ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E-1850

I.Friš, P.Winternitz

INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES AND SUBGROUPS OF THE PROPER LORENTZ GROUP

ЯФ, 1965, т.1, в5, с889-901

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1964

E-1850

I.Friš, P.Winternitz

INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES AND SUBGROUPS OF THE PROPER LORENTZ GROUP

1. Introduction

Vilenkin and Smorodinsky[1] have considered the problem of expanding functions (scattering amplitudes) into sums and integrals over the eigenfunctions of the Laplace operator on a hyperboloid. These eigenfunctions form a basis for a representation of the proper homogeneous Lorentz group (further just Lorentz group). Four coordinate systems were choesn among these allowing the separation of variables in the Laplace equation; the explicit form of the basis functions and formulas for direct and inverse expansions were found in these coordinates. Geometrically the considered coordinate systems are characterized by the fact, that they are axially symmetrical and have one centre (i.e. all the coordinate surfaces can be obtained by motions of direct lines, circles, horocycles and equidistants). It has been shown in[2] that these systems can be obtained group-theoretically, considering certain subgroups of the Lorentz group and demanding that the basis functions of the representation should be eigenfunctions not only of the Laplacian in the Lobachevsky space, but also of the invariants of the corresponding subgroups.

In this paper we develop the group theoretical approach from a more general point of view. We find all (mutually not conjugated) continuous subgroups of the Lorentz group and their invariants. We prove the following statement: a coordinate system, allowing variable separation, corresponds to every nonequivalent mode of picking out subgroups of the Lorentz group, containing invariants. These exhaust just all coordinate systems having one geometrical centre[x]. The subgroup invariants together with the Laplace operator form a complete system of commuting observables and their common eigenfunctions form the basis of a representation. If there is a group of one-dimensional space rotations among the picked out subgroups, then we obtain the systems considered previously[1,2].

Further we consider a simpler example - the group of motions of a Euclidean plane - and show how coordinate systems with two centres (elliptical type coordinates) can be approached, considering expressions quadratical in the infinitesimal operators. To throw some light on the physical meaning of these expressions, we show the connection of one of them with the Laplace-Lenz vector (the additional integral of motion, conserved only in a Coulomb field[4]).

---

[x] All orthogonal coordinate systems, for which the variables separate in the Laplace equation in spaces with constant curvature, were found by Olevsky[3].

II. Subgroups of the proper Lorentz group

Further we denote the proper homogeneous Lorentz group L and the corresponding Lie algebra (infinitesimal algebra) – $\mathfrak{L}$. We shall also make use of the infinitesimal group ring of L (the universal envelopping algebra). The generators of $\mathfrak{L}$ corresponding to space and hyperbolic rotations, will be denoted by A_i and B_i (i = 1,2,3). We shall call elements of the centre of the infinitesimal group ring invariants[5].

First of all we must enumerate all (mutually non-conjugated) continuous subgroups of the Lorentz group. To our knowledge this question has so far not been considered in the literature, so we shall look at it in some detail.

Two (continuous) subgroups and their algebras are conjugated if an inner automorphism exists transforming one into the other. A convenient method of investigating conjugation questions makes use of the so-called adjoint representation. Dropping the details we put a six-dimensional vector

$$C = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{where} \quad a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1}$$

into correspondence with the element $C = a_i A_i + b_i B_i$ (summation from 1 to 3). On the other hand a 6 x 6 matrix

$$\mathcal{T} = \text{adj } T = \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \quad \text{where} \quad R = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix} \tag{2}$$

corresponds to every element $T = r_i A_i + s_i B_i$. It is easy to prove that the relation $[T, C_1] = C_2$ is equivalent to the relation $C_2 = \text{adj } T \, C_1$ in the adjoint representation.

It follows from the general theory (e.g.[6]) that two elements C, C' are conjugated, if and only if such an element $T \in \mathfrak{L}$, i.e. such a matrix $\mathcal{T}$ of the type (2) exists that

$$C' = (\exp \mathcal{T})\, C \tag{3}$$

To obtain a more convenient criterion of conjugation, we form a three-dimensional vector $\tilde{\tilde{C}} = a + i b$ corresponding to each six-dimensional vector c and a 3x3 matrix $\tilde{\tilde{\mathcal{T}}} = R + iS$ corresponding to $\mathcal{T}$ of (2). It is easy to see that this correspondence is an isomorphism and that $\exp \tilde{\tilde{\mathcal{T}}}$ is a complex orthogonal 3 x 3 matrix (and any complex 3 x 3 matrix can be written in this form[7]).

Thus we have : two elements of $\mathfrak{L}$, $C = a_i A_i + b_i B_i$ and $C' = a'_i A_i + b'_i B_i$

are conjugated if and only if a complex orthogonal matrix $\mathcal{O}$ exists, satisfying

$$\begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \end{pmatrix} = \mathcal{O} \begin{pmatrix} a_1' + ib_1' \\ a_2' + ib_2' \\ a_3' + ib_3' \end{pmatrix} \qquad (4)$$

It is proved in Appendix I that this statement implies the following criterion:

The necessary and sufficient condition for two elements C and C' of $\mathfrak{L}$ to be conjugated is that

$$\delta_1 \equiv a_i^2 - b_i^2 = a_i'^2 - b_i'^2 , \qquad \delta_2 \equiv 2 a_i b_i = 2 a_i' b_i' \qquad (5)$$

except for the case $\delta_1 = \delta_2 = 0$ when we obtain two classes: a trivial one with $a_i = b_i = 0$ $(i = 1,2,3)$ and a nontrivial one in which at least one a_i or (and) b_i is non-zero.

Using this criterion, we shall now enumerate all classes of continuous subgroups of the Lorentz group. Each of these subgroups is conjugated to a subgroup, determined by one of the following algebras:

1) One-parametrical subgroups

a) $C_\alpha = \cos\alpha\ A_1 + \sin\alpha\ B_1 \qquad 0 \leq \alpha < \pi$

b) $C = A_1 + B_2$

The first is a continuum of classes, depending on one parameter α. Each class C_α is characterized by the value α, where $\delta_1 = \cos 2\alpha$, $\delta_2 = \sin 2\alpha$. The second is one class of conjugated algebras, for which we have $\delta_1 = \delta_2 = 0$ Of course, every one-parameter group has an invariant (equal to its generator).

2) Two-parametrical subgroups.

It is well known that only two types of two-dimensional Lie algebras exist: an abelian one $[K, L] = 0$ and a non-abelian $[K, L] = K$. Both types are contained in the Lorentz group. It can be simply verified, that every two-dimensional Lie algebra in $\mathfrak{L}$ is conjugated to one of the following:

a) A_1, B_1

b) $A_1 + B_2$, $A_2 - B_1$

c) $A_1 + B_2$, $-B_3$

The first two are abelian and isomorphous, but not conjugated, since $\delta_1 = \delta_2 = 0$ only for unity in a) but for every element of b). Moreover, the corresponding groups are isomorphous only locally: the first is the group of translations on a cylinder, the second on a Euclidean plane.

The third algebra is non-abelian and we shall prove in Appendix II, that it has no invariant (naturally every element of an abelian algebra is an invariant).

3) Three-parametrical subgroups.

An infinite number of non-isomorphous three-dimensional real Lie algebras exists, but they can all be reduced to eight types [8]. All of them, that are contained in $\mathfrak{L}$, are conjugated to one of the following :

a) A_1, A_2, A_3 $\qquad [A_i, A_k] = \epsilon_{ik\ell} A_\ell$

b) B_1, B_2, A_3 $\qquad [B_1, B_2] = -A_3, \quad [A_3, B_1] = B_2, \quad [B_2, A_3] = B_1$

c) $A = A_1 + B_2, \quad B = A_2 - B_1, \quad C_\alpha = \cos\alpha\ A_3 + \sin\alpha\ B_3 \qquad 0 \leq \alpha < \pi$

$[A, B] = 0 \quad [B, C_\alpha] = \cos\alpha\ A - \sin\alpha\ B, \quad [C_\alpha, A] = \cos\alpha\ B + \sin\alpha\ A$

The first algebra corresponds to the three-dimensional rotation group. Its only independent invariant is

$$L^2 = A_1^2 + A_2^2 + A_3^2 \tag{6}$$

The algebra b) defines the three-dimensional Lorentz group with the invariant

$$H^2 = B_1^2 + B_2^2 - A_3^2 \tag{7}$$

In c) we have a continuum of classes of algebras. It is proved in Appendix II that such an algebra has an invariant only if $\alpha = 0$. In this case we obtain the group of motions of an Euclidean plane (a horosphere) with the invariant

$$O^2 = (A_1 + B_2)^2 + (A_2 - B_1)^2 \tag{8}$$

4) Four-parametrical subgroups.

All four-dimensional Lie algebras are classified in [8]. Only one of them is contained in $\mathfrak{L}$:

$$A = A_1 \qquad B = B_1 \qquad C = A_3 + B_2 \qquad D = A_2 - B_3$$

$$[A B] = 0 \qquad [A C] = -D \qquad [B C] = -C$$

$$[C D] = 0 \qquad [A D] = C \qquad [B D] = -D$$

Such an algebra has no invariants (cf. Appendix II).

The Lorentz group, similarly as any Lie group of higher dimension than three, has no subgroup of index one [6], i.e. no five-parametrical subgroup.

We have exhausted all the subalgebras of $\mathfrak{L}$ and hence all the continuous subgroups of the Lorentz group. We completely ignore the existence of discrete subgroups of L, the existence of which does not influence the Lie algebra.

III Subgroups of the Lorentz group and variable separation in the Laplace equation on a hyperboloid

In this chapter we shall prove the statement formulated in the introduction. In our case the group invariant[x)] is a differential operator (on a hyperboloid). We construct complete sets of commuting operators out of the invariants of the Lorentz group and its subgroups. We find the common eigenfunctions of each set of operators (we shall say that these operators are diagonal in the given representation) and show that just one coordinate system, in which all the (common) eigenfunctions are separated (i.e. can be written in the form $\psi = \psi_1 \psi_2 \psi_3$ where each ψ_i depends on just one variable), can be put into correspondence with each set of commuting operators (i.e. with each mode of picking out subgroups of the Lorentz group). By comparison with /3/ we see that we exhaust all coordinate systems with no elliptical-type coordinate surfaces.

Further we shall consider only subgroups having invariants. We introduce the following graphs[xx)]. A semicircle (hyperbola) corresponds to hyperbolic type groups (the four- and three-dimensional Lorentz group and any one-dimensional group conjugated to B_1) ; a square to a Euclidean group (E_2 or a one-parameter subgroup, conjugated to $A_1 + B_2$); a circle to compact subgroups and a triangle to the group of motions of a cylinder (A_1 , B_1). The graphs, corresponding to the breaking up of L into subgroups, are shown on figures 1-4. The first (lowest) part of each figure illustrates the whole group L, the second, one of its maximal subgroups (taking only groups with invariants into account), the third, one-dimensional subgroups. Using these graphs we can directly write down the eigenfunctions in the corresponding coordinate system and give a geometrical description of it.

Let us prove our assertion by listing all possible sets of subgroups:

1. The rotation group R_3 (fig.1). The set of operators $A^2 = A_1^2 + A_2^2 + A_3^2$ and A_1 leads to the spherical system S /1,2/ and the eigenfunction

---

x) In this paper we consider only representations in which the second invariant of the Lorentz group $\Lambda' = A_i B_i$ equals zero identically, i.e. we restrict ourselves to spin zero particles.

xx) A similar graphical method illustrating the introduction of various types of orthogonal coordinates for the n -dimensional rotation group, was used by N.Ya.Vilenkin.

$$\psi_{p\ell m}(a,\theta,\phi) = \frac{1}{\sqrt{\mathrm{sh}\,a}} P^{-(\ell+1/2)}_{-1/2+ip}(\mathrm{ch}\,a)\, P^{m}_{\ell}(\cos\theta)\, e^{im\phi} \qquad (9)$$

2. The three-dimensional Lorentz group L_3 (fig.2). There are three mutually non-conjugated one-parametrical subgroups in L_3 and their generators are A_3, B_1, $B_2 - A_3$. Diagonalizing the invariant $B_1^2 + B_2^2 - A_3^2$ and one of the mentioned generators, we obtain three coordinate systems. The first two have been considered in /1,2/, the third is new and could be called "hyperbolic-translational" - HT. The eigenfunctions are:

a) Hyperbolical system H (A_3 diagonal)

$$\psi_{pqm}(a,b,\phi) = \frac{1}{\mathrm{ch}\,a} P^{ip}_{-1/2+iq}(\mathrm{th}\,a)\, P^{m}_{-1/2+iq}(\mathrm{ch}\,b)\, e^{im\phi} \qquad (10)$$

b) Lobachevsky system L (B_1 diagonal)

$$\psi_{pq\mu}(a,b,c) = \frac{1}{\mathrm{ch}a} P^{ip}_{-1/2+iq}(\mathrm{th}a) \frac{1}{\sqrt{\mathrm{ch}b}} P^{iq}_{-1/2+i\mu}(\mathrm{th}b)\, e^{i\mu c} \qquad (11)$$

c) HT- system ($B_2 - A_3$ diagonal)

$$\psi_{pq\mu}(a,b,x) = \frac{1}{\mathrm{ch}\,a} P^{ip}_{-1/2+iq}(\mathrm{th}a)\, e^{b/2} K_{iq}(\mu e^{b})\, e^{i\mu x} \qquad (12)$$

where $K_\nu(z)$ is a MacDonald Function.

3) The Euclidean group E_2 (fig.3).

The only two non-conjugated subgroups of E_2 are determined by A_3 (rotations) or by $A_1 + B_2$ and $A_2 - B_1$ (translations). Diagonalizing the invariant $\mho^2 = (A_1 + B_2)^2 + (A_2 - B_1)^2$ and the corresponding subgroup generators, we obtain two coordinate systems (one of them is new).

a) Horospherical systems 0 (A_3 -diagonal)

$$\psi_{p\kappa m}(a,r,\phi) = e^{-a} K_{ip}(\kappa e^{-a})\, J_m(\kappa r)\, e^{im\phi} \qquad (13)$$

where $J_m(x)$ is a Bessel function.

b) Horospherical-translational system OT ($A_1 + B_2$ and $A_2 - B_1$ diagonal)

$$\psi_{p\mu\nu}(a,x,y) = e^{a} K_{ip}(\sqrt{\mu^2+\nu^2}\, e^{a})\, e^{i\mu x + i\nu y} \qquad (14)$$

4) The cylinder subgroup (fig.4.). The diagonality of A_1 and B_1 gives the cylindrical system C /1/ in which

$$\psi_{p\tau m}(a,b,\phi) = e^{i(\tau a + m\phi)} (\mathrm{sh}\, b)^{m} (\mathrm{ch}\, b)^{-m-1-ip} \cdot$$

$$\cdot F\left(\frac{m+1+ip+i\tau}{2}, \frac{m+1+ip-i\tau}{2}, m+1, \mathrm{th}^2 b\right) \tag{15}$$

where $F(\alpha, \beta, \gamma, z)$ is the hypergeometrical series. The second two-dimensional abelian subgroup $(A_1 + B_2, \; A_2 - B_1)$ is not maximal, since it is contained in E_2 and gives the OT systems again.

We can abstract the following rules for the eigenfunctions from the above considerations:

Each end of the chain on fig. 1-4 corresponds to an exponential, discrete for a circle, otherwise continuous.

Roughly speaking, a Legendre polynomial corresponds to each arrow from circle to circle, various spherical functions to arrows from semi-circles to circles or semi-circles, MacDonald functions to arrows from semicircles to squares, Bessel functions to arrows from squares to circles and hypergeometrical series to arrows from semicircles to triangles.

The rules for the coordinate systems are:

A family of planes corresponds to each end of the chain: a pencil with a common axis to a circle, a family of planes perpendicular to a given axis - to a semicircle and a family of planes, parallel to a given one corresponds to a square.

A set of spheres, corresponds to a circle on the second place, a set of hyperspheres to a semicircle, a set of horospheres to a square.

A set of circular cylinders corresponds to an arrow from a semi-circle in the middle to a circle, a set of equidistant cylinders to an arrow from semicircle to semicircle and a set of horospherical cylinders to an arrow from semicircle to square. The generalization of the graphical method to the n -dimensional Lorentz group is straight-forward, but further symbols will be necessary.

The group theoretical origin of the elliptical type coordinates, allowing separation, has not been considered. However, they can doubtlessly be obtained by a more detailed study of the infinitesimal group ring of the proper Lorentz group, e.g. by investigating all mutually non conjugated pairs of expressions, quadratic in the group generators. This question will be considered in a separate paper. Here we shall discuss the analogous but considerably simpler case of the group of motions of a Euclidean plane.

IV. Group of motions of a Euclidean plane and elliptical coordinate systems

It is well-known[9], that the coordinates in the equation

$$\Delta \psi ' = \lambda \psi \qquad (16)$$

where Δ is the two-dimensional Laplace operator, can be separated in four types of coordinate systems - cartesian, polar, parabolic and elliptical. The group E_2 of motions of a Euclidean plane is just the group of plane transformations, leaving equation (16) invariant[x)]. Its Lie algebra is determined by

$$[P_1, P_2] = 0 \qquad [P_2 A] = P_1 \qquad [A P_1] = P_2 \qquad (17)$$

and the invariant of E_2 is just $\Delta = P_1^2 + P_2^2$.

Let us prove the statement: A one - to-one correspondence can be established between the set of all linear self-adjoint operators L_k, being homogeneous quadratical polynomials in the infinitesimal operators of E_2, and coordinate systems K, in which the variables separate. The condition that the operator L_k should be diagonal on a system of functions separated in K, determines L_K uniquely (except for a linear combination with Δ). Similarly as for the Lorentz group, operators which are invariants of subgroups of E_2 (translations or rotations) correspond to coordinate systems with one geometrical centre. As K runs through all coordinate systems allowing separation, L_K runs through all linear self-adjoint second order differential operators, commuting with Δ .

Note that:

1. A linear differential operator commutes with Δ if and only if it is a polynomial in A, P_1, P_2 and is self-adjoint, if it is symmetrical in its (non-commuting) variables.

2. The coordinate systems K and K' are equivalent from the point of view of variable separating, if they are connected by a transformation belonging to E_2. In such a case L_K and $L_{K'}$ are conjugated. Hence we can restrict ourselves to mutually non-conjugated operators. The proof of our statement will be given in two steps. Firstly let us enumerate all coordinates systems and the corresponding operators L_K .

a) Cartesian coordinates. The operator P_1^2 is diagonal (together with

---

x) In general E_2 also contains inversions, but we shall not consider them here. However, they are important, since they are necessary to eliminate the remaining degeneracy in the parabolic and elliptical systems.

$P_2^2 = \Lambda - P_1^2$), where P_1 and P_2 are invariants of the translation subgroup).

b) Polar coordinates. The operator A^2 is diagonal (where Λ is the invariant of the one-dimensional rotation subgroup).

c) Parabolic coordinates.

$$x = \tfrac{1}{2} (\xi^2 - \eta^2) \qquad y = \xi \eta$$

The operator

$$P = A P_2 + P_2 A = \frac{1}{\xi^2 + \eta^2} \left(\eta^2 \frac{\partial^2}{\partial \xi^2} - \xi^2 \frac{\partial^2}{\partial \eta^2} \right) \tag{18}$$

is diagonal.

d) Elliptical coordinates

$$x = \ell \xi \eta \qquad y = \ell \sqrt{(\xi^2 - 1)(1 - \eta^2)}$$

(where $\ell > 0$ is the focus distance).

The diagonal operator is

$$E_\ell = A_3^2 - \frac{\ell^2}{2} (P_2^2 - P_1^2) = \frac{1}{\xi^2 - \eta^2} \left\{ (\xi^2 - 1) \eta^2 \frac{\partial^2}{\partial \xi^2} + \xi^2 (1 - \eta^2) \frac{\partial^2}{\partial \eta^2} + \xi \eta^2 \frac{\partial}{\partial \xi} - \xi^2 \eta \frac{\partial}{\partial \eta} \right\} \tag{19}$$

Thus a definite linear operator[x)] corresponds to each "separating" coordinate system. We shall prove in Appendix III that these operators exhaust all symmetrical second order polynomials in the generators of E_2 , i.e. that any such polynomial is conjugated to one of P_1^2 (or equivalently $P_1 P_2$) A^2 , P , E_ℓ (or a combination of one of them with Λ).

V. On the physical meaning of the diagonal operators

In chapter III we have constructed various complete sets of commuting operators from the invariants of the Lorentz group and its subgroups. Loosely speaking, certain physical quantities, quantum numbers etc. should correspond to these sets. The connection between such invariants and relativistic angular momentum theory was considered in /2/. Classical integrals of motion corresponding to the subgroup invariants were constructed and electromagnetic fields were discussed in which these integrals are conserved. We shall not go into these questions here.

x) Writing P and E_ℓ explicitly in cartesian coordinates, it is easy to see that they, similarly as P_1^2 and A, are self-adjoint.

To clarify the physical meaning of the quadratic polynomials introduced in IV, let us consider the parabolic coordinates (it is of course evident that P_1, P_2 and A correspond to linear and angular momentum respectively). Besides $P = AP_2 + P_2A \equiv L_1$ we introduce $L_2 = AP_1 + P_1A$. It is easy to see that

$$[L_1, L_2] = 4\Delta A \qquad [L_2, A] = -L_1 \qquad [A, L_1] = -L_2 \tag{20}$$

Considering a definite representation of the E_2 group, we can put $\Delta = E = \text{const}$ (energy of two-dimensional free motion). Putting $R_i = \frac{L_i}{\sqrt{4|E|}}$, $i = 1,2$; $R_3 = -A$ we see that R_k realize the algebra of the three-dimensional rotation group for $E < 0$ and of the three-dimensional Lorentz group for $E > 0$. Thus we obtain a new "higher" symmetry, not contained in E_2. It is well known that the Coulomb interaction conserves this symmetry. In the Coulomb field this leads, in classical mechanics, to the conservation of a typical integral of motion (the Laplace-Lenz vector) and to closed (elliptical) orbits, in quantum mechanics - to the additional degeneracy of the (two-dimensional) "hydrogen atom".

Note that the situation is quite analogous in the three-dimensional case. The components of the Laplace-Lenz vector

$$\vec{L} = \tfrac{1}{2}(\vec{p} \times \vec{M} - \vec{M} \times \vec{p}) + m\,a\frac{\vec{r}}{r} \tag{21}$$

(where $\vec{p}$ and $\vec{M}$ are the linear and angular momenta, m the mass and a the constant in the Coulomb potential) together with those of $\vec{M}$ form the generators of the four-dimensional rotation group (or Lorentz group). The consequences of this symmetry are analogous to those in the two-dimensional case (cf./10/).

The diagonality of one of the components of $\vec{L}$ ($a = 0$) leads to the separation of variables in the equation $\Delta\psi = \lambda\psi$ (Δ - three-dimensional Euclidean Laplace operator) in parabolic coordinates.

The authors thank Ja.A.Smorodinsky, who initiated this investigation, for his constant interest, encouragement and helpful discussions.

APPENDIX I. Proof of the conjugation criterion.

Let $C_i = a_{i,k}A_k + b_{i,k}B_k$, $i = 1,2$ (here and further we sum from 1 to 3 over repeated indices) be two elements of $\mathfrak{L}$. We introduce the functions

$$\delta_1(C) = a_k^2 - b_k^2 \qquad \delta_2(C) = 2a_k b_k \tag{22}$$

We consider three possibilities:

1. At least one of the expressions $\delta_1(C_1)$, $\delta_2(C_1)$ differs from zero. Then C_2 is conjugated to C_1 if and only if

$$\delta_1(C_1) = \delta_1(C_2) \qquad \delta_2(C_1) = \delta_2(C_2) \tag{23}$$

2. $\delta_1(C_1) = \delta_2(C_1) = 0$ but $C_1 \neq 0$. Then C_2 is conjugated to C_2 if it satisfies (23) and also $C_2 \neq 0$.

3. $C_1 \equiv 0$ - conjugated only to itself.

Proof: a) Necessity. We have shown that two elements C_1 and C_2 are conjugated if a complex orthogonal matrix $\mho$ exists, for which

$$\mho \,\tilde{\tilde{c}}_1 = \tilde{\tilde{c}}_2 \tag{24}$$

where

$$\tilde{\tilde{c}} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} a_1 + i\, b_1 \\ a_2 + i\, b_2 \\ a_3 + i\, b_3 \end{pmatrix} \tag{25}$$

It follows from $\mho^T \mho = E$ that the transformation (24) conserves the "length"

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \delta_1(C) + i\,\delta_2(C) = \delta(C) \tag{26}$$

i.e. both its real and imaginary part.

b) Sufficiency. Let $\tilde{\tilde{c}}$ be given by (25) and $\delta(C) \neq 0$. The relation

$$\tilde{\tilde{c}} = 0\, \tilde{\tilde{c}}_1 \qquad \text{where} \qquad \tilde{\tilde{c}}_1 = \begin{pmatrix} \sqrt{\delta} \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

is then satisfied for any orthogonal matrix of the type

$$\mho = \begin{pmatrix} \frac{\gamma_1}{\sqrt{\delta}} & \cdot & \cdot \\ \frac{\gamma_2}{\sqrt{\delta}} & \cdot & \cdot \\ \frac{\gamma_3}{\sqrt{\delta}} & \cdot & \cdot \end{pmatrix}. \tag{28}$$

Thus any vector $\tilde{\tilde{c}}$ with $\delta(C) \neq 0$ is conjugated to the vector $\tilde{\tilde{c}}_1$ (we must chose the half-plane in which we take the square root).

Now let $\delta(C) = 0$, but $\tilde{\tilde{c}} \neq 0$, i.e. at least one $\gamma_i \neq 0$ (say $\gamma_1 \neq 0$).

The matrix

$$\mho = \begin{pmatrix} \frac{\gamma_1(\gamma_1^2 + 1)}{2\gamma_1^2}, & i\,\frac{\gamma_1(\gamma_1^2 - 1)}{2\gamma_1^2}, & 0 \\ \frac{\gamma_2(\gamma_1^2 - 1)}{2\gamma_1^2}, & i\,\frac{\gamma_2(\gamma_1^2 + 1)}{2\gamma_1^2}, & -i\,\frac{\gamma_3}{\gamma_1} \\ \frac{\gamma_3(\gamma_1^2 - 1)}{2\gamma_1^2}, & i\,\frac{\gamma_3(\gamma_1^2 + 1)}{2\gamma_1^2}, & i\,\frac{\gamma_2}{\gamma_1} \end{pmatrix} \tag{29}$$

is orthogonal and satisfies the relation

$$\tilde{\tilde{c}} = \mathcal{O}\, \tilde{\tilde{c}}_2 \qquad \text{with} \qquad \tilde{\tilde{c}}_2 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{30}$$

Thus all $\tilde{\tilde{c}} \neq 0$ with $\delta(C) = 0$ are conjugated to $\tilde{\tilde{c}}_2$.

APPENDIX II. Subgroups of the Lorentz group with no invariants.

a) A real Lie algebra A has an invariant if and only if its complex extansion A^* has one. This condition is evidenly necessary, since $A \subset A^*$ and the condition $[f, e_k] = 0$ implies $[f, i e_k] = 0$. Let us prove that it is sufficient. Let f be an invariant of A^* i.e. $[f, e_k] = 0$ for $e_k \in A^*$. It follows that $[f, e_k] = \overline{[\bar{f}, \bar{e}_k]} = 0$ Hence $f + \bar{f}$ if an invariant and $f + \bar{f} \in A$.

b) Let us prove the following lemma:

Let A be a Lie algebra (possibly complex) with the generators $e_0, \ldots, e_n$ satisfying

$$[e_k, e_\ell] = 0 \qquad \text{for} \qquad k \neq 0, \ell \neq 0$$

$$[e_k, e_0] = \lambda_k e_k \qquad \text{for} \qquad k \neq 0 \tag{31}$$

The algebra A has an invariant, if and only if non negative integer numbers $m_1, \ldots, m_n$, not all equal to zero, exist such that

$$\Sigma \lambda_k m_k = 0 \tag{32}$$

We shall need the lemma only for $n = 1,2$. Let us prove it for $n = 2$ (the proof for general n is analogous). We exclude the trivial case $\lambda_1 = \ldots = \lambda_n = 0$. Any element of the universal algebra over A can be uniquely written as

$$f = \sum_{k_1, k_2} P_{k_1 k_2}(e_0)\, e_1^{k_1}\, e_2^{k_2} \tag{33}$$

where $P_{k_1 k_2}(x)$ is a polynomial.

We have

$$[f, e_1^s] = \Sigma\, (P_{k_1 k_2}(e_0) - P_{k_1 k_2}(e_0 - \lambda_1 s))\, e_1^{k_1 + s}\, e_2^{k_2} \tag{34}$$

and an analogous formula for $[f, e_2^s]$.

$$[f, e_0] = \Sigma\, P_{k_1 k_2}(e_0)(\lambda_1 k_1 + \lambda_2 k_2)\, e_1^{k_1}\, e_2^{k_2} \tag{35}$$

Let us prove relations (34), (35). It follows from the linearity of the commutators, that we can limit ourselves to the case $f = e_0^{k_0}\, e_1^{k_1}\, e_2^{k_2}$. It is easy to verify by induction that

$$e_k^r\, e_0 = (e_0 + r\lambda_k)\, e_k^r \tag{36}$$

Really $e_k e_0 = e_0 e_k + \lambda_k e_k$

$e_k^r e_0 = e_k(e_k^{r-1} e_0) = e_r(e_0 + (r-1)\lambda_k) e_k^{r-1} = (e_0 e_k + \lambda_k e_k + (r-1)\lambda_k e_k) e_k^{r-1} = (e_0 + r\lambda_k) e_k^r \qquad k = 1,2$

Thus we obtain $e_0^{k_0} e_1^{k_1} e_2^{k_2} e_0 = e_0^{k_0} e_1^{k_1} (e_0 + k_2\lambda_2) e_2^{k_2} = (e_0 + k_1\lambda_1 + k_2\lambda_2) e_0^{k_0} e_1^{k_1} e_2^{k_2}$ which proves (35). The relation $e_0^s e_1^r = (e_0 + s\lambda_1)^r e_0^s$ proving (34) can also be verified by induction.

Now let $f = \Sigma P_{k_1 k_2}(e_0) e_1^{k_1} e_2^{k_2}$ be an invariant. We have $0 = [f, e_0] = \Sigma P_{k_1 k_2}(e_0)(\lambda_1 k_1 + \lambda_2 k_2) e_1^{k_1} e_2^{k_2}$ and hence $P_{k_1 k_2} = 0$ if $k_1\lambda_1 + k_2\lambda_2 \neq 0$. Further we have $0 = [f, e_1] = \Sigma(P_{k_1 k_2}(e_0) - P_{k_1 k_2}(e_0 - \lambda_1) e_1^{k_1+1} e_2^{k_2})$ and hence for $\lambda_1 \neq 0$ (and similarly for $\lambda_2 \neq 0$) all the polynomials $P_{k_1 k_2}$ must be constants. Since λ_1 and λ_2 cannot be zero simultaneously, f does not depend on e_0. Thus $e_1^{k_1} e_2^{k_2}$ is an invariant if $\lambda_1 k_1 + \lambda_2 k_2 = 0$. If no such k_1, k_2 exist, the algebra has no invariants. This proves the lemma.

c) Let us consider those algebras of chapter II, for which we have asserted, that they have no invariants. Algebra 2b) satisfies the conditions of the lemma with $n = 1, \lambda = 1$. Since $k\lambda = 0$ cannot be satisfied by any $k > 0$ the algebra has no invariants.

Algebras 3c) can be complexly extended, written as

$$[A', B'] = 0, \quad [A', C'] = (\cos\alpha + i\sin\alpha) A', \quad [B' C'] = (-\cos\alpha + i\sin\alpha) B'.$$

and they satisfy the conditions of the lemma. We have $0 = k_1\lambda_1 + k_2\lambda_2 = i(k_1 + k_2)\sin\alpha + (k_1 - k_2)\cos\alpha$. This can be satisfied only for $\alpha = 0$ and hence of all groups of the type 3c) only E_2 has an invariant. Algebra 4). The conditions $[C B] = C$ and $[D B] = D$ imply $[C^n, B] = n C^n$, $[D^n, B] = n D^n$ Let us write the possible invariant as

$$f = \sum_{k,\ell} P_{k\ell}(A, B) C^k D^\ell \tag{37}$$

We obtain $0 = [f, B] = \sum_{k,\ell} P_{k\ell}(A, B)(k + \ell) C^k D^\ell$ and we see, that $P_{k\ell} \neq 0$ implies $k = \ell = 0$. It follows that the invariant must be a polynomial in A and B. Put $E = C + iD$. We have $[A B] = 0$, $[A, E] = iE$, $[B, E] = -E$. It follows from the lemma that the algebra A,B,E has no invariant, since $m_1 i - m_2 = 0$ cannot be satisfied by real m_1, m_2. Still less can a function of only A and B be an invariant of the whole group.

APPENDIX III. Symmetrical quadratic polynomials in the generators of the group E_2.

We shall consider non commutative polynomials of the type

$$f = a A^2 + b_1(A P_1 + P_1 A) + b_2(A P_2 + P_2 A) + c_1 P_1^2 + 2 c_2 P_1 P_2 + c_3 P_2^2 \tag{38}$$

We call two polynomials f_1 and f_2 equivalent if $f_2 = \lambda f_1 + \mu \Delta$ where $\lambda \neq 0$ and $\Delta = P_1^2 + P_2^2$ As mentioned in chapter IV, equivalent polynomials define the same coordinate system. Let us denote:

$$\begin{aligned} A(f) &= a \\ B(f) &= b_1^2 + b_2^2 - a(c_1 + c_3) \\ C(f) &= (c_1 - c_3)^2 + 4c_2^2 \\ D(f) &= b_1^2 c_3 - 2b_1 b_2 c_2 + b_2^2 c_1 - a(c_1 c_3 - c_2^2) \end{aligned} \tag{39}$$

Every polynomial f is equivalent to one of the following:

α) Let $A(f) \neq 0$. Put $\lambda = \dfrac{1}{A(f)}$, $\mu = \dfrac{B(f)}{2A^2(f)}$.

We obtain an equivalent polynomial with

$$A(f_2) = 1, \qquad B(f_2) = 0 \tag{40}$$

β) Let $A(f) = 0$, $B(f) \neq 0$. Put $\lambda = [B(f)]^{-1/2}$, $\mu = -D(f)[B(f)]^{-3/2}$.

We obtain an equivalent polynomial with

$$A(f_2) = 0, \qquad B(f_2) = 1, \quad D(f_2) = 0 \tag{41}$$

γ) Let $A(f) = B(f) = 0$ (which implies also $D(f) = 0$) and $C(f) \neq 0$. Put $\lambda = [C(f)]^{-1/2}$, $\mu = -\dfrac{c_1 + c_3}{2\sqrt{C(f)}}$ For the equivalent polynomial we have

$$A(f_2) = B(f_2) = D(f_2) = 0 \qquad C(f_2) = 1 \qquad c_1 + c_3 = 0 \tag{42}$$

We shall not consider the trivial class of polynomials equivalent to 0. Polynomials satisfying (40), (41), or (42) will be called normal (every polynomial is equivalent to one of the normal ones). Now we shall classify equivalent polynomials with respect to conjugacy (the operator Λ is conjugated only to itself and hence conjugation does not violate equivalence). We now prove:

The necessary condition for two classes of equivalent polynomials $\{f_1\}$ and $\{f_2\}$ to be conjugated is that both must belong to the same type α), β) or γ). This condition is also sufficient for classes β) and γ); for the class α) a further condition must be fulfilled:

$\ell^4(f_1) = \ell^4(f_2)$ where

$$\ell^4(f) = \frac{4[b_1 b_2 - a c_2]^2 + [b_1^2 - b_2^2 - a(c_1 - c_3)]^2}{a^4} \tag{43}$$

Proof. 1) Necessity. Any internal automorphism in E_2 is determined by three parameters x, y, ϕ and the relations

$$A \to A + x P_1 + y P_2 , \quad P_1 \to \cos\phi \, P_1 + \sin\phi \, P_2 , \quad P_2 \to -\sin\phi \, P_1 + \cos\phi \, P_2 \qquad (44)$$

Putting (44) into (38) we see that f goes over into the conjugated polynomial f' with the coefficients

$$\begin{aligned}
a' &= a \\
b_1' &= ax + b_1 \cos\phi - b_2 \sin\phi \\
b_2' &= ay + b_1 \sin\phi + b_2 \cos\phi \qquad (45)\\
c_1' &= ax^2 + 2x (b_1 \cos\phi - b_2 \sin\phi) + c_1 \cos^2\phi - 2 c_2 \cos\phi \sin\phi + c_3 \sin^2\phi \\
c_2' &= axy + x (b_1 \sin\phi + b_2 \cos\phi) + y (b_1 \cos\phi - b_2 \sin\phi) + (c_1 - c_3) \cos\phi \sin\phi + c_2 \cos 2\phi \\
c_3' &= ay^2 + 2 y (b_1 \sin\phi + b_2 \cos\phi) + c_1 \sin^2\phi + 2 c_2 \cos\phi \sin\phi + c_3 \cos^2\phi \, .
\end{aligned}$$

The invariance of a implies that conjugation conserves the type α). For $a = a' = 0$ we have $b_1'^2 + b_2'^2 = b_1^2 + b_2^2$ and the type (β) is conserved. For $a = b_1 = b_2 = 0$ we have $C(f') = C(f)$ and the type γ) is also conserved. It can also be directly verified that $\ell^4(f)$ is invariant with respect to conjugation.

2) Sufficiency. We shall prove that any polynomial f is conjugated to a certain "canonical" polynomial, which we choose in the following form for the considered types:

Type α) $\quad E_\ell = A^2 + \frac{\ell^2}{2}(P_1^2 - P_2^2)$ (various $\ell \geq 0$ imply mutually non-conjugated f)

Type β) $\quad P = A P_2 + P_2 A \qquad (46)$

Type γ) $\quad P_1 P_2$

For the type α) we have

$$\ell^4 = 4 (b_1 b_2 - c_2)^2 + [(b_1^2 - c_1) - (b_2^2 - c_3)]^2 = 4 [(b_1 b_2 - c_2)^2 + (b_1^2 - c_1)^2] .$$

1) Let $\ell \neq 0$. Putting

$$\cos 2\phi = \frac{2}{\ell^2} (c_1 - b_1^2) , \qquad \sin 2\phi = \frac{2}{\ell^2} (c_2 - b_1 b_2)$$

we see that an automorphism with the parameters b_1 b_2 and ϕ connects E_ℓ with a general polynomial f satisfying (40).

2) Let $\ell = 0$, i.e. $b_1 b_2 = c_2$, $b_1^2 = c_1$, $b_2^2 = c_3$

The general f of this type can be written as

$$f = (A + b_1 P_1 + b_2 P_2)^2$$

and is obtained from A^2 by an automorphism with $x = b_1$, $y = b_2$ and ϕ arbitrary.

For type β) we put

$$-\sin\phi = b_1 , \cos\phi = b_2 , \qquad x = \frac{c_1}{2b_1} , \ y = \frac{c_3}{2b_2}$$

The corresponding automorphism transforms P into the general polynomial satisfying (41).

For type γ) we put

$$\cos 2\phi = 2c_2 , \qquad \sin 2\phi = 2c_3 , \qquad x,y \text{ - arbitrary}$$

and thus transform $P_1 P_2$ into the general expression satisfying (42).

This completes the proof.

References

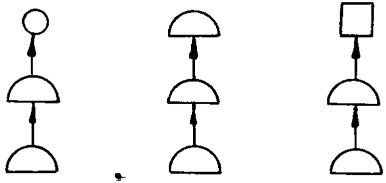
1. Н.Я. Виленкин, Я.А. Смородинский. ЖЭТФ 46, 1793 (1964).
2. Ja.A.Smorodinsky, M.Uhlř, P.Winternitz, JINR E-1591 and P-1690, Dubna, 1964.
3. М.Н. Олевский. Математический сборник 27, 379 (1950).
4. Л.Д. Ландау, Е.М. Лифшиц. Механика ФМ, Москва, 1958.
5. И.М. Гельфанд. Математический сборник 26, 103 (1950).
6. Н.Г. Чеботарев. Теория групп Ли. Гостехиздат, М.Л., 1940.
7. C.Chevalley. Theory of Lie groups. I, Princeton Univ. Press. New Jersey, 1946. Russian transl.: К. Шевалле. Теория групп Ли, 1, ИЛ, М., (1948).
8. Г.М. Мубаракзянов. Изв. вузов, Матем. 1 (32), 113 (1963).
9. P.M.Morse, H.Feshbach. Methods of Theoretical Physics, I, (1953).McGraw-Hill N.Y.

   Russian Translation: Ф.М. Морс, Г. Фешбах, Методы теоретической физики, 1, ИЛ., 1958.
10. V.A.Fock, Zs. f. Phys. 98, 145 (1935).

Received by Publishing Department
on October 20, 1964.

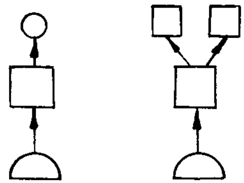
Fig.1

Fig.2



a) b) c)

Fig.3



a) b)

Fig.4