ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E - 1894

V.V. Babikov

ON A CALCULATION OF THE EFFECTIVE RANGE EXPANSION PARAMETERS FOR LOW-ENERGY NN POTENTIAL SCATTERING

ЯФ, 1965, т 1, в 5, с 793-802.

ЛАБОРАТОРИЯ ЯДЕРНЫХ РЕАКЦИЙ

1964

E - 1894

V.V. Babikov

ON A CALCULATION OF THE EFFECTIVE RANGE EXPANSION PARAMETERS FOR LOW-ENERGY NN POTENTIAL SCATTERING

Submitted to "Ядерная физика"

I. Introduction

It is well-known that the effective-range expansion parameters, especially the scattering length, are very sensitive to small variations of the nucleon-nucleon potential. This fact is due to small values of the energy of a real ($^3S_1$ state) and a virtual ($^1S_0$ state) levels in the two-nucleon system. On the other hand, the experimental two-nucleon data are most accurate for the low-energy region.

Therefore many important features of nuclear forces, e.g. the charge dependence and the softness of the repulsive core are often discussed in terms of the scattering length a_0, the effective range r_0, and the shape parameter P. The parameters are defined by the well-known formulae:

$$k \operatorname{ctg} \delta_0^{s,t} = -1/a_{s,t} + \tfrac{1}{2} r_{s,t} k^2 - P_{s,t} r_{s,t}^3 k^4 + O(k^6) \qquad (1)$$

for the singlet np and nn $^1S_0$ state and the triplet np $^3S_1$ -state,

$$2\pi\eta (e^{2\pi\eta} - 1)^{-1} k \operatorname{ctg} \delta_0 + h(\eta) R^{-1} = -1/a_p + \tfrac{1}{2} r_p k^2 - P_p r_p^3 k^4 + O(k^6) \qquad (2)$$

for the $^1S_0$ pp scattering. Here $\eta = (2kR)^{-1}$, $R = \hbar^2/2m z_1 z_2 e^2 = 28{,}8\,f$, for $m = \frac{m_N}{2}$, $z_1 = z_2 = 1$, $h(\eta) = \eta^2 \sum_{n=1}^{\infty} n^{-1} (n^2 + \eta^2)^{-1} - 0.577\ldots - \ln \eta$.

In this paper some methods of an accurate numerical calculation of a_0, r_0, P for the given nucleon-nucleon potential are briefly described.* The Coulomb potential is taken into account exactly for low-energy pp scattering. Exact equations are obtained, which include effects of nuclear tensor forces[1,2] in the case of scattering in triplet mixed states. The effect of a bound state in the $^3S_1$ -state (deuteron) is also discussed.

Several examples of calculation of a_s, r_s, P_s for the $^1S_0$ np (nn) scattering,** of a_p, r_p, P_p for the $^1S_0$ pp scattering and of a_t, r_t for $^3S_1$ np scattering are given in the paper. The NN potentials giving a good fit for high-energy scattering data are used. Two potentials including a hard core, namely those of Hamada-Johnston[4] and of Breit et al.[5] are considered. The potential of Babikov et al.[6] with a soft core of Yukawa type is also investigated.

---

* A detailed discussion of the methods is made in an unpublished paper of the author[1].

** The equations suitable for calculation of the low-energy $^1S_0$ np scattering parameters were also obtained in the paper of Levy and Keller[3].

2. Equations for the $^1S_0$ np (nn) Scattering Parameters

In the singlet $^1S_0$ neutron-proton state only the central two-nucleon potential should be considered.

Using the phase function method [1,3] we define the series expansion of the function

$$\mathrm{tg}\,\delta_0(r,k) = -k\sum_{n=0}^{\infty} a_n(r)k^{2n} . \qquad (3)$$

The phase shift function $\delta_0(r,k)$ has a meaning of the phase shift due to the potential $V(r)\theta(r'-r)$ so its asymptotic value $\delta_0(\infty,k)$ equals the phase shift produced by the whole potential $V(r)$.

It may be shown [1,3] that the functions a_n satisfy a recurrent system of the first-order differential equations. For $a_0(r)$, $a_1(r)$, $a_2(r)$ the equations and corresponding initial values are of the form ($\hbar = 2m = 1$)

$$a_0' = V(r-a_0)^2 , \quad a_0(0) = 0 , \qquad (4a)$$

$$a_1' = -2V(r-a_0)a_1 - \frac{1}{3}r^2V(r^2-4ra_0+a_0^2), \quad a_1(0) = 0, \qquad (4b)$$

$$a_2' = -2V(r-a_0)a_2 + V(\frac{2}{45}r^6 - \frac{4}{15}r^5a_0 + \frac{1}{3}r^4a_0^2 + \frac{4}{3}r^3a_1 - 2r^2a_0a_1 + a_1^2), a_2(0) = 0. \qquad (4c)$$

The following formulae relate the functions a_n to the effective-range expansion coefficients (1):

$$a_s = \lim_{r\to\infty} a_0(r), \qquad (5a)$$

$$r_s = \lim_{r\to\infty} r_s(r) , \quad r_s(r) = 2a_1(r)/a_0^2(r), \qquad (5b)$$

$$P_s = \lim_{r\to\infty} P_s(r) , \quad P_s(r) = \frac{1}{8}\{a_0^3(r)/a_1^3(r)\}[a_1^2(r) - a_0(r)a_2(r)] . \qquad (5c)$$

The funotions $a_0(r')$, $r_s(r')$, $P_s(r')$ have a meaning of the oorresponding parameters of low-energy scattering on the potential $V(r)\theta(r'-r)$.

If the potential includes a hard oore, i.e. $V(r) = \infty$, $r \le r_0$ one has to integrate the equations (4) starting from the point $r = r_0$ with new initial values [1]

$$a_0(r_0) = r_0 , \quad a_1(r_0) = \frac{1}{3}r_0^3 , \quad a_2(r_0) = -\frac{2}{15}r_0^5 . \qquad (6)$$

The equations (4) are of use when none of the sequent potentials $V(r)\theta(r'-r)$ has a bound state. This is due to the faot that the scattering length a_0 tends to an unlimited value if there is a zero energy level in the potential. The second condition supposes that the potential V is of a short range, decreasing at least exponentially with $r\to\infty$. This is a real case for nuclear potentials. There are no bound levels in the neutron-proton and neutron-neutron $^1S_0$ states. Therefore equations (4) are suitable for calculation of the parameters a_s, r_s, P_s.

Equations (4) were integrated for a number of potentials which give a good fit for the $^1S_0$ high-energy scattering data [4-6]. The solutions $a_0(r)$, $a_1(r)$, $a_2(r)$ and the functions $r_s(r)$, $P_s(r)$ for the potential of Hamada-Johnston [4] ($\mu_\pi^{-1}r_0 = 0.343$) are shown by solid lines in Fig. 1. The numerical results are presented in the Table.

3. Equations for the $^1S_0$ pp Scattering Parameters

The presence of the long-range Coulomb potential leads to a rather complicated analytical behaviour of the nuclear phase shift $\delta_0(k)$ so that the expansion (3) is not valid now. However we can write:

$$\mathrm{tg}\,\delta_0(r,k,\eta) = -2\pi\eta(e^{2\pi\eta}-1)^{-1}k\sum_{n=0}^{\infty}A_n(r,h(\eta))k^{2n} . \qquad (7)$$

Here η is considered to be an independent parameter. We can introduce new functions only of r, $a_0(r)$, $a_1(r)$, $a_2(r)$ if:

$$A_0(r,h) = a_0(r)[1 + a_0(r)hR^{-1}]^{-1} , \qquad (8a)$$

$$A_1(r,h) = a_1(r)[1 + a_0(r)hR^{-1}]^{-2} , \qquad (8b)$$

$$A_2(r,h) = b_2(r,h)[1 + a_0(r)hR^{-1}]^{-2} , \quad a_2(r) = b_2(r,0) . \qquad (8c)$$

It can be shown [1] that the functions a_n satisfy the equations:

$$a_0' = V(rL_1 - a_0H_1)^2 , \quad a_0(0) = 0 , \qquad (9a)$$

$$a_1' = -V(rL_1 - a_0H_1)(2H_1a_1 - r^2Ma_0 + \frac{1}{3}r^3L_2), \quad a_1(0) = 0, \qquad (9b)$$

$$a_2' = -V(rL_1 - a_0H_1)[2H_1a_2 - r^2Ma_1 + \frac{1}{12}r^4Na_0 - \frac{1}{60}r^5(\frac{10}{9}L_3 - \frac{1}{9}L_4)]$$
$$+ V(H_1a_1 - \tfrac{1}{2}r^2Ma_0 + \frac{1}{6}r^3L_2)^2 , \quad a_2(0) = 0 . \qquad (9c)$$

In equations (9) $V(r)$ is the nuclear potential (without the Coulomb one) and the following notations of Jackson and Blatt [7] are used

$$L_n(\frac{r}{R}) = n!(\frac{r}{R})^{-\frac{1}{2}n} I_n(2\sqrt{\frac{r}{R}}) , \qquad (10a)$$

$$H_n(\frac{r}{R}) = \frac{2}{(n-1)!}(\frac{r}{R})^{\frac{1}{2}n} K_n(2\sqrt{\frac{r}{R}}) , \qquad (10b)$$

$$M(\frac{r}{R}) = \frac{2}{3}(\frac{r}{R})^{-1}[L_1(\frac{r}{R}) - H_2(\frac{r}{R})] , \qquad (10c)$$

$$N(\frac{r}{R}) = \frac{4}{3}(\frac{r}{R})^{-1}\{L_2 + 2(\frac{r}{R})^{-1}H_3 + \frac{12}{5}(\frac{r}{R})^{-2}[H_4 - L_1]\} . \qquad (10d)$$

In the absence of the Coulomb interaction ($R = \infty$) the functions (10) are equal to unity and equations (9) coincide with equations (4).

If there is a hard core in the potential $V(r)$ the initial conditions for equations (9) are:

$$a_0(r_0) = r_0 \frac{L_1(r_0/R)}{H_1(r_0/R)}, \quad a_1(r_0) = \frac{r_0^3}{3!}\left(\frac{3M}{H_1} - \frac{L_2}{L_1}\right),$$
$$a_2(r_0) = \frac{r_0^5}{5!}\left(\frac{10}{9}\frac{L_3}{L_1} - \frac{1}{9}\frac{L_4}{L_1} - 5\frac{N}{H_1} + 30\frac{M^2}{H_1^2} - 10\frac{L_2 M}{L_1 H_1}\right). \tag{11}$$

Usually $\frac{r_0}{R} \ll 1$ holds, so the formulae (11) can be reduced to simpler expressions (6).

Equations (5) relate the functions a_0, a_1, a_2 to the corresponding parameters a_p, r_p, P_p of the expansion (2).

The potentials [4-6] analysed in the previous section were used to compute the scattering length a_p, the effective range r_p and the shape parameter P_p for low-energy $^1S_0$ pp-scattering by means of equations (9). The results are shown in the Table. For the case of Hamada-Johnston [4] potential ($x_0 = 0.343$) the functions $a_0(r)$, $r_p(r)$, and $P_p(r)$ are shown in Fig. 1 by dashed lines.

4. Equations for the $^3S_1$ np Scattering Parameters

As was mentioned above equations (4) do not hold for the case of the triplet $^3S_1$-state when the tensor forces and a bound level (deuteron) are present.

Let us consider first the effect of the bound states. At the point r_1 where a zero energy bound state appears the value of the function $a_0(r_1)$ and those of the others $a_n(r_1)$ become infinite. Therefore we must reform equations (4) in such a way that all expressions are finite. This can be done by the following substitutions [1]:

$$a_0(r) = \operatorname{tg}\alpha(r), \tag{12a}$$

$$a_1(r) = \beta_2(r)a_0^2(r) + \beta_1(r)a_0(r) + \beta_0(r), \tag{12b}$$

$$a_2(r) = \beta_2^2(r)a_0^3(r) + \gamma_2(r)a_0^2(r) + \gamma_1(r)a_0(r) + \gamma_0(r). \tag{12c}$$

The functions α, β, γ are finite everywhere and satisfy the equations:

$$\alpha' = V(r\cos\alpha - \sin\alpha)^2, \quad \alpha(0) = 0. \tag{13}$$

$$\beta_2' = V(2r\beta_2 + \beta_1 - r^2), \quad \beta_2(0) = 0, \tag{14a}$$

$$\beta_1' = 2V(-r^2\beta_2 + \beta_0 + \tfrac{2}{3}r^3), \quad \beta_1(0) = 0, \tag{14b}$$

$$\beta_0' = -rV(r\beta_1 + 2\beta_0 + \tfrac{1}{3}r^3), \quad \beta_0(0) = 0. \tag{14c}$$

$$\gamma_2' = V(2r\gamma_2 + \gamma_1 + \tfrac{1}{3}r^4 + \tfrac{4}{3}r^3\beta_2 - 3r^2\beta_2^2 - 2r^2\beta_1 + \beta_1^2 + 2\beta_2\beta_0), \quad \gamma_2(0) = 0, \tag{15a}$$

$$\gamma_1' = 2V(-r^2\gamma_2 + \gamma_0 - \tfrac{2}{15}r^5 + \tfrac{2}{3}r^3\beta_1 - r^2\beta_0 + \beta_1\beta_0), \quad \gamma_1(0) = 0, \tag{15b}$$

$$\gamma_0' = -V(r^2\gamma_1 + 2r\gamma_0 - \tfrac{2}{45}r^6 - \tfrac{4}{3}r^3\beta_0 - \beta_0^2), \quad \gamma_0(0) = 0. \tag{15c}$$

If the potential $V(r)$ contains a hard core the initial conditions for equations (15) are:

$$\alpha(r_0) = \operatorname{arctg} r_0,$$
$$\beta_2(r_0) = \tfrac{1}{2}r_0, \quad \beta_1(r_0) = 0, \quad \beta_0(r_0) = -\tfrac{1}{6}r_0^3,$$
$$\gamma_2(r_0) = -\tfrac{1}{24}r_0^3, \quad \gamma_1(r_0) = 0, \quad \gamma_0(r_0) = -\tfrac{3}{40}r_0^5. \tag{16}$$

If the nuclear tensor forces are taken into account there will be a mixture of two triplet states $^3S_1$ and $^3D_1$.

So the low-energy scattering parameters for the $^3S_1$ state are connected with the parameters for the $^3D_1$ state.

The expansions for two phase shifts functions $\delta_{j,\ell}$ and for a mixing parameter function ϵ_j can be written as follows:

$$\operatorname{tg}\delta_{1,0}(r,k) = -k\sum_{n=0}^{\infty} a_n(r)k^{2n}, \tag{17a}$$

$$\operatorname{tg}\epsilon_1(r,k) = -\frac{1}{3}k^3\sum_{n=0}^{\infty} b_n(r)k^{2n}, \tag{17b}$$

$$\operatorname{tg}\delta_{1,2}(r,k) = -\frac{1}{45}k^5\sum_{n=0}^{\infty} c_n(r)k^{2n}. \tag{17c}$$

When there is no bound level, one will have the following equations [1,2] which are generalizing equations (4a) and (4b) in case the tensor forces are present.

$$a_0' = V_1(r - a_0)^2 - 2Tr^{-2}(r - a_0)b_0 + V_2 r^{-4}b_0^2, \quad a_0(0) = 0, \tag{18a}$$

$$b_0' = \tfrac{1}{5}Tr^{-2}(r - a_0)(r^5 - c_0) + Tr^{-2}b_0^2 - V_1(r - a_0)b_0 - \tfrac{1}{5}V_2 r^{-4}(r^5 - c_0)b_0, \quad b_0(0) = 0, \tag{18b}$$

$$c_0' = \tfrac{1}{5}V_2 r^{-4}(r^5 - c_0)^2 - 2Tr^{-2}(r^5 - c_0)b_0 + 5V_1 b_0^2, \quad c_0(0) = 0. \tag{18c}$$

$$a_1' = -V_1(r - a_0)(2a_1 - a_0 r^2 + \tfrac{1}{3}r^3) + 2Tr^{-2}(a_1 b_0 + a_0 b_1 - \tfrac{1}{3}a_0 b_0 r^2 - b_1 r)$$
$$+ V_2 r^{-4}(2b_1 + \tfrac{1}{3}b_0 r^2)b_0, \quad a_1(0) = 0, \tag{19a}$$

$$b_1' = -\tfrac{1}{5}Tr^{-2}[(r^5 - c_0)a_1 + (r - a_0)c_1 + \tfrac{1}{3}a_0 c_0 r^2 - \tfrac{4}{7}a_0 r^7 + \tfrac{5}{21}r^8] + Tr^{-2}(2b_1 - \tfrac{1}{3}b_0 r^2)b_0 \tag{19b}$$
$$+ V_1(a_1 b_0 + a_0 b_1 - a_0 b_0 r^2 - b_1 r + \tfrac{2}{3}b_0 r^3) + \tfrac{1}{5}V_2 r^{-4}(c_1 b_0 + c_0 b_1 + \tfrac{1}{3}c_0 b_0 r^2 - b_1 r^5 - \tfrac{2}{21}b_0 r^7), \quad b_1(0) = 0,$$

$$c_1' = -\tfrac{1}{5}V_2 r^{-4}(r^5 - c_0)(2c_1 + \tfrac{1}{3}c_0 r^2 + \tfrac{1}{7}r^7) + 2Tr^{-2}(c_1 b_0 + c_0 b_1 - \tfrac{1}{3}c_0 b_0 r^2 - b_1 r^5 + \tfrac{4}{7}b_0 r^7) \tag{19c}$$
$$+ 5V_1(2b_1 - b_0 r^2)b_0, \quad c_1(0) = 0.$$

Here the notations are used:

$$V_1(r) = V_c(r), \quad T(r) = 2\sqrt{2}\,V_t(r), \tag{20}$$

$$V_2(r) = V_c(r) - 2V_t(r) - 3V_{\ell s}(r) - 3V_{\ell\ell}(r).$$

The presence of a hard core in the central potential leads to the following initial values:

$$a_0(r_0) = r_0\,, \quad b_0(r_0) = 0\,, \quad c_0(r_0) = r_0^5\,,$$

$$a_1(r_0) = \frac{1}{3} r_0^3\,, \quad b_1(r_0)\,, \quad c_1(r_0) = -\frac{5}{21} r_0^7\,. \tag{21}$$

The scattering length and the effective range can be calculated by means of (5a) and (5b).

The equations are considerably complicated as both the tensor forces and the presence of a bound state are taken into account. In the case [1,2] one has to introduce several new functions

$$a_0 = \operatorname{tg}\alpha\,, \tag{22a}$$

$$b_0 = \beta_1 a_0 + \beta_0\,, \tag{22b}$$

$$c_0 = 5\beta_1^2 a_0 + 5\beta_1\beta_0 + \gamma_0\,, \tag{22c}$$

$$a_1 = A_2 a_0^2 + A_1 a_0 + A_0\,, \tag{23a}$$

$$b_1 = A_2\beta_1 a_0^2 + (A_1\beta_1 + B_1) a_0 + (A_0\beta_1 + B_0)\,, \tag{23b}$$

$$c_1 = 5A_2\beta_1^2 a_0^2 + 5(A_1\beta_1 + 2B_1)\beta_1 a_0 + 5(A_0\beta_1^2 + B_1\beta_0 + B_0\beta_1 + C_0)\,. \tag{23c}$$

The corresponding equations are of the form [1,2]

$$\alpha' = V_1(r\cos\alpha - \sin\alpha)^2 - 2Tr^{-2}(r\cos\alpha - \sin\alpha)(\beta_1\sin\alpha + \beta_0\cos\alpha) + V_2 r^{-4}(\beta_1\sin\alpha + \beta_0\cos\alpha)^2\,, \quad \alpha(0) = 0\,, \tag{24a}$$

$$\beta_1' = Tr^{-2}\beta_1(r\beta_1 + \beta_0) - \frac{1}{5}Tr^{-2}(r^5 - \gamma_0) + V_1(r\beta_1 + \beta_0) - \frac{1}{5}V_2 r^{-4}\beta_1(r^5 - \gamma_0)\,, \quad \beta_1(0) = 0\,, \tag{24b}$$

$$\beta_0' = Tr^{-2}\beta_0(r\beta_1 + \beta_0) + \frac{1}{5}Tr^{-1}(r^5 - \gamma_0) - V_1 r(r\beta_1 + \beta_0) - \frac{1}{5}V_2 r^{-4}\beta_0(r^5 - \gamma_0)\,, \quad \beta_0(0) = 0\,, \tag{24c}$$

$$\gamma_0' = \frac{1}{5}V_2 r^{-4}(r^5 - \gamma_0)^2 - Tr^{-2}(r^5 - \gamma_0)(r\beta_1 + \beta_0)\,, \quad \gamma_0(0) = 0\,. \tag{24d}$$

$$A_2' = V_1(2A_2 r + A_1 - r^2) + 2Tr^{-2}\left(A_2\beta_1 r - A_2\beta_0 + A_1\beta_1 + B_1 - \frac{1}{3}\beta_1 r^2\right) + V_2 r^{-4}\beta_1\left(-2A_2\beta_0 + A_1\beta_1 + 2B_1 + \frac{1}{3}\beta_1 r^2\right)\,, \quad A_2(0) = 0\,, \tag{25a}$$

$$A_1' = 2V_1\left(-A_2 r^2 + A_0 + \frac{2}{3}r^3\right) + 2Tr^{-2}\left(2A_2\beta_0 r + 2A_0\beta_1 - B_1 r + B_0 - \frac{1}{3}\beta_0 r^2\right) + 2V_2 r^{-4}\left(-A_2\beta_0^2 + A_0\beta_1^2 + B_0\beta_1 + B_1\beta_0 + \frac{1}{3}\beta_1\beta_0 r^2\right)\,, \quad A_1(0) = 0\,, \tag{25b}$$

$$A_0' = -V_1 r\left(A_1 r + 2A_0 + \frac{1}{3}r^3\right) + 2Tr^{-2}(A_1\beta_0 r + A_0\beta_0 - A_0\beta_1 r - B_0 r) + V_2 r^{-4}\beta_0\left(-A_1\beta_0 + 2A_0\beta_1 + 2B_0 + \frac{1}{3}\beta_0 r^2\right)\,, \quad A_0(0) = 0\,, \tag{25c}$$

$$B_1' = V_1\left(B_1 r + B_0 - \beta_0 r^2 - \frac{2}{3}\beta_1 r^3\right) + Tr^{-2}\left(B_1\beta_0 + 2B_1\beta_1 r + B_0\beta_1 + C_0 - \frac{1}{3}\beta_1\beta_0 r^2 - \frac{1}{15}\gamma_0 r^2 + \frac{4}{35}r^7\right) + V_2 r^{-4}\left(\frac{1}{5}B_1\gamma_0 - \frac{1}{5}B_1 r^5 + C_0\beta_1 + \frac{1}{15}\beta_1\gamma_0 r^2 - \frac{2}{105}\beta_1 r^7\right)\,, \quad B_1(0) = 0\,, \tag{25d}$$

$$B_0' = V_1\left(-B_1 r^2 - B_0 r + \frac{2}{3}\beta_0 r^3 + \frac{1}{3}\beta_1 r^4\right) + Tr^{-2}\left(B_1\beta_0 r + B_0\beta_1 r + 2B_0\beta_0 - C_0 r - \frac{1}{3}\beta_0^2 r^2 - \frac{1}{21}r^8\right) + V_2 r^{-4}\left(\frac{1}{5}B_0\gamma_0 - \frac{1}{5}B_0 r^5 + C_0\beta_0 + \frac{1}{15}\beta_0\gamma_0 r^2 - \frac{2}{105}\beta_0 r^7\right)\,, \quad B_0(0) = 0\,, \tag{25e}$$

$$C_0' = -\frac{1}{5}V_2 r^{-4}(r^5 - \gamma_0)\left(2C_0 + \frac{1}{15}\gamma_0 r^2 + \frac{1}{35}r^7\right) - \frac{1}{5}Tr^{-2}(r^5 - \gamma_0)\left(B_1 r + B_0 - \frac{1}{3}\beta_0 r^2\right) + Tr^{-2}(\beta_1 r + \beta_0)\left(C_0 + \frac{1}{21}r^7\right)\,, \quad C_0(0) = 0. \tag{25f}$$

If there is a hard core in the central potential, the initial values are the following:

$$\alpha(r_0) = \operatorname{arctg} r_0\,, \quad \beta_1(r_0) = \beta_0(r_0) = 0\,, \quad \gamma_0(r_0) = r_0^5\,, \tag{26}$$

$$A_2(r_0) = \tfrac{1}{2} r_0\,, \quad A_1(r_0) = 0\,, \quad A_0(r_0) = -\frac{1}{6} r_0^3\,,$$

$$B_1(r_0) = B_0(r_0) = 0\,, \quad C_0(r_0) = -\frac{1}{21} r_0^7\,.$$

Equations (24), (25) were solved for two potentials [4,5] possessing hard cores. The results for the Hamada-Johnston potential are shown in Fig. 2. The numerical values obtained for the scattering length and the effective range are presented in the Table.

5. Scattering in States with $l > 0$

In this section we shall give the equations which are useful for computing the first coefficients of low-energy expansions of the phase shifts ($\ell > 0$).

For scattering in the singlet and non-mixing triplet states with an arbitrary orbital angular momentum of the neutron-proton (neutron-neutron) system the expansion can be written as follows [1]:

$$\operatorname{tg}\delta_\ell(r,k) = -\frac{k^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}\sum_{n=0}^{\infty} k^{2n} A_{\ell n}(r). \tag{27}$$

Then the first function $A_{\ell 0}(r)$ satisfies a simple equation

$$A_{\ell 0}' = \frac{1}{2\ell+1} V\left(r^{\ell+1} - r^{-\ell} A_{\ell 0}\right)^2\,, \quad A_{\ell 0}(0) = 0. \tag{28}$$

In the presence of a hard core the initial condition is $A_{\ell 0}(r_0) = r_0^{2\ell+1}$.

If the Coulomb potential is taken into account for the same states of the proton-proton system the expansion and the corresponding equation are of the form [1]

$$\operatorname{tg}\delta_\ell(r,k,\eta) = -(2\ell+1)C_\ell^2(\eta)k^{2\ell+1}\sum_{n=0}^{\infty} A_{\ell n}(r,\eta)k^{2n}\,,$$

$$C_\ell^2(\eta) = 2^{2\ell}[(2\ell+1)!]^{-2}[\ell^2+\eta^2][(\ell-1)^2+\eta^2]\ldots[1+\eta^2]\,2\pi\eta\,(e^{2\pi\eta}-1)^{-1}\,; \tag{29}$$

$$A_{\ell 0}'(r,\infty) = \frac{1}{2\ell+1} V(r)\left[r^{\ell+1} L_{2\ell+1}\!\left(\frac{r}{R}\right) - r^{-\ell} H_{2\ell+1}\!\left(\frac{r}{R}\right) A_{\ell 0}(r,\infty)\right]^2\,, \quad A_{\ell 0}(0,\infty) = 0\,. \tag{30}$$

In equation (30) a transition $k \to 0$, $\eta \to \infty$ is performed. The initial value of $A_{\ell 0}(r_0,\infty)$ for the case of a hard core potential can be obtained by equalizing the expression in brackets in equation (30) to zero.

When the tensor forces are present in mixing triplet neutron-proton states one has to deal with the expansions[1,2]

$$\mathrm{tg}\,\delta_{j,j-1}(r,k) = -\frac{k^{2j-1}}{(2j-1)!!\,(2j-3)!!}\sum_{n=0}^{\infty} A_{jn}(r)k^{2n}, \tag{31a}$$

$$\mathrm{tg}\,\epsilon_j(r,k) = -\frac{k^{2j+1}}{(2j+1)!!\,(2j-3)!!}\sum_{n=0}^{\infty} B_{jn}(r)k^{2n}, \tag{31b}$$

$$\mathrm{tg}\,\delta_{j,j+1}(r,k) = -\frac{k^{2j+3}}{(2j+3)!!\,(2j+1)!!}\sum_{n=0}^{\infty} C_{jn}(r)k^{2n}. \tag{31c}$$

The system of equations for coefficients is the following:

$$A'_{j0} = \frac{1}{2j-1} V_{j,j-1}(r^{j} - A_{j0} r^{-j+1})^2 - 2T_j (r^{j} - A_{j0} r^{-j+1}) B_{j0} r^{-j-1} + (2j-1) V_{j,j+1} B_{j0}^2 r^{-2j-2}, \tag{32a}$$

$$B'_{j0} = \frac{1}{(2j-1)(2j+3)} T_j (r^{j} - A_{j0} r^{-j+1})(r^{j+2} - C_{j0} r^{-j-1}) + T_j B_{j0}^2 r^{-2j}$$
$$- \frac{1}{2j-1} V_{j,j-1}(r^{j} - A_{j0} r^{-j+1}) B_{j0} r^{-j+1} - \frac{1}{2j+3} V_{j,j+1}(r^{j+2} - C_{j0} r^{-j-1}) B_{j0} r^{-j-1}, \tag{32b}$$

$$C'_{j0} = \frac{1}{2j+3} V_{j,j+1}(r^{j+2} - C_{j0} r^{-j-1})^2 - 2T_j (r^{j+2} - C_{j0} r^{-j-1}) B_{j0} r^{-j+1} + (2j+3) V_{j,j-1} B_{j0}^2 r^{-2j+2}. \tag{32c}$$

Here the effective potentials for mixing partial waves are*:

$$V_{j,j-1} = V_c - \frac{2(j-1)}{2j+1} V_t + (j-1) V_{\ell s} + (j-1) V_{\ell\ell},$$
$$V_{j,j+1} = V_c - \frac{2(j+2)}{2j+1} V_t - (j+2) V_{\ell s} - (j+2) V_{\ell\ell}, \tag{33}$$
$$T_j = \frac{6\sqrt{j(j+1)}}{2j+1} V_t.$$

The initial values for the functions in (32) are zeros. If there is a hard core in the central potential, then

$$A_{j0}(r_0) = r_0^{2j-1}, \quad B_{j0}(r_0) = 0, \quad C_{j0}(r_0) = r_0^{2j+3}. \tag{34}$$

6. Results and Discussion

The equations of the previous sections were integrated by means of the numerical Runge-Kutta method for three potentials[4-6] at an electronic computer. For two of the potentials a slight variation of the hard core[4] and of the soft repulsive core[6] was also considered. The resulting values of the scattering length, the effective range and the shape parameter are presented in the

* The quadratic ℓs potential $V_{\ell\ell}\,\ell_{12}$ is defined here as in the Hamada-Johnston paper[4]: $\ell_{12} = [\delta_{j\ell} + (\vec{\sigma}_1\vec{\sigma}_2)]\vec{\ell}^{\,2} - (\vec{\ell}\vec{s})^2$.

Table, where the experimental data are shown as well. In Figs. 1 and 2 one can see the behaviour of the functions involved.

As is seen in the Figures the asymptotic values of the functions are reached within a sufficiently short range. Note that the functions a_0, r_e we are interested in become constants sooner than auxiliary functions, for example β_0, γ_0.

Therefore one can discontinue integrating the equations before all the functions reach their asymptotic values. However, in order to be sure that these values are reached we computed the quantities up to the point $\mu_\pi^{-1} r = 100$. Then all the functions become constants within a high degree of accuracy. The table contains the results obtained for a_0, r_e, P.

Comparing the $^1S_0$ np parameters of the Table ($r_0 = 0.343$) with those obtained by Hamada and Johnston[4]: $a_s = -17.0f$, $r_s = 2.83f$, $P_s = 0.016$ one can see a slight descrepancy in a_s, r_s values and a very large difference in P_s values (about a factor 2). We believe that this fact demonstrates the advantage of our method of computation. The usual procedure consists of computing at first phase shifts $\delta_0(k)$ for a number of k values and then searching the parameters a_0, r_e, P using the expansion (1). Small uncertainties in a_s, r_s values give rise to a relativily large uncontrolled error in P_s in such a procedure. It should be noted that there is a definite loss of accuracy in computing the difference $a_1^2 - a_0 a_2$ [eq.(5c)] of two nearly equal quantities. But this operation can be controlled in the process of computing. Similar discrepancies are seen in the case of a smaller radius of the hard core $r_0 = 0.341$. The H-J results are $a_s = -23.7f$, $r_s = 2.73f$.

An approximate formula of Jackson and Blatt[7] for the a_p evaluation

$$\frac{1}{a_p} = \frac{1}{a_s} + \frac{1}{R}\ln\frac{r_s}{R} + \frac{0.330}{R}$$

is widely used, but a possible deviation from the actual value is not definitely estimated.

The possibility of the direct computation of the proton-proton scattering parameters a_p, r_p, and P_p permits us to make a more accurate comparison with the experimental values.

It can be seen in the Table that all potentials under consideration do not fit the experimental data both for $^1S_0$ np and $^1S_0$ pp scattering. The two types of the potentials with a hard core and a soft one show that there can be a weak charge dependence of the nuclear forces.

The equations of Section 4 allow us exactly to take into account the effects of the tensor forces and to pass out the limits of the shape-independent approximation.

The results obtained for the $^3S_1$ np - parameters show that the H-J potential [4] is much more suitable for the description of low-energy np phenomena than that of Breit et al. [5], though both potentials are equally satisfactory in terms of the high-energy scattering phenomena. Our results for $r_t = 1.740f$ are to be compared with the H-J value [4] $\rho(-\epsilon,-\epsilon) = 1.77f$. The difference is the measure of the inaccuracy of the shape-independent approximation. It appears to be not very large.

Finally, it should be mentioned that the formulae given in this paper can be also used in the case of low-energy scattering of particles other than nucleons.

The author thanks J. Bystricky and L.A. Smirnova for performing the numerical calculations.

References

1. V.V. Babikov, Preprint JINR, J-1728, Dubna (1964).
2. V.V. Babikov. Nucl. Physics (Moscow), 1, N 2, (1965) (in press).
3. B.R. Levy and J.B. Keller, J. Math. Phys., 4, 54 (1963).
4. T. Hamada and I.D. Johnston. Nucl. Phys., 34, 382 (1962).
5. K.E. Lassila, M.H. Hull Jr., H.M. Ruppel, F.A. McDonald and G. Breit. Phys.Rev., 126, 881 (1962).
6. V.V. Babikov, I. Bystricky and F. Lehar. Report at XII Int.Conf. on High Energy Physics, Dubna (1964).
7. J.D. Jackson and J.M. Blatt, Rev. Mod. Phys., 22, 77 (1950).
8. H.P. Noyes. Phys. Rev., 130, 2025 (1963).
9. H.P. Noyes. Phys.Rev. Lett., 12, 171 (1964).

Received by Publishing Department
on November 16, 1964.

Table

The singlet and triplet scattering lengths, effective ranges and shape parameters calculated for various potentials

Potential	^1S_o np			^1S_o pp			$^3S_1$ np		Ref. Notes
	a_s (Fermis)	r_s	P_s	a_p	r_p (Fermis)	P_p	a_t	r_t (Fermis)	
Hamada-Johnston	−16.711	2.857	0.0315	−7.729	2.749	0.0478	5.371	1.740	/4/ $x_o=0.343$
Hamada-Johnston	−21.720	2.767	0.0316	−8.542	2.664	0.0499	5.136	1.708	/4/ $x_o=0.341$
Breit et al.	−13.531	2.965	0.0201	−7.078	2.829	0.0372	1.638	1.356	/5/ $x_o=0.350$
Babikov et al.	−22.794	2.807	0.0278	−8.710	2.721	0.0371			/6/ $g_\omega^2-2f_\omega^2=29.2$
Babikov et al.	−15.834	2.931	0.0269	−7.572	2.840	0.0357			/6/ $g_\omega^2-2f_\omega^2=29.6$
Exp. data	−23.678 ± 0.028	2.51 ±0.15		−7.8163 ±0.0048	2.746 ±0.014		5.396 ±0.011	1.726 ±0.014	/8/,/9/ shape indep.
Exp. data				−7.8284 ±0.0080	2.794 ±0.026	0.026 ±0.014			/9/ shape dep.