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COMMUTATOR OF THE SCALAR FIELD IN THE TWO-DIMENSIONAL PSEUDO-RIEMANNIAN SPACE-TIME MODEL

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Recently a series of papers [1-3] has appeared which are devoted to the construction of quantum field theory in the external gravitational field. However, in these papers the gravitational field is assumed to be either weak or satisfying fairly strong special requirements. In the present work the two-dimensional model of an arbitary pseudo-Riemannian space-time is considered and an explicit expression for the scalar field commutator is found.

Let us first consider the true case of the four-dimensional pseudo-Riemannian space-time. In accordance with the flat-space-time case we have to solve the Cauchy problem for the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}\,g^{\alpha\beta}\frac{\partial\psi}{\partial x^{\beta}}\right)+m^{2}\psi=0 \qquad (1)$$

provided that on a certain space-like hypersurface Σ one specifies the function $\psi(x)$ and its derivative in the direction of the normal to Σ. On the hypersurface Σ the function $\psi(x)$ is a operator obeying conditions

$$[\psi(M_1),\psi(M_2)]_{M_1,M_2\in\Sigma}=0,$$

$$[n^{\alpha}(M_1)\psi_{\alpha}(M_1),\, n^{\beta}(M_2)\psi_{\beta}(M_2)]_{M_1,M_2\in\Sigma}=0, \qquad (2)$$

$$\int_{\Sigma} f(M)[\psi(M_1),\psi_{\alpha}(M)]_{M_1\in\Sigma}\,d\sigma^{\alpha}(M)=i f(M_1),$$

where $\psi_{\alpha}=\frac{\partial\psi}{\partial x^{\alpha}}$, $n^{\alpha}(M)$ is the normal to Σ and $f(M)$ is an arbitrary function[x)].

After solving the Cauchy problem we can calculate the commutator

$$D(M_1,M_2)=i[\psi(M_1),\psi(M_2)]; \qquad (3)$$

here M_1 and M_2 are the arbitrary space-time points.

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x) In the general case of $n+1$ - dimensional space

$$\psi_{\alpha}\,d\sigma^{\alpha}=\sqrt{-g}\begin{vmatrix}\psi^{0} & \psi^{1} & \dots & \psi^{n}\\ d_1x^{0} & d_1x^{1} & \dots & d_1x^{n}\\ \dots & \dots & \dots & \dots\\ d_nx^{0} & d_nx^{1} & \dots & d_nx^{n}\end{vmatrix}, \quad \psi^{\alpha}=g^{\alpha\beta}\psi_{\beta}.$$

In the two-dimensional case the problem in question can be explicitly solved by the Riemann method. In this case any metric form of the space-time can be written as

$$ds^2 = 4a^2(x,y)\,dx\,dy\,. \tag{4}$$

The coordinates x , y are called isotropic; the "future" with respect to the point $M_0(x_0, y_0)$ is defined by the condition

$$(x-x_0)(y-y_0) > 0, \quad x+y > x_0+y_0\,. \tag{!5}$$

The Klein-Gordon equation in the isotropic coordinates is written as follows

$$\frac{\partial^2\psi}{\partial x\,\partial y} + a^2(x,y)m^2\,\psi = 0\,. \tag{6}$$

Now we have to find the solution for eq. (6) under the condition that on the curve

$$y = \mu(x), \quad \mu' < 0 \tag{7}$$

$\psi = \phi(x)$ and $\left[\frac{\partial\psi}{\partial x} - \mu'\frac{\partial\psi}{\partial y}\right]_{y=\mu(x)} = \pi(x)$ are given. In accordance with (2) the operators $\phi(x)$ and $\pi(x)$ obey the commutation relations

$$[\phi(x_1), \phi(x_2)] = 0, \quad [\pi(x_1), \pi(x_2)] = 0,$$
$$[\phi(x_1), \pi(x_2)] = i\delta(x_1 - x_2)\,, \tag{8}$$

since on the curve (7) $\psi_a\, d\sigma^a = \pi(x)\,dx$.

The Riemann method is based on the Green formula

$$A = \oint_\Sigma d\sigma^a (v\psi_a - \psi v_a) = \int_V dV\,(v\nabla_a\psi^a - \psi\,\nabla_a v^a)\,; \tag{9}$$

here $_a$ is the symbol of the covariant differentiation and dV is the invariant element of the volume of region V . If ψ and v obey the Klein-Gordon equation, then the integrand in the last integral vanishes and, consequently, for such ψ and v

$$A = \oint_\Sigma d\sigma^a (v\psi_a - \psi v_a) = 0\,, \tag{10}$$

In the two-dimensional case we apply this formula to the contour $P_0 Q_0 M_0$ plotted in Fig. 1, where $M_0(x_0, y_0)$ is an arbitrary point. We have

(11)

$$A = \int_{Q_0}^{M_0} (v\psi_y - \psi v_y)\big|_{x=x_0} dy + \int_{M_0}^{P_0} (\psi v_x - v\psi_x)\big|_{y=y_0} dx + \int_{P_0}^{Q_0} (\psi v_n - v\psi_n)\big|_{y=\mu(x)} dx = 0, \tag{11}$$

where we introduce the notation $a_n = a_x - \mu' a_y$.

We choose for $v = v(x,y;x_0,y_0) = v_0(x,y)$ the Riemann function being the solution for the Goursat problem:

$$\frac{\partial^2 v_0(x,y)}{\partial x \partial y} + a^2(x,y) m^2 v_0(x,y) = 0, \tag{12}$$

$$v_0\big|_{x=x_0} = v_0\big|_{y=y_0} = 1.$$

Then the formula (11) yields

$$\psi(M_0) = \tfrac{1}{2}\psi(Q_0) + \tfrac{1}{2}\psi(P_0) + \tfrac{1}{2}\int_{P_0}^{Q_0} (v_0\psi_n - \psi v_{0n})\big|_{y=\mu(x)} dx. \tag{13}$$

Thus, we have expressed the function obeying the Klein-Gordon equation in terms of its value and the value of its normal derivative on the curve $y=\mu(x)$. In particular, the field operator at the point $M_0(x_0, y_0)$ is

$$\psi(M_0) = \tfrac{1}{2}\phi(x_0) + \tfrac{1}{2}\phi(\mu^*(y_0)) + \tfrac{1}{2}\int_{\mu^*(y_0)}^{x_0} dx\,\{v_0(x,\mu(x))\pi(x) - v_{0n}(x,\mu(x))\phi(x)\}, \tag{14}$$

where $x=\mu^*(y)$ is the function inverse to that $y=\mu(x)$.

Putting in (13) $\psi(M_0) = v(M_0;M_1)$ we get an identity for the function $v(M_0;M_1)$

$$2v(M_0;M_1) = v(Q_0;M_1) + v(P_0;M_1) + \int_{P_0}^{Q_0} dx \begin{vmatrix} v(M;M_0) & v(M;M_1) \\ v_n(M;M_0) & v_n(M;M_1) \end{vmatrix}_{y=\mu(x)}. \tag{15}$$

Using eq. (11) it is not difficult to prove one more identity for the function $v(M_1;M_2)$. Let in (11) the point M_0 possess the coordinates $x_0 = x_2$, $y_0 = y_1$. Then the point P_0 coincides with the point P_1, and the point Q_0 with Q_2. Assuming further $\psi(M) = v(M;M_1)$, $v(M) = v(M;M_2)$ we find

$$v(Q_2;M_1) - v(P_1;M_2) = \int_{P_1}^{Q_2} dx \begin{vmatrix} v(M;M_1) & v(M;M_2) \\ v_n(M;M_1) & v_n(M;M_2) \end{vmatrix}_{y=\mu(x)} \qquad (16)$$

From (15) and (16) it follows that the Riemann function is symmetrical:

$$v(M_1;M_2) = v(M_2;M_1). \qquad (17)$$

The obtained results allow one to find the commutator $D(M_1,M_2)$. In calculating it one needs to commute the operators of the form

$$A_i = \phi(a_i) + \phi(b_i) + \int_{a_i}^{b_i} dx \{ p_i(x)\pi(x) + q_i(x)\phi(x) \} \qquad (18)$$

$$(i = 1,2);$$

without loss of generality it may be assumed that $b_i > a_i$. The three cases are possible.

In the first case intercepts $[a_1, b_1]$ and $[a_2, b_2]$ are not overlapped and then, obviously, $[A_1, A_2] = 0$.

In the second case these intercepts are overlapped partially; assuming $a_2 > a_1$ we get

$$i[A_1(M_1), A_2(M_2)] = \qquad (19)$$

$$= p_1(a_2) - p_2(b_1) + \int_{a_2}^{b_1} dx \begin{vmatrix} p_1(x) & p_2(x) \\ q_1(x) & q_2(x) \end{vmatrix}.$$

In the third case one of the intercepts, e.g. $[a_1, b_1]$, contains the whole another. Then

$$i[A_1(M_1), A_2(M_2)] = \qquad (20)$$

$$= p_1(a_2) + p_1(b_2) + \int_{a_2}^{b_2} dx \begin{vmatrix} p_1(x) & p_2(x) \\ q_1(x) & q_2(x) \end{vmatrix}.$$

Using the expresion of the form (19) and (20) obtained by an immediate commutation of $\psi(M_1)$ and $\psi(M_2)$, and on the other hand, applying the identities (15), (16), (17) for the Riemann function, it is not difficult to get that

$$D(M_1,M_2) = \tfrac{1}{2}\epsilon(M_1,M_2)v(M_1;M_2), \qquad (21)$$

where

$$\epsilon(M_1,M_2) = \begin{cases} 1, & \text{if the point } M_1 \text{ is in "the future" with respect to } M_2; \\ -1, & \text{if the point } M_2 \text{ is in "the future" with respect to } M_1; \\ 0, & \text{if the points } M_1 \text{ and } M_2 \text{ are space-like with respect to one another.} \end{cases}$$

In isotropic coordinates $\epsilon(M_1,M_2)$ can be expressed in terms of the function

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

in the following way:

$$\epsilon(M_1,M_2) = \theta(x_1-x_2)\theta(y_1-y_2) - \theta(x_2-x_1)\theta(y_2-y_1). \qquad (22)$$

By differentiating (21) it may be shown that not only on the curve $y=\mu(x)$ but on any space-like curve $y=\tilde{\mu}(x)$ the operators $\tilde{\pi}(x)=\left[\frac{\partial\psi}{\partial x}-\mu'\cdot\frac{\partial\psi}{\partial y}\right]_{y=\tilde{\mu}(x)}$ and $\tilde{\phi}(x)=\psi(x),\tilde{\mu}(x))$ satisfy the commutation relations (8).

In conclusion we note that the Riemann function, as it follows from (9), is the solution of the integral equation

$$v(x,y;x_0,y_0) = 1 - m^2\int_{x_0}^{x} d\xi \int_{y_0}^{y} d\eta\, a^2(\xi,\eta)v(\xi,\eta;x_0,y_0). \qquad (23)$$

Solving this equation by iterations we get an expression for $v(x,y;x_0,y_0)$ in the form of the convergent series

$$v(x,y;x_0,y_0) = \qquad (24)$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n m^{2n}\int_{x_0}^{x} d\xi \int_{y_0}^{y} d\eta_1\, a^2(\xi_1,\eta_1)\int_{x_0}^{\xi_1} d\xi_2 \int_{y_0}^{\eta_1} d\eta_2\, a^2(\xi_2,\eta_2)\ldots\int_{x_0}^{\xi_{n-1}} d\xi \int_{y_0}^{\eta_{n-1}} d\eta_n\, a^2(\xi_n,\eta_n).$$

In the flat space-time case $a^2(x,y)=1$ we obtain from (24)

$$v(x,y;x_0,y_0) = J_0\left(2m\sqrt{(x-x_0)(y-y_0)}\right), \qquad (25)$$

where J_0 is the Bessel function. In this case the expression $4(x-x_0)(y-y_0)$ is the squared distance between the points M and M_0.

Note that for $m^2=0$ the obtained results entirely coincide with analogous ones for the flat two-dimensional case. This is due to the fact that for $m^2=0$ the function $a^2(x,y)$ fall out of eq. (9). In other words, in two-dimensional model the curvature of space does not affect zero mass particles.

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