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ON AN INVESTIGATION OF THE ANALYTIC PROPERTIES OF SCATTERING AMPLITUDES IN THE NONRELATIVISTIC THREE-BODY PROBLEM

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1. Introduction

Great attention has been recently paid to the study of the analytic properties of the nonrelativistic three-body scattering amplitudes in the complex l -plane[1-4].

There exist several methods for separating the total angular momentum in the three-body problem. The first one suggested by Newton[1] consists in separating the two relative orbital momenta and combining them in order to get the total angular momentum with the aid of the Clebsch-Gordan coefficients. Difficulties arising in the analytic continuation of the partial scattering amplitude obtained by this method have been discussed in [2,4].

Another method of introduction of the total angular momentum has been suggested in Omnes's papers[2]. The merit of this method is demonstrated in investigating the three-body problem in the absence of the bound states.

In the following we present a recipe of introducing the total angular momentum which is convenient to study the problems of scattering on the bound state. Notice that this method is easily generalized to the $N(N>3)$ body problem where one can use, e.g., the Weinberg equations.

2. The kinematics of the three-body system

Let us consider the problem of scattering of a particle with mass m_1 on the bound state of two other particles with masses m_2 and m_3 . The particles interact through the two-body spherically-symmetrical potentials.

The state of the three-body system can be characterized by the three momenta $\vec{k}_1, \vec{k}_2, \vec{k}_3$. In place of these it is convenient to introduce momenta corresponding to the well-known Jacobi coordinates: $(\vec{K}, \vec{k}_{23}, \vec{p}_1)$

$$\vec{K} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3$$
$$\vec{k}_{23} = \frac{m_3 \vec{k}_2 - m_2 \vec{k}_3}{m_{23}} \tag{2.1}$$
$$\vec{p}_1 = \frac{1}{M} \{ m_1 (\vec{k}_2 + \vec{k}_3) - m_{23} \vec{k}_1 \}$$

where $\vec{K}$ is the total momentum of the system, $\vec{k}_{23}$ is the relative momentum of particles 2 and 3, $\vec{p}_1$ is the momentum of the particle 1 with respect to the center-of-mass of the two other particles, M is the total mass of the system

$$M = m_1 + m_2 + m_3 \qquad (2.2)$$

and

$$m_{23} = m_2 + m_3 \qquad (2.3)$$

Other possible sets of variables, $(\vec{K}, \vec{k}_{31}, \vec{p}_2)$ and $(\vec{K}, \vec{k}_{12}, \vec{p}_3)$ are determined in a similar way.

In the center of mass system we have

$$\vec{K} = 0$$
$$\vec{k}_{23} = \frac{m_3 \vec{k}_2 - m_2 \vec{k}_3}{m_{23}} \qquad (2.4)$$
$$\vec{p}_1 = \vec{k}_2 + \vec{k}_3$$

and the same for sets of variables $(\vec{K}, \vec{k}_{31}, \vec{p}_2)$ and $(\vec{K}, \vec{k}_{12}, \vec{p}_3)$.

Thus, in the center-of-mass system the state of the three-body system can be described by the total momentum of two particles (which is equal to the momentum of a remaining particle but with opposite sign) and by the momentum of the relative motion of these two particles[x)].

We shall describe the state of the system by the state vectors $|\Psi>$ which form the Hilbert space. In this space as a basis we can choose the set of vectors $|\vec{k}_{23}, \vec{p}_1>$ possessing the orthonormality and completeness properties:

$$<\vec{k}'_{23}\, \vec{p}'_1 | \vec{k}_{23}\, \vec{p}_1> = \delta(\vec{k}_{23} - \vec{k}'_{23})\, \delta(\vec{p}_1 - \vec{p}'_1) \qquad (2.5)$$

$$\int |\vec{k}_{23}\, \vec{p}_1><\vec{k}_{23}\, \vec{p}_1|\, d\vec{k}_{23}\, d\vec{p}_1 = 1 \qquad (2.6)$$

As another basis a complete set of vectors $|I M \ell_{23} m_{23} k_{23} p_1>$ can be chosen where I, M are the total angular momentum of the system and its projection on an arbitrary axis, ℓ_{23} and m_{23} are the orbital momentum, of the

x) The Dalitz variables /7/ which are specified in two different inertial systems are convenient for relativistic problems /8,9/. In the nonrelativistic case they are identical to the Jacobi coordinates used by us.

relative motion of particles 2 and 3 and its projection on $\vec{p}_1$ respectively, and $k_{23} = |\vec{k}_{23}|$, $p_1 = |\vec{p}_1|$. The orthonormality and completeness relations for these vectors are of the form

$$< I'M'\ell'_{23}\, m'_{23}\, k'_{23}\, p'_1 \,|\, I M\, \ell_{23} m_{23} k_{23} p_1 > = \delta_{II'}\, \delta_{MM'}\, \delta_{\ell_{23}\ell'_{23}} \times \tag{2.7}$$
$$\times\, \delta_{m_{23} m'_{23}} \frac{\delta(k_{23}-k'_{23})}{k^2_{23}} \frac{\delta(p_1-p'_1)}{p^2_1}$$

$$\sum_{IM\ell m} \int |\, I M\, \ell_{23}\, m_{23} k_{23}\, p_1 >< I M\, \ell_{23}\, m_{23}\, k_{23}\, p_1 |\, k^2_{23} p^2_1\, dk_{23}\, dp_1 = 1 \tag{2.8}$$

These bases are linked by the following transformation function

$$< I\, M\, \ell_{23}\, m_{23}\, k'_{23}\, p'_1 |\, \vec{k}_{23}\, \vec{p}_1 > = \sqrt{\frac{2I+1}{4\pi}}\; D^I_{Mm_{23}}(\vec{p}_1) \times \tag{2.9}$$
$$\times\, Y_{\ell_{23} m_{23}}(\vec{k}_{23}) \frac{\delta(k_{23}-k'_{23})}{k^2_{23}} \frac{\delta(p_1-p'_1)}{p^2_1}$$

where $D^I_{Mm_{23}}(\vec{p}_1)$ is the Wigner function.

The transformation functions $< I M\, \ell_{31}\, m_{31}\, k'_{31}\, p'_2 | \vec{k}_{31} \vec{p}_2 >$ and $< I M \ell_{12}\, m_{12}\, k'_{12}\, p'_3 | \vec{k}_{12} \vec{p}_3 >$ can be written analogously.

3. Expansion of the Faddeev equations for the wave function in partial waves

Let us consider the Faddeev equations for scattering of particle 1 on the bound state of other particles:

$$|\Psi^{(1)}> = |\Phi_{23}> - G_0(z)\, T_{23}(z) \{ |\Psi^{(2)}> + |\Psi^{(3)}> \}$$
$$|\Psi^{(2)}> = -\, G_0(z)\, T_{31}(z) \{ |\Psi^{(1)}> + |\Psi^{(3)}> \} \tag{3.1}$$
$$|\Psi^{(3)}> = -\, G_0(z)\, T_{12}(z) \{ |\Psi^{(1)}> + |\Psi^{(2)}> \}$$

Here $G_0(z)$ is the free Green function

$$G_0(z) = \{ H_0 - z \}^{-1}, \tag{3.2}$$

where H_0 is the total kinetic energy which is expressed in terms of the Jacobi variables

$$H_0 = -\frac{1}{2\mu_{23}}\Delta_{23} - \frac{1}{2\mu_1}\Delta_1 = -\frac{1}{2\mu_{31}}\Delta_{31} - \frac{1}{2\mu_2}\Delta_2 = -\frac{1}{2\mu_{12}}\Delta_{12} - \frac{1}{2\mu_3}\Delta_3 \quad (3.3)$$

$$\mu_{ik} = \frac{m_i \; m_k}{m_{ik}} \; ; \quad \mu_\ell = \frac{m_\ell \; m_{ik}}{M} \; ; \quad i, k \neq \ell \quad (3.4)$$

and

$$z = \frac{p_1^2}{2\mu_1} - \epsilon_{23} + i\eta \; ; \quad \eta \to 0 \quad (3.5)$$

where ϵ_{23} is the binding energy of particles 2 and 3. The operators T_{ik} satisfy the equations

$$T_{ik}(z) = V_{ik} - V_{ik} \, G_0(z) \, T_{ik}(z) \quad (3.6)$$

where V_{ik} denotes the two-body potentials.

The initial state is characterized by the state vector

$$|\Phi_{23}\rangle = |\Phi_{23}\rangle_{\vec{p}_1^{\,o}\,\ell^o_{23}\, m^o_{23}\, \epsilon^o_{23}} = \int d\vec{q}\; \phi_{\ell^o_{23}\, m^o_{23}\, \epsilon_{23}}(\vec{q})\,|\vec{q}\;\vec{p}_1^{\,o}\rangle \quad (3.7)$$

where

$$\phi_{\ell^o_{23}\, m^o_{23}\, \epsilon^o_{23}}(\vec{q}) = \phi_{\ell^o_{23}\, \epsilon^o_{23}}(q)\, Y_{\ell^o_{23}\, m^o_{23}}(\vec{q}) \quad (3.8)$$

is the wave function of the bound state of particles 2 and 3 in the momentum representation.

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To expand the set of equations (3.1) in partial wave we need the following matrix elements

$$\langle I' M' \ell'_{23} \; m'_{23} \, k'_{23} \, p'_1 \,|\, G_0(z) \,|\, I M \; \ell_{23} \; m_{23} k_{23} \; p_1 \rangle =$$

$$= \delta_{II'} \; \delta_{MM'} \, \delta_{\ell_{23}\,\ell'_{23}} \, \delta_{m_{23}\, m'_{23}} \frac{\delta(k_{23}-k'_{23})}{k^2_{23}} \; \frac{\delta(p_1 - p'_1)}{p_1^2} \; G_0(k_{23}\, p_1\, z) \quad (3.9)$$

where

$$G_0(k_{23}\, p_1\, z) = \left(\frac{k^2_{23}}{2\mu_{23}} + \frac{p_1^2}{2\mu_1} - z\right)^{-1} \quad (3.10)$$

and

$$\langle I'M'\ell'_{23}\, m'_{23}\, k'_{23}\, p'_1 |\, T_{23}(z)\, | IM\, \ell_{23}\, m_{23} k_{23}\, p_1 \rangle =$$

$$= \delta_{II'}\; \delta_{MM'}\; \delta_{\ell_{23}\,\ell'_{23}}\, \delta_{m_{23}\, m'_{23}} \frac{\delta(p_1 - p'_1)}{p_1^2} \times \tag{3.11}$$

$$\times \langle k'_{23} \,|\, t_{23}^{\ell_{23}} \left(z - \frac{p_1^2}{2\mu_1} \right) |\, k_{23} \rangle$$

here $t_{23}^{\ell_{23}}(\xi)$ is a partial two-body scattering amplitudes outside the energy surface and satisfies the equation

$$\langle k'_{23} \,|\, t_{23}^{\ell_{23}}(\xi) \,|\, k_{23} \rangle = \langle k'_{23} \,|\, V_{23}^{\ell_{23}} \,|\, k_{23} \rangle - \tag{3.12}$$

$$- \int \frac{d q_{23}\, q_{23}^2 \langle k'_{23} \,|\, V_{23} \,|\, q_{23} \rangle \langle q_{23} \,|\, t_{23}^{\ell_{23}} \,|\, k_{23} \rangle}{\frac{q_{23}^2}{2\mu_{23}} - \xi}$$

in this case

$$\langle k'_{23} \,|\, V_{23}^{\ell_{23}} \,|\, k_{23} \rangle = \frac{2}{\pi} \int_0^\infty j_{\ell_{23}}(k'_{23} r)\, j_{\ell_{23}}(k_{23} r)\, V(r)\, r^2 dr \tag{3.13}$$

where $j_{\ell_{23}}$ are the Bessel spherical functions.

The matrix elements of the operators $T_{31}(z)$ and $T_{12}(z)$

$$\langle I'M'\ell'_{31}\, m'_{31}\, k'_{31} p'_2 \,|\, T_{31}(z) \,|\, IM\, \ell_{31} m_{31}\, k_{31}\, p_2 \rangle \tag{3.14}$$

$$\langle I'M'\ell'_{12}\, m'_{12} k'_{12}\, p'_3 \,|\, T_{12}(z) \,|\, IM\, \ell_{12} m_{12}\, k_{12}\, p_3 \rangle$$

can be written in a similar way.

In expanding the set (3.1) in partial waves it is convenient to write the first equation in the basis $|IM\, \ell_{23}\, m_{23}\, k_{23}\, p_1\rangle$, the second one in $|IM\, \ell_{31}\, m_{31}\, k_{31} p_2\rangle$ and the third equation in the basis $|IM\, \ell_{12} m_{12} k_{12} p_3\rangle$ Multiplying the set of the equations (3.1) by the appropriate bra vectors we get

$$\langle IM\ell_{23}\, m_{23} k_{23}\, p_1 \,|\, \Psi^{(1)} \rangle = \langle IM\ell_{23}\, m_{23}\, k_{23}\, p_1 \,|\, \Phi \rangle_{\vec{p}^{\,\circ}_1 \ell^{\circ}_{23} m^{\circ}_{23} \epsilon^{\circ}_{23}} -$$

$$- \langle IM\, \ell_{23}\, m_{23}\, k_{23}\, p_1 \,|\, G_0\, T_{23} \{ |\, \Psi^{(2)} \rangle + |\, \Psi^{(3)} \rangle \} \tag{3.15}$$

$$<\{M\, \ell_{31}\, m_{31}\, k_{31}\, p_2 | \Psi^{(2)}> = -< \{M\, \ell_{31}\, m_{31} k_{31} p_2 |\; G_0\; T_{31} \{ |\Psi^{(1)}> + |\Psi^{(3)}> \}$$

$$<\{M\, \ell_{12}\, m_{12}\, k_{12}\, p_3 | \Psi^{(3)}> = -<\{M\; \ell_{12}\; m_{12} k_{12} p_3 |\; G_0\; T_{12} \{ |\Psi^{(1)}> + |\Psi^{(2)}> \} \quad .$$

Making use of (3.17) and (3.8) for the initial state in the basis $<\{M\, \ell_{23}\, m_{23}\, k_{23}\, p_1|$ we have

$$\Phi_{\{M \ell_{23} m_{23}}^{\vec{p}_1^{\,o}\, \ell^o_{23}\, m^o_{23}\, \epsilon^o_{23}} (k_{23}\, p_1) = <\{M\; \ell_{23}\; m_{23}\; k_{23}\; p_1 |\; \Phi>_{\vec{p}_1^{\,o}\, \ell^o_{23}\, m^o_{23}\, \epsilon^o_{23}} =$$
$$= \sqrt{\frac{2I+1}{4\pi}}\; D^{I}_{M m_{23}} (\vec{p}_1^{\,o})\, \delta_{\ell_{23}\, \ell^o_{23}}\, \delta_{m_{23}\, m^o_{23}}\, \frac{\delta(p_1 - p_1^{\,o})}{p_1^2}\, \phi_{\ell^o_{23}\, \epsilon^o_{23}} (k_{23}) \qquad (3.16)$$

Introducing the following notation

$$\Psi^{(i)}_{\{M \ell m} (kp) = <\{M\; \ell\; m\; k\; p\; |\; \Psi^{(i)}> \qquad (3.17)$$

and using (2.6), (2.8), (3.9), (3,11), (3.16) and (3.13) we obtain

$$\Psi^{(1)}_{\{M \ell_{23} m_{23}} (k_{23} p_1) = \Phi_{\{M\, \ell_{23}\; m_{23}}^{\vec{p}_1^{\,o}\, \ell^o_{23}\, m^o_{23}\, \epsilon^o_{23}} (k_{23} p_1) - G_0 (p_1\, k_{23} z) \;\times$$
$$\times \int k'^2_{23}\, dk'_{23}\; t_{23}^{\ell_{23}} (k_{23}, k'_{23}, z - \frac{p_1^2}{2\mu_1}) \{ \Psi^{(2)}_{\{M \ell_{23} m_{23}} (k'_{23}\, p_1) + \Psi^{(3)}_{\{M \ell_{23} m_{23}} (k'_{23} p_1) \}$$

$$\Psi^{(2)}_{\{M \ell_{31}\; m_{31}} (k_{31} p_2) = - G_0 (p_2\, k_{31}\, z)\; \int\; d\, k'_{31}\; k'^2_{31} \;\times$$
$$\times\; t_{31}^{\ell_{31}}\; (k_{31}, k'_{31}, z - \frac{p_2^2}{2\mu_2}) \{ \Psi^{(1)}_{\{M \ell_{31}\; m_{31}} (k'_{31}, p_2) + \Psi^{(3)}_{\{M\, \ell_{31}\, m_{31}} (k'_{31}, p_2) \} \qquad (3.18)$$

$$\Psi^{(3)}_{\{M\; \ell_{12}\, m_{12}} (k_{12} p_3) = - G_0 (p_3\, k_{12} z)\; \int dk'_{12} k'^2_{12} \times$$
$$\times\; t_{12}^{\ell_{12}}\; (k_{12}, k'_{12}\;, z - \frac{p_3^2}{2\mu_3}) \{ \Psi^{(1)}_{\{M \ell_{12} m_{12}} (k'_{12} p_3) + \Psi^{(2)}_{\{M \ell_{12} m_{12}} (k'_{12}\, p_3) \}$$

To link the functions of (3.18) written in different bases we employ the formula

$$\psi^{(1)}_{IM\ell_{23}m_{23}}(k_{23}p_1) = \int dk_{31}dp_2\, k_{31}^2 p_2^2 \times \tag{3.19}$$

$$\times \sum_{\ell_{31}m_{31}} < IM\ \ell_{23}\ m_{23}\ k_{23}p_1 \mid IM\ \ell_{31}m_{31}k_{31}p_2 > \psi^{(1)}_{IM\ell_{31}m_{31}}(k_{31}\ p_2)$$

and so on, where the quantities like $< I'M'\ell_{23}m_{23}k_{23}p_1| IM\ell_{31}m_{31}k_{31}p_2>$ are referred to as recoupling coefficients /10/ and given by

$$< I'M'\ell_{23}\,m_{23}\,k_{23}\,p_1| IM\ \ell_{31}\,m_{31}k_{31}p_2 > = A\,\delta_{II'}\quad \delta_{MM'} \quad \times$$

$$\times\ \delta\left(\frac{k_{23}^2}{2\mu_{23}} + \frac{p_1^2}{2\mu_1} - \frac{k_{31}^2}{2\mu_{31}} - \frac{p_2^2}{2\mu_2}\right)\sqrt{(2\ell_{23}+1)(2\ell_{31}+1)}\ \times \tag{3.20}$$

$$\times\ d^{I}_{m_{23}m_{31}}(\chi)\, d^{\ell_{23}}_{m_{23}\,\bullet}(\theta_{23})\, d^{\ell_{31}}_{m_{31}\,o}(\theta_{31})$$

Here χ is the angle between $\vec{k}_1$ and $\vec{k}_2$, θ_{23} is the angle between $\vec{k}_1$ and $\vec{k}_{23}$, θ_{31} is the angle between $\vec{k}_2$ and $\vec{k}_{13}$ and A is definite constant.

Thus, using (3.19) we obtain the set of integral equations for partial waves.

The merit of these equations is that their kernels contain no Clebsch-Gordan coefficients but are expressed in terms of the Wigner D -function possessing the well-known analytic properties in the total angular momentum.

The latter circumstance will play an essential role in investigating the analytic continuation of the Faddeev functions in the total angular momentum /11/.

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