_i=0} + \dots \tag{2.6}$$

(summation over dummy indices understood).

The coefficients of $\phi_i(x)$, $\phi_i(x)\ \phi_k(y)$, ... are the irreducible n- point functions ( $\rho$ - functions) of the normal solution, which by assumption are renormalizable.

The stationarity condition or self-consistency condition for $F$ (serving to determine $\phi_i(x)$ ) then can be written as:

$$\frac{\delta F}{\delta \phi_i(x)} = \rho_i^{(1)}(x) + \int dy\, \phi_k(y)\, \rho_{ik}^{(2)}(x,y) + \dots = 0,$$

where

$$\rho^{(n)}_{ik\dots\ell}(x,y,\dots,z) = \left[\frac{\delta^n F}{\delta\phi_i(x)\, \delta\phi_k(y)\dots\delta\phi_\ell(z)}\right]_{\phi_i=0}.$$

If we perform a renormalization transformation transformation for the normal solution, $\rho^{(n)}$ will be multiplied by a factor $Z_2^{-\frac{1}{2}n}$ where $Z_2^{\frac{1}{2}}$ is the wave function renormalization constant.

Writing then $\rho^{(n)} = Z^{-\frac{1}{2}n} \rho_R^{(n)}$ $\phi = Z_2^{\frac{1}{2}} \phi_R$ ( $\phi_R$, $\rho_R$ are the renormalized



quantities), we obtain the self-consistent equations between renormalized quantities. By the same taken, if the infinities of the $\rho$ function can be removed by renormalization, we obtain an equation for $\phi$ in terms of finite quantities only.

Similarly, the expression for $G_{ik}^{-1}$ etc. will contain finite (renormalized) quantities even if we choose the symmetry breaking solution ($\phi_i \neq 0$).

The case of a breakdown of the second kind can be treated exactly in the same way: $S$ should be expanded in powers of of $G_{ik}(x,y) - \delta_{ik} G^{(0)}(x,y)$ , where $\delta_{ik} G^{(0)}(x,y)$ is the complete Green's function corresponding to the normal solution. In what follows we shall work exclusively with these renormalized functionals, so the suffix $R$ will be omitted.

## 3. Orthogonal Group, Goldstone Theorem

We consider a self-interacting spinless field of $n$ components. $F$ is a functional of the only orthogonal invariant, we can form of $\phi_i(x)$

$$f(x,y) = \phi_i(x)\, \phi_i(y). \tag{3.1}$$

Assuming a symmetry breaking of the first kind, $\phi_i$ can be determined from

$$\frac{\delta F}{\delta \phi_i(x)} = 2\int \frac{\delta F}{\delta f(x,y)}\, \phi_i(y)\, dy = 0. \tag{3.2}$$

Because of translation invariance $\phi_i = \text{const}$ , so either

$$\phi_i = 0 \quad \text{(normal solution)} \tag{3.3a}$$

or

$$\int \frac{\delta F}{\delta f(x,y)}\, dy = 0 \quad \text{(symmetry breaking solution)}. \tag{3.3b}$$

The inverse Green's function is:

$$G_{ik}^{-1}(x,y) = \frac{\delta^2 F}{\delta\phi_i(x)\, \delta\phi_k(y)} =$$

$$= 2\left(\frac{\delta F}{\delta f(x,y)}\,\delta_{ik} + 2\phi_i\,\phi_k \int \frac{\delta^2 F}{\delta f(x,u)\,\delta f(y,v)}\,du\,dv\right).$$

Defining the transversal and longitudinal parts of $G^{-1}_{ik}$ by writing:

$$G^{-1}_{ik}(x-y) = \frac{\phi_i\,\phi_k}{\phi^2}\,G^{-1}_L(x,y) + \left(\delta_{ik} - \frac{\phi_i\,\phi_k}{\phi^2}\right) G^{-1}_T(x-y) \tag{3.4}$$

$$(\phi^2 = \phi_i\,\phi_i).$$

We find:

$$G^{-1}_L(x-y) = 2\left[\frac{\delta F}{\delta f(x,y)} + 2\phi^2 \int \frac{\delta^2 F}{\delta f(x,u)\,\delta f(y,v)}\,du\,dv\right], \tag{3.5a}$$

$$G^{-1}_T(x-y) = 2(n-1)\,\frac{\delta F}{\delta f(x,y)}\,. \tag{3.5b}$$

By observing that $\delta F/\delta f(x,y)$ depends on $x-y$ only and hence the left-hand side of (3.3b) is just the Fourier transform of $\delta F/\delta f$ at $p=0$, we obtain $G^{-1}_T(p)|_{p=0} = 0$, which is Goldstone's theorem. (The reader will immediately recognise that our proof of the Goldstone theorem is very similar to that given in ref. /4/ ).

## 4. Unitary Symmetry Breakdown of the First Kind

Consider an octet of spinless particles (e.g. the octet of pseudoscalar mesons) interacting with a hypothetical octet of scalar particles. Assume that the vacuum expectation value of the scalar and pseudoscalar fields; $S_i(x) \to S_i \neq 0$, $P_i(x) \to 0$ in the physical limit; $F$ is a functional of the following basic unitary invariants

$$f_1(x,y) = S_i(x)\,S_i(y)\,,$$



$$\begin{aligned}
f_2(x,y,z) &= D_{ik\ell}\,S_i(x)\,S_k(y)\,S_\ell(z),\\
f_3(x,y) &= P_i(x)\,P_i(y),\\
f_4(x,y,z) &= D_{ik\ell}\,P_i(x)\,P_k(y)\,P_\ell(z),\\
f_5(x,y) &= S_i(x)\,P_i(y),\\
f_6(x,y,z) &= D_{ik\ell}\,S_i(x)\,S_k(y)\,P_\ell(z),\\
f_7(x,y,z) &= D_{ik\ell}\,P_i(x)\,P_k(y)\,S_\ell(z),\\
f_8(x,y,z,u) &= S_i(x)\,S_k(y)\,D_{ik\ell}\,D_{\ell mn}\,P_m(u)\,P_n(v).
\end{aligned} \tag{4.1}$$

Here the external sources are different from zero (otherwise $P_i = 0$ by assumption), the indices $i$, $k$, ... run from 1 to 8, $D_{ik\ell}$ is the usual symmetric matrix. The first self-consistency condition reads:

$$\frac{\delta F}{\delta S_i(x)} = 2\int \frac{\delta F}{\delta f_1(x,y)}\,S_i(y)\,dy + 3\int \frac{\delta F}{\delta f_2(x,y,z)}\,D_{ik\ell}\,S_k(y)\,S_\ell(z)\,dy\,dz = 0\,. \tag{4.2}$$

In order to write down the second self-consistency condition: $\delta F/\delta P_i(x) = 0$, one has to notice that because of parity conservation, $F$ depends on $f_4$, $f_5$, $f_6$ quadratically. Therefore if $P_i(x) \to 0$, the second self consistency condition reduces to a trivial identity $0 = 0$.

Construct again the inverse Green functions:

$$G^{-1}_{ik}(x-y) = \frac{\delta^2 F}{\delta S_i(x)\,\delta S_k(y)}\,, \qquad H^{-1}_{ik}(x-y) = \frac{\delta^2 F}{\delta P_i(x)\,\delta P_k(y)}$$

(the mixed derivatives vanish by parity conservation).

We find:

$$G^{-1}_{ik}(x-y) = 2\,\frac{\delta F}{\delta f_1(x,y)}\,\delta_{ik} + \int \frac{\delta F}{\delta f_2(x,y,z)}\,D_{ik\ell}\,S_\ell(z)\,dz +$$

$$+ 4\int \frac{\delta^2 F}{\delta f_1(x, x_1)\,\delta f_1(y, y_1)} S_i(x_1) S_k(y_1)\, dx_1\, dy_1 \; +$$

$$+ 9\int \frac{\delta^2 F}{\delta f_2(x, x_1, x_2)\,\delta f_2(y, y_1, y_2)} \times$$

$$\times D_{i\ell m}\, S_\ell(x_1) S_m(x_2) D_{knp}\, S_n(y_1)\, S_p(y_2)\, dx_1 \ldots dy_2 \; + \qquad (4.3)$$

$$+ 6\int \frac{\delta^2 F}{\delta f_1(x, x_1)\,\delta f_2(y, y_1, y_2)} S_i(x_1) D_{k\ell m} \times$$

$$\times S_\ell(y_1) S_m(y_2)\, dx_1 dy_1\, dy_2 \, .$$

$$+ 6\int \frac{\delta^2 F}{\delta f_2(x, x_1, x_2)\,\delta f_1(y, y_1)} D_{i\ell m}\, S_\ell(x_1)$$

$$\times S_m(x_2)\, S_k(y_1)\, dx_1\, dx_2\, dy_1 \, .$$

$$H^{-1}_{ik}(x-y) = 2\,\frac{\delta F}{\delta f_3(x, y)}\,\delta_{ik} \; +$$

$$+ 2\int \frac{\delta F}{\delta f_7(x, y, z)} D_{ik\ell}\, S_\ell(z)\, dz \qquad (4.4)$$

$$+ 2\int \frac{\delta^2 F}{\delta f_8(x, x_1, y, y_1)} D_{ikr}\, D_{rmn}\, S_m(x_1) S_n(y_1)\, dx_1\, dy_1 \; +$$

$$+ \int \frac{\delta^2 F}{\delta f_5(x, x_1)\,\delta f_5(y, y_1)} S_i(x_1) S_k(y_1)\, dx_1\, dy_1$$

$$+ \int \frac{\delta^2 F}{\delta f_5(x, x_1)\,\delta f_6(y, y_1, y_2)} S_i(x_1) D_{k\ell m}\, S_\ell(y_1) S_m(y_2) \times dx_1 dy_1 dy_2 \; +$$

$$+ \int \frac{\delta^2 F}{\delta f_5(y, y_1)\,\delta f_6(x, x_1, x_2)} S_k(y_1) D_{i\ell m}\, S_\ell(x_1) \cdot S_m(x_2)\, dx_1\, dx_2\, dy_1 \; +$$

$$+ \int \frac{\delta^2 F}{\delta f_6(x, x_1, x_2)\,\delta f_6(y, y_1, y_2)} D_{i\ell m}\, D_{kpq} \times$$

$$\times S_\ell(x_1) S_m(x_2) S_p(y_1) S_q(y_2)\, dx_1\, dx_2\, dy_1\, dy_2 \, .$$



So far all our equations are covariant under unitary transformations. However, if $G_{ik}$ and $H_{ik}$ are to represent physical particles, they must be diagonal matrices in unitary space. If $S_i \neq 0$, eq. (4.2) can be satisfied together with the diagonality condition (with an appropriate choice of axes), only if $S_1 = \ldots = S_7 = 0$, $S_8 = S \neq 0$.

Then (4.2) goes over to

$$\int \frac{\delta F}{\delta f_1(x, y)}\, dy = 3^{1/2} S \int \frac{\delta F}{\delta f_2(x, y, z)}\, dy\, dz$$

and (4.3) and (4.4) are simplified correspondingly.

Now in eq. (4.3) all terms but the first two differ from zero only for $i = k = 8$. Comparing with (4.2), one finds

$$G^{-1}_{44}(p=0) = G^{-1}_{55}(p=0) = G^{-1}_{66}(p=0) = G^{-1}_{77}(p=0) = 0 \, .$$

which means that the mass of the "scalar $\overset{*}{K}$-meson" is zero, so we again find a kind of Goldstone theorem.

What concerns the PS octet, the masses of the particles belonging to the same isotopic multiplet are equal. This, of course, does not necessarily mean that the mass splitting is described by a Gell-Mann-Okubo formula. If $\delta F/\delta f_5 = \delta F/\delta f_6 = 0$ which excludes representations other than 8, then the Fourier transform of $H^{-1}_{ik}$ can be written in the form:

$$H^{-1}_{ik}(p^2) = \delta_{ik}\,(p^2 + A(p^2)) + S D^8_{ik}\, B(p^2) \, .$$

The functions $A(p^2)$ and $B(p^2)$ can be written in the form of a dispersion integral, e.g.

$$A(p^2) = \frac{1}{\pi} \int_0^\infty d\kappa^2\, \frac{\alpha(\kappa^2)}{\kappa^2 - p^2} \, .$$

An approximate Gell-Mann-Okubo formula for the masses results if we assume that $1^0)$ $A(p^2)$, $B(p^2)$ are slowly varying functions of the masses involved in the intermediate states, and $2^0)$ are slowly varying functions of $p^2$ in the neighbourhood of the root of $H^{-1}_{ik}$.

## 5. Unitary Symmetry. Breakdown of the Second Kind

As an example for a spontaneous breakdown of the second kind of symmetry, we consider a unitary triplet of fields.

We proceed as before, assuming now that the vacuum expectation value of the fild vanishes.

The Green function $G_\alpha^\beta$ can be decomposed into singulet and octet parts:

$$G_\alpha^\beta = \delta_\alpha^\beta G^{(0)} + \lambda_\alpha^{i\beta} G^i \tag{5.1}$$

$$(\alpha, \beta = 1,2,3).$$

The generating functional $S$ depends on the invariants:

$$G^{(0)}(x,y),$$

$$g_1(x,y,u,v) = G^i(x,y)\, G^i(u,v),$$

$$g_2(x,y,u,v,w,z) = \tag{5.3}$$

$$= D_{ik\ell}\, G^i(x,y)\, G^k(u,v)\, G^\ell(w,z).$$

The self-consistency condition now reads:

$$\frac{\delta S}{\delta G^{(0)}(x,y)} = 0, \tag{5.3a}$$

$$\frac{\delta S}{\delta G^i(x,y)} = 2\int \frac{\delta S}{\delta g_1(x,y,u,v)} G^i(u,v)\, du\, dv$$

$$+ 3 D_{ik\ell} \int \frac{\delta S}{\delta g_2(x,y,u,v,w,z)} G^k(u,v)\, G^\ell(w,z) \times \tag{5.3b}$$

$$\times du\, dv\, dw\, dz = 0.$$

Considerations, completely analogous to those of the previous section show that only $i = 3,8$ are allowed in eq. (5.1). In order to extract further information from eq. (5.3b) we have to consider the external octet source $h_\alpha^\beta - \frac{1}{3}\delta_\alpha^\beta h_\gamma^\gamma$ switched on. (The presence of the external source



furnishes the symmetry breaking term necessary for the calculation of symmetry breaking solution, cf. ref./3/).

The physically reasonable choice for the source is a function with constant "direction" in unitary space, i.e.

$$h_\alpha^\beta(x) - \frac{1}{3}\delta_\alpha^\beta h_\gamma^\gamma(x) = (a_\alpha^\beta - \frac{1}{3}\delta_\alpha^\beta a_\gamma^\gamma)\, h(x),$$

where $a_\alpha^\beta$ is a constant matrix and $h(x)$ a scalar function. (This choice, in a certain sense, corresponds to a minimal violation of the symmetry). Then, there certainly exist a solution for $G_\alpha^\beta$ of the form:

$$G_\alpha^\beta(x) = \delta_\alpha^\beta G^{(0)}(x) + \lambda_\alpha^{i\beta} a^i G^{(8)}(x),$$

where $a^i$ is a constant vector. Then eq. (5.3b) takes on the form

$$2a^i \int \frac{\delta S}{\delta g_1} G^{(8)}\, du\, dv +$$

$$+ 3 D_{ik\ell}\, a^k a^\ell \int \frac{\delta S}{\delta g_2} G^{(8)} G^{(8)}\, du\, dv\, dw\, dz. \tag{5.4}$$

Therefore, again as in the previous section, the "vectors" $a^i$ and $D_{ik\ell}\, a^k a^\ell$ should be parallel in order to eq. (5.4) have a solution. This can be satisfied only if

$$a_1 = \ldots = a_7 = 0, \quad a_8 \neq 0.$$

We obtain again the result that the spontaneous breaking of symmetry splits off the isodublet from the isosingulet in the representation 3.

## 6. Example: Self-Coupled Scalar Field with Orthogonal Symmetry

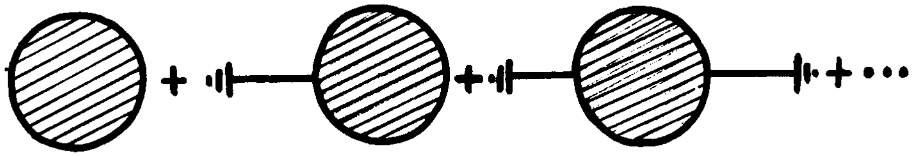
In order to illustrate, how the general procedure outlined in the previous sections works in concrete cases, we consider the well-known guinea- pig of field theory: a self-coupled scalar field with orthogonal symmetry. We write down the unrenormalized Lagrangian formally as follows:

$$L = \tfrac{1}{2}\vec{\Phi}_{,\mu}\vec{\Phi}_{,\mu} - \tfrac{1}{2}\mu_0^2\vec{\Phi}\vec{\Phi} - \tfrac{1}{4}\lambda(\vec{\Phi}\vec{\Phi})^2 - \vec{j}\vec{\Phi}\ , \qquad (6.1)$$

with $\lambda > 0$ , $\vec{j}$ is the external source, $\vec{\Phi}$ is a vector in the $n$ dimensional "isotopic" space.

Our aim is the approximate calculation of the generating functional $F$ and finding the possible values of $\langle \Phi_i \rangle$ .

To this end, let us observe that our formal expansion (2.6) in the limit $j_i \to 0$ admits a simple graphical interpretation. A glance at eq. (2.6) shows namely that $F$ is the sum of the following diagrams (fig. 1).



F i g. 1. Diagrams contributing to $F$ .

where the "black boxes" mean irreducible $\rho$ -functions of the normal solution (in particular the first box the sum of vacuum diagrams), a "grounded" external line stands for the vacuum expectation value of the field operator. (NB that no propagator is to be inserted between the "black box" and the "grounding").

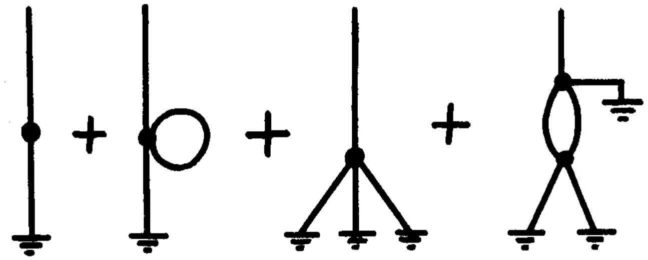
Similarly, the functional derivatives of $F$ can be obtained from the diagrams of fig. 1 by removing a corresponding number of "groundings" in all possible ways.

Therefore, if we have any approximation technique to calculate the $\rho$ - functions of the normal solution, we can calculate those of the symmetry breaking one as well. For the sake of simplicity, we chose a very primitive approximation in our model:

1 We approximate the normal $\rho$ - functions by some simple terms of their perturbation series, and

2° . We break off the Volterra series of $F$ after the first few terms. (For the Lagrangian (6.1) the $\rho$ -functions with an odd number of external lines, of course, vanish). We break off the Volterra series after the fourth order term, and choose the simplest nontrivial approximation for the self-energy and vertex parts. Then denoting $\langle \Phi_i \rangle = \phi_i$ the approximation we have chosen for $\delta F/\delta \phi_i$ is shown on fig. 2.

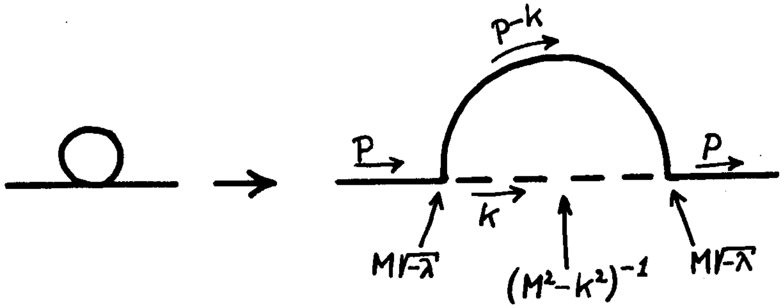




F i g. 2. Approximate expression of $\dfrac{\delta F}{\delta \chi_i}$ .

In fig. 2 the two-point vertex corresponds to the inverse free propagator: $(\mu_0^2 - k^2)\delta_{ik}$ however, because of translation invariance only the $k = 0$ component contributes. In order to calculate the contribution of the self-energy diagram (the second diagram on fig. 2) we define the four-point vertex as the limit of a nonlocal one, by inserting the propagator of a fictious particle of mass $M$ and coupling constant $iM\lambda^{1/2}$ .

The procedure is illustrated in fig. 3.




F i g. 3 . Regularization of the four-point vertex.

After having performed the calculation, we let $M^2$ tend to infinity. (This procedure is chosen just because it is convenient, the final, renormalized expression should not depend on the regularization procedure chosen).

The calculation of the contribution of the diagram 3b is straightforward. We find $(s = p^2)$

$$\Pi_{ik}(s) = (\Pi_0 + \frac{s}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{ds' \cdot \frac{1}{16}\frac{\lambda}{\pi}}{s'^2 (s' - s)} \times$$

$$\times [(s' - (M+\mu)^2)(s' - (M-\mu)^2)]^{1/2}) \delta_{ik}$$

( $\Pi_0$ is a subtraction constant), which for $M^2/\mu^2 \gg 1$ gives

$$\Pi_{ik}(s) \approx (\Pi_0 - \frac{1}{16}\frac{\lambda s}{\pi}) \delta_{ik} \quad .$$

We introduce renormalized quantities:

$$\mu^2 = (\mu_0^2 + \Pi_0)(1 + \frac{1}{16}\frac{\lambda}{\pi^2})^{-1} \quad ,$$

$$G_{R_{ik}}(s) = \frac{\delta_{ik}}{\mu^2 - s} = \frac{z_2 \delta_{ik}}{\mu_0^2 - s - \Pi(s)} \quad ,$$

$$Z_2 = (1 + \frac{1}{16}\frac{\lambda}{\pi})^{-1} \quad .$$

Leu us remark the following. We have chosen $\lambda > 0$ because in the classical theory this gives a positive definite energy. We see now that in the quantized theory at least in our approximation, $\lambda > 0$ is necessary, if we want to satisfy the condition $0 \leq Z_2 \leq 1$.

Similarly, the four-point function, apart from its tensor structure will be given by the expression:

$$\lambda + a + \frac{1}{16}\frac{\lambda s}{\pi^2}\int_{4\mu^2}^{\infty} \frac{ds'}{(s' - s)s'} \left(\frac{s' - 4\mu^2}{s'}\right)^{1/2} \quad .$$

Again, as usual, the subtraction constant $a$ can be included into a multiplicative renormalization of the coupling constant, $\lambda \to Z_1 \lambda$.

Hence, by introducing the renormalized vacuum expectation value:

$\chi_{Ri} = Z_2^{1/2} \chi_i$ and by noting that all the diagrams contribute to the equation symbolized by fig. 2. at the point $s = 0$ we obtain the equation determining $\chi_{Ri}$

$$\frac{\delta F}{\delta \chi_{Ri}} = \chi_{Ri} (\mu^2 + \lambda_R \chi_{Rk} \chi_{Rk}) = 0 \qquad (6.3)$$



with

$$\lambda_R = Z_2^{-2} Z_1 \lambda \quad .$$

Let us notice the remarkable fact that the equation we obtained in terms of the renormalized quantities is exactly the same as that obtained in the Hartree-Fock approximation /2/. Similarly, we obtain the same expressions for the propagators as well. Thus the higher order terms do not affect Goldstone's conclusion, in conforming with our general proof.

We have, of course, to admit that our exposition of the calculation procedure was rather sketchy. The reader, familiar with renormalization theory will notice immediately that we should introduce $\lambda_R$ , $\mu^2$ into the self-energy part as well, we defined $\lambda_R$ in an unconventional way by subtracting at $s = 0$ , etc. However, the same reader will immediately check that it is just the consequent renormalization procedure which leads to eq. (6.3) and the choice of the normalization point can affect but the numerical value of $\lambda_R$ .

Let us add a final remark to the calculations of this section about the Goldstone theorem. There exist several general proofs of it /2,4/ at different levels, There were, however, two "votes" against it /6,7/; both based on concrete calculations. This situation is rather discomforting as neither of the general proofs presented so far can claim at complete rigorousity. We believe that our calculation sheds some light on the problem. Both the work of Kamefuchi and Umezawa /6/ and Marx and Kuti /7/ get rid of the ultraviolet infinities by means of a cut-off, while we applied the familiar renormalization procedure. The latter does not violate nay of the principles of local quantum field theory, while a cut-off procedure necessarily violates at least one of them, thereby in violating some premises of the Goldstone theorem.

## 7. D i s c u s s i o n

Let us summarise what we have achieved.

We outlined a formalism which seems to be flexible enough to treat both "normal" and "anomalous" solutions of field theory. We have shown further that the infinities of the symmetry breaking solution cause essentially no more (but probably, no less) problem, than those of the normal one.

Nevertheless, our proof of the symmetry breaking solution was based on a formal Volterra expansion around the normal one. It is quite possible that such an expansion diverges and then our "proof", of course, does not work. At present, however, we have no other tool at hand to treat the problem.

Assuming that our steps are justified, Goldstone's results follow essentially from the stationarity condition of certain functionals and simple invariance arguments. Concerning the SU (3) symmetry, now - a - days - believed to govern strong interactions, we reach a nontrivial and beautiful conclusion, scil. that if the breakdown of SU (3) is spontaneous, then the preferred direction in unitary space is uniquely determined and astronishingly enough - the preferred direction is exactly what Nature seems to choose.

What we cannot obtain in this way is the electromagnetic mass splitting.

The breakdown of the first kind of SU (3) , discussed in sect. 4 leads to the existence of massless, strongly interacting particles. Such objects, unless they are very weakly coupled to other particles, would lead e.g. to an apparent strong violation of strangeness conservation. On the other hand, if they were very weakly coupled, it seems rather difficult to understand the large mass splitting they cause. Therefore we are inclined to believe that is broken spontaneously, it is a breakdown of the second kind. If we think in terms of a triplet model, a breakdown of the second kind together with a unitary invariant "binding force" between the triplets leads naturally to the approximate Gell-Mann-Okubo mass formula.

Convergent or not, our expansion (2.6) leads to a reasonable calculation procedure for anomalous solutions in field theory, as illustrated in sect. 6. The procedure can be easily generalized for more realistic models and symmetry breaking of the second kind as well.

Last but not least, there are difficult questions: does Nature really choose the "anomalous" solution of field equations? If so, what is the physical meaning of the quantities entering the expressions? As to the first question, no answer can be given at present. Even if we supposed that the world could be described by a field theory, and supposed we can find several solutions of the field equations - which is the "physical" one of the latter? Guided again by the analogy with the nonrelativistic many-body problem, one suspects that there must exist some sort of stability condition which selects the right solution. So far no such satisfactory condition is known to us. Concerning the second question, the analogy with the many-body problem seems to break down. If one considers an interacting Bose-gas, the vacuum expectation value of the field operator has a very simple meaning - it is the square root of the density of the condensate.

However, in a relativistic field theory there is, in fact, no way to measure $\langle \Phi \rangle$ . (It does not enter explicitly in either of Green's functions). Further, taking the example of sect. 6 we see that the renormalized mass is connected



with the physical mass of the anomalous solution, but not identical with it. (In the Hartree-Fock approximation one can even show that the anomalous solution is unstable unless $\mu^2 < 0$ ).

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