ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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ACAUSALITY AND DISPERSION RELATIONS

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1. Introduction

The "ultraviolet catastrophe" in modern field theory is essentially that the vacuum expectation values of the most important physical quantities have singularities on the light cone.

The origin of these singularities may be sought for in the form of causality which is the basis of the physical space-time metric: $s^2 = t^2 - \vec{x}^2$.

On the other hand, there are no grounds to believe that the form of causality in the microworld should coincide with that in the macroworld, as it is adopted in modern theory /1,2,3/. However, in Einstein-Minkowski space there exists no notion of the neighbourhood of the two points $\mathcal{P}(x')$ and $\mathcal{P}(x'')$ since the distance $x^2 = (x' - x'')^2$ is indefinite. For this reason any attempts to introduce the "universal" length which would characterize the small space-time region in Einstein-Minkowski space are doomed.

One may postulate, of course, that not only Einstein-Minkowski metric but also the very notion of space-time continuum are not exact enough, and are not at all valid in the microworld. In this case, since we preserve the notion of space-time continuum the theoretical scheme we are developing will be only a model of reality. In this case also one has to define the notion of the "microworld region"; intuitively we are apt to mean by that elementary particle physics – the region of high energies and small distances.

In the region of large distances and low energies there seem to be no reasons to doubt the validity of the conventional concepts of space-time and causality. At any rate experiment yields no grounds for this.

Thus, whatever possible changes of causality would be on a "small scale" the notion of "smallness" must be defined and so that there would exist a transition to the "large" space-time regions where it is natural to keep old metric relations.

Since the distance $x^2 = t^2 - r^2$ in Einstein-Minkowski space is indefinite, in order to make a transition to large distances it is insufficient to have a scalar universal length a which would provide this transition. It is necessary to have a certain time-like vector n (without restrictions one can consider $n^2 = 1$, $n_0 > 0$)

For the time being we will treat this vector purely formally. The introdnction of this vector allows us, besides the invariant $x^2 = t^2 - r^2$ to introduce the invariant $I_x = (x, n) = tn_0 - \vec{r}\vec{n}$.
Using these two invariants it is possible to form a positive-definite quantity

$$R^2 = 2I_x^2 - x^2 \geq 0 , \tag{1}$$

which permits to define the notion of the neighbourhood of two particles in the four-dimensional space-time in a invariant form /3,4/.

In the proper coordinate system where $n = (1,0,0,0)$, $R^2 = t^2 + r^2$ we are able, besides R^2_x to introduce the invariant

$$L = \frac{1}{\sqrt{2}} [\sqrt{R^2 - I_x^2} \pm I_x] , \tag{1'}$$

which determines the neighbourhood of the point to the light cone ($-$ for the cone of the absolute future and $+$ for the cone of the absolute past); in the proper coordinate system

$$L = \frac{1}{\sqrt{2}} (r \pm t).$$

The second reason which does not make it possible to restrict to the invariant $x^2 = t^2 - r^2$ is that if the signal propagation is allowed in the spatial region ($x^2 < 0$) , as it is supposed in /5,6/, then such a violation of causality is symmetrical with respect to the past and future.

Meanwhile, causality must be violated (apart) independently for advanced and retarded interactions. Indeed, in the conventional theory the general propagation function F may be represented in the form

$$F = a F^{ret} + b F^{adv} , \qquad (2)$$

where a and b are arbitrary constants. The violation of causality (which may be weak) must not put a bound on the arbitrariness of the constants a and b .

So, we suppose that there is, besides the invariant $x^2 = t^2 - r^2$ the invariant $I_x = (x, n)$ as well. Further we note that there are two principally different possibilities for the choice of the unit vector n :

a) the vector n is exterior with respect to the system of interacting particles. A similar possibility is treated in papers /7,8/. Under such an assumption concerning the vector n there exists an explicit dependence of the scattering amplitude on the frame of reference (see., e.g. /7/). This means that the scattering amplitude may be different in the laboratory system and in the centre-of-mass system. In other words, a possibility is allowed that Mickelson's experiment gives a positive result in the high energy region. This seems to be very attractive, but still very little studied.

Therefore, we will treat another possibility b) when the vector n is connected with the very system of interacting particles (see /3/ and /4/). It is supposed in this case that the violation of the metric relations takes place not in vacuum, but in a medium formed by the matter of colliding particles. As a vector n one may take any unit vector directed along the momenum of one or several particles participating in the collision*. However, it is more reasonable to take the vector n which is more symmetrical with respect to the particles or their states. Such a symmetrical vector in the case of the pairing collision may be, for instance, the centre-of-mass momentum of colliding particles $P = (p + k)$ of the Breit vector $P = (p + p')$:

$$n = \frac{P}{\sqrt{p}} , \qquad P = (p + k) \qquad \text{or} \qquad P = (p + p') \qquad (3)$$

(here p is the nucleon momentum , k is the meson momentum before the collision, p', k' are the same quantities after collisions). By such a choice of n the scattering amplitude $\mathfrak{M}$ for the process

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* In this case of many particles each subgroup of the interacting particles may have its internal vector n .

$a + b \to c + d$ will be, as in the conventional theory, a function of only the invariants $s = (p+k)^2$ and $t = (p + p')^2 = (k' - k)^2$ and of some universal length a which characterizes the acausality region: $\mathfrak{M} = \mathfrak{M}(s, t, a)$

If n is a vector exterior with respect to the system of colliding particles, then in the amplitude $\mathfrak{M}$ there will hold an explicit dependence on the coordinate system so that besides s and t there will be present, at least one invariant $I = (p+k, n)$ which does not reduce to s and t.

2. Retarded and Advanced Amplitudes.

We assume that there exist asymptotic incoming and outgoing waves $\phi_{in}(x)$ and $\phi_{out}(x)$ (see, e.g., /9/), which are related through the unitary matrix S:

$$\phi_{out}(x) = S\, \phi_{in}(x)\, S^{-1}. \tag{4}$$

Then the retarded and advanced matrix elements of the scattering amplitudes $\mathfrak{M}$ for the two-body process $p+k \to p' + k'$ (where p is the nucleon momentum, k is the meson momentum before the collision, p', k' the same quantities after the collision) may be written in the form /9/:

$$\mathfrak{M}^{ret}(p',k'; p,k) = i \int \exp \frac{i}{2}(k+k', x) < p' | \frac{\delta}{\delta\, \phi(\frac{x}{2})} \left[\frac{\delta S}{\delta\, \phi(-\frac{x}{2})} S^{+} \right] |p>, \tag{5}$$

$$\mathfrak{M}^{adv}(p',k'; p,k) = i \int \exp \frac{i}{2}(k+k', x) < p' | \frac{\delta}{\delta\, \phi(-\frac{x}{2})} \left[\frac{\delta S}{\delta\, \phi(\frac{x}{2})} S^{+} \right] |p>, \tag{5'}$$

These processes do not yet imply the causality of the processes.

Denoting the one-particle matrix elements by

$$\Phi^{ret}_{p'p}(x) = i < p' | \frac{\delta}{\delta\, \phi(\frac{x}{2})} \left[\frac{\delta S}{\delta\, \phi(-\frac{x}{2})} S^{+} \right] |p> \tag{6}$$

and

$$\Phi^{adv}_{p'p}(x) = i < p' | \frac{\delta}{\delta\, \phi(-\frac{x}{2})} \left[\frac{\delta S}{\delta\, \phi(\frac{x}{2})} S^{+} \right] |p> \tag{6'}$$

we notice that

$$\Phi^{ret}_{p'p}(x) = \Phi^{adv}_{pp'}(-x), \qquad \dot{\Phi}^{ret}_{p'p}(x) = \Phi^{ret}_{pp'}(x) \tag{7}$$

(In the following, for the notational simplicity we shall often omit the indices p and p'. Instead of $\Phi_{p'p}(x)$ we shall write $\Phi(x)$). It follows from (7) that the Fourier transforms of the corresponding functions possess the properties:

$$\tilde{\Phi}^{ret}(Q) = \tilde{\Phi}^{adv}(-Q), \qquad \dot{\tilde{\Phi}}^{ret}(Q) = \tilde{\Phi}^{ret}(-Q). \tag{8}$$

Now we consider possible types of the causality violation which are compatible with the usual form of the causality for large distances r and large time intervals t.

We will be concerned at first with the usual retarded $F^{ret}(x)$ and advanced $F^{adv}(x)$ propagation functions.

In Fig. 1 the shaded area shows the space-time region where these functions may be different from zero. At the same time

$$F^{adv}(x) = F^{ret}(-x). \tag{9}$$

The corresponding acausal functions will be designated by $\Phi(x)$. The causality violation is supposed to be that these functions may be different from zero outside the shaded area as well. However, they must decrease sufficiently rapidly as we go into the "forbidden" region:

$$\Phi^{ret}(x) \to 0 \qquad \text{at} \qquad L = \frac{1}{\sqrt{2}}(r - t) \to \infty, \tag{10}$$

$$\Phi^{adv}(x) \to 0 \qquad \text{at} \qquad L = \frac{1}{\sqrt{2}}(r + t) \to \infty. \tag{10'}$$

A more special case would have taken place if causality has been violated only near the vertex of the light cone. Here in (10) and (10^1) we should mean $R \to \infty$ instead of $L \to \infty$.

The remaining functions may be constructed in the usual manner out of $\Phi^{ret}(x)$ and $\Phi^{adv}(x)$. The acausal analogue of the causal commutator $\Phi(x)$ is equal to

$$\Phi(x) = \Phi^{ret}(x) - \Phi^{adv}(x) = \Phi^{+}(x) + \Phi^{-}(x) = \tag{11}$$
$$= \langle p' | [\, j(\tfrac{x}{2}),\; j(-\tfrac{x}{2}) \,] | p \rangle ,$$

where $j(x) = \dfrac{\delta S}{\delta \phi(x)} S^{+}$ and $\Phi^{\pm}$ mean the positive and negative-frequency parts of the commutator $\Phi(x)$. Similarly, the acausal analogue of the causal function $\mathfrak{D}_c(x)$ is:

$$\Phi_c(x) = \tfrac{1}{2}[\Phi^{ret}(x) + \Phi^{adv}(x)] - \tfrac{1}{2}[\Phi^{+}(x) - \Phi^{-}(x)]. \tag{12}$$

The second requirement which we impose on the acausal propagation functions consists in the conservation of the usual spectrality condition.

It follows naturally from the assumption that the acausality which manifest itself at small distances does not affect the spectrum of free particles. The spectrality condition states that the Fourier transform of the acausal function $\tilde{\Phi}(Q)$:

$$\tilde{\Phi}(Q) = \int \Phi(x)\, e^{iQx}\, d^4x \tag{13}$$

must vanish in some region $\mathfrak{R}(Q)$ which is the same as that for the corresponding causal function $\tilde{F}(Q)$.

If we put $\tfrac{1}{2}(p+p') = (a,0,0,0)$ and denote by m_1, m_2 the masses of the lowest intermediate states which may contribute to the terms of the commutator then the region $\mathfrak{R}(Q)$ will be determined by the inequality

$$\alpha - \sqrt{\vec{Q}^2 + m_2^2} < Q_0 < -\alpha + \sqrt{\vec{Q}^2 + m_1^2} \tag{14}$$

i.e., this is the region outside two hyperboloids. In the case $\alpha > \frac{m_1 + m_2}{2}$ these hyperboloids intersect. For pion-nucleon scattering, we have:

$$m_1 = 3m, \qquad m_2 = M + m.$$

3. Interaction with an Indefinite Signal Propagation.

In what follows we will consider a model of the acausal theory in which the signal propagates not quite along the light cone.

To start, we take the simplest example which is a direct generalization to the relativistic region of the acausality case treated in papers /9/ and /10/.

Let $F^{ret}(x)$ be a retarded propagation function of the conventional local theory. We assume that in the acausal theory the interaction may propagate inside the shifted light cone (see Fig.1).

Suppose that the magnitude of the shift is equal to

$$\xi = \alpha n \sigma, \tag{15}$$

where α is a certain small length, n is a characteristic time vector, σ is the invariant parameter (the "proper time").

Then the true acausal propagation function will be

$$\Phi^{ret}(x) = F^{ret}(x - \xi). \tag{16}$$

Regarding ξ as a function of σ and introducing the propagation function of the shifts $f(\sigma)$, we can write (1.6) in a more general form

$$\Phi^{ret}(x) = \int F^{ret}[x - \xi(\sigma)]\, f_1(\sigma)\, d\sigma. \tag{17}$$

For the advanced functions we shall have, respectively

$$\Phi^{adv}(x) = \int F^{adv}[x - \xi(\sigma)]\, f_2(\sigma)\, d\sigma. \tag{18}$$

In virtue of condition $f_1(\sigma) = f_2(-\sigma) = f(\sigma)$.

The Fourier-transform of these functions states

$$\tilde{\Phi}^{ret}(Q) = \tilde{F}^{ret}(Q)\, \tilde{f}(Qn\alpha) \tag{19}$$

and

$$\tilde{\Phi}^{adv}(Q) = \tilde{F}^{adv}(Q)\, \tilde{f}(-Qn\alpha), \tag{19'}$$

where

$$\tilde{f}(Qn\alpha) = \int e^{iQn\alpha\sigma} f(\sigma)\, d\sigma. \tag{20}$$

It follows from (11), (19) and (19$'$) that

$$\tilde{\Phi}(Q) = \tilde{F}^{ret}(Q)\,\tilde{f}(Qn\alpha) - F^{adv}(Q)\,\tilde{f}(-Qn\alpha) \qquad (21)$$

One can just see from here that the spectrality condition is fulfilled if $\tilde{f}(Qn\alpha) = \tilde{f}(-Qn\alpha)$. Besides, since the equality $\tilde{f}(-Qn\alpha) = \tilde{f}^{*}(Qn\alpha)$ must be also fulfilled, the function $\tilde{f}(Qn\alpha)$ must be even and real.

Further, in the proper coordinate system $n = (1,0,0,0)$, $L = \sqrt{2}\,\xi$ (cf. Fig. 2); therefore $L \to \infty$ means that $\sigma \to +\infty$. It follows from here that the condition of the macroscopic causality will be fulfilled (the "anomalous" signal will be whatever small), if $f(\sigma)$ is a sufficiently rapidly decreasing function at $\sigma \to \infty$

It is seen from the fomulae (19) and (19$'$) that for the quantities $\tilde{\Phi}^{ret}(Q)/\tilde{f}(Qn\alpha)$ and $\tilde{\Phi}^{adv}(Q)/\tilde{f}(-Qn\alpha)$ there will hold ordinary dispersion relations. The additional singularities of the acausal functions Φ^{ret} and Φ^{adv} coincide with the singularities of the functions $\tilde{f}(\pm Qn\alpha)$.

Note, that if $f(\sigma)$ falls off very sharply with the growth of σ, then in the Q plane appears a singularity on a circle of infinitely large radius. For example:

$$f(\sigma) = \delta(\sigma - 1), \qquad \tilde{f}(Qn\alpha) \approx e^{iQna}, \qquad (22)$$

$$f(\sigma) = e^{-\sigma^2}, \qquad \tilde{f}(Qn\alpha) \approx e^{-a^2(Qn)^2}. \qquad (22')$$

For a more smooth, exponential decrease there arises a pole

$$f(\sigma) = e^{-\sigma}, \quad \sigma > 0, \quad \tilde{f}(Qn\alpha) = \frac{1}{1 - iQn\alpha}. \qquad (22'')$$

However, in virtue of what has been said above the spectrality conditions are satisfied by the function (22^1) only.

A. Consider now a more general case of the acausal propagation function

$$\Phi^{ret}(x) = \int F^{ret}(x - \xi)\,\rho_1(\xi, n)\,d^4\xi, \qquad (23)$$

Here the propagation function $F^{ret}(x)$ is again taken over from the conventional causal theory, while the weight function $\rho_1(\xi, n)$ vanishes at $R \to \infty$. Note that condition (9) requires that $\rho_1(\xi, n) = \rho_2(-\xi, n)$ Therefore, further we omit indices 1 and 2. Due to the vanishing of ρ at $R \to \infty$ macroscopic causality is fulfilled.

Indeed, the signal Φ^{ret} may be regarded as the one from a certain source $\rho(x)$ extended near the coordinate origin $r, t \sim 0$ (see Fig.2). Further the Fourier transform states

$$\tilde{\Phi}^{ret}(Q) = \tilde{F}^{ret}(Q)\,\tilde{\rho}(Q, n), \qquad (24)$$

$$\tilde{\Phi}^{adv}(Q) = \tilde{F}^{adv}(Q)\,\tilde{\rho}(-Q, n), \qquad (24')$$

where $\tilde{\rho}(Q,n)$ is the Fourier-transform of the function $\rho(\xi,n)$. The symmetry conditions (8) require that

$$\tilde{\rho}(-Q,n) = \tilde{\rho}(Q,n), \quad \tilde{\rho}(-Q,n) = \tilde{\rho}^*(Q,n). \tag{24''}$$

Then

$$\tilde{\Phi}(Q) = [\tilde{F}^{ret}(Q) - \tilde{F}^{adv}(Q)]\,\tilde{\rho}(Q,n). \tag{24'''}$$

These functions evidently vanish in the region $\mathfrak{R}(Q)$ and, hence, the spectral condition is fulfilled.

Note, that the analytic properties of the functions $\tilde{\Phi}(Q)/\tilde{\rho}(Q,n)$ coincide with the analytic properties of these functions in causal field theory.

As we have pointed out above the appearance of essential singularities (at infinity) of the function $\tilde{\rho}(I_Q, Q^2)$ is rather an anomaly than a usual situation.

Indeed, for this the space-time region of acausality should be sharply bounded (sharper than by an exponent). In particular, by a sharp cut off $\rho(x,n) = \frac{4}{\pi^2 a^4} \int_0^{a^2} d\xi\, \delta(\xi - R^2)$ we shall have

$$\tilde{\rho}(I_Q, Q^2) = \frac{4}{\pi^2 a^4} \int d^4x\, e^{iQx} \int_0^{a^2} d\xi\, \delta(\xi - R^2) = \tag{25}$$

$$\frac{8}{a^2[2(Qn)^2 - Q^2]}\, J_2(a\sqrt{2(Qn)^2 - Q^2}),$$

where $J_2(z)$ is the Bessel function. Since asymptotically $J_2(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z - \frac{5}{4}\pi)$, then $\tilde{\rho}(I_Q, Q^2)$ will have a singularity at infinity. This is clearly seen in the proper coordinate system $n = (1,0,0,0)$, where the invariant

$$\sqrt{2(Qn)^2 - Q^2} = \sqrt{Q_0^2 + \vec{Q}^2}.$$

In particular, if this is a Breit system then $\sqrt{Q_0^2 + \vec{Q}^2} = \sqrt{2\omega^2 - m^2 - \vec{q}^2}$, where ω is the meson energy, m is its mass, q is the momentum transfer. When $\vec{q} = 0$, $|\omega| \gg m$ the function $\tilde{\rho}$ will contain the factor $\exp(\pm i\sqrt{2}\,\omega a)$. For the Caussian distribution

$$\rho(x) = \frac{1}{a^4} \exp - \frac{R^2}{a^2}, \quad \tilde{\rho}(I_Q, Q^2) = \exp\{-\frac{a^2}{4}[2(Qn)^2 - Q^2]\}$$

and the essential singularity is due to the factor $\exp - \frac{a^2\omega^2}{2}$.

Now we consider in more detail the case when $\rho(x)$ decreases exponentially, or in a more general form

$$\rho(x) \approx R^m \exp(-\frac{R}{a}). \tag{26}$$

In this case there appear additional poles in the plane ω.

For the sake of definiteness, we will be concerned with the case

$$\rho(x) = \frac{1}{8\pi a^2 R^2} \exp\left(-\frac{R}{a}\right) \tag{27}$$

(the factor a^n is chosen so that $\tilde{\rho}(Q) \to 1$ when $a \to 0$). Then

$$\tilde{\rho}(Q) = \frac{1}{1 + a^2[2(Qn)^2 - Q^2]} . \tag{28}$$

Or in the Breit system

$$\tilde{\rho}(Q) = \frac{1}{1 + a^2(2\omega^2 - m^2 - \vec{q}^{\,2})} . \tag{29}$$

As far as there is no essential singularity at infinity, the dispersion relations with the necessary subtractions may be written for the observed matrix element $\mathfrak{M}(k', p'; k, p)$.

Note that the case (A) treated above is formally obtained from (17), if we put

$$\rho(\xi, n) = \int \delta(\xi - an\sigma) f(\sigma) d\sigma$$

and integrate over ξ .

B. Now we consider the case when the causality is violated only near the vertex of the light cone. Here one can suppose:

$$\Phi^{ret}(x) = F^{ret}(x) + \phi^{ret}(x, n) , \tag{30}$$

where $\phi^{ret}(x, n)$ is an acausal addition to the causal function $F^{ret}(x)$ vanishing as we go away from the coordinate origin. We assume that $\phi^{ret}(x, n) = \phi^{ret}(R^2, xn)$ and that

$$\phi^{ret}(R^2, xn) \underset{R \to \infty}{\to} 0 . \tag{31}$$

Analogously one can introduce

$$\Phi^{adv}(x) = F^{adv}(x) + \phi^{adv}(x, n)$$

and hence

$$\Phi(x) = \Phi^{ret}(x) - \Phi^{adv}(x) = F(x) + \phi(R^2, xn) .$$

At the same time

$$\phi(R^2, xn) = \phi^+(R^2, xn) - \phi^-(R^2, xn) .$$

Then

$$\tilde{\phi}^{\pm}(Q) = \int \phi^{\pm}(R^2, xn) \exp iQx \, d^4x =$$

$$= -\tfrac{1}{4} \int \phi^{\pm}(a^2, \beta) e^{i\beta\xi} \exp i\left[\eta a^2 + \frac{R^2(Q - \xi n)}{4\eta}\right] da^2 d\beta \, d\eta \, d\xi \tag{32}$$

$$= \int \phi^{\pm}(\alpha^2, \beta)\, e^{i\beta\xi} \frac{J_1[\alpha R(Q-\xi n)]}{R(Q-\xi n)} \alpha^2 d\alpha\, d\beta\, d\xi .$$

Here $R^2(Q-\xi n)$ has the same meaning as in (1) with the substitution of x by $(Q-\xi n)$: in the system where $n = (1, 0, 0, 0)$, $R^2 = (Q_0 - \xi)^2 + \vec{Q}^2$. In this system our expression is of the form

$$\tilde{\phi}^{\pm}(Q) = \int \phi^{\pm}(\alpha^2, \beta)\, e^{i\beta\xi} \frac{J_1[\alpha\sqrt{(Q_0-\xi)^2 + \vec{Q}^2}]}{\sqrt{(Q_0-\xi)^2 + \vec{Q}^2}} \alpha^2 d\alpha\, d\beta\, d\xi .$$

In virtue of the spectrality conditions $\tilde{\phi}^{+}(Q) = 0$ for all Q satisfying the inequality $Q_0 > -a + \sqrt{\vec{Q}^2 + m_1^2}$ Similarly $\tilde{\phi}^{-}(Q) = 0$ for all Q, satisfying the inequality $Q_0 > a - \sqrt{\vec{Q}^2 + m_2^2}$
The expression

$$(Q_0 - \xi)^2 + \vec{Q}^2 = R^2(\xi) \tag{33}$$

is a family of the circumferences of radius R and the coordinates of the centre $(\xi, 0, 0, 0)$. We choose $R(\xi) = R_0(\xi)$ so that the hyperballs (14) would be envelopes for our family of the circumferences. Then for the upper hyperboloid

$$R_0^{+}(\xi) = \frac{(\xi + a)^2}{2} - m_1^2 \tag{34}$$

and ξ must change within the interval $[+\infty\ ,\ 2m_1 - a\ ]$. Here the lower boundary is found from the requirement that $Q = \pm\sqrt{\frac{(\xi + a)^2}{4} - m_1^2}$ be a real value). Similarly for the lower hyperboloid

$$R_0^{-}(\xi) = \frac{(\xi - a)^2}{2} - m_2^2 \tag{35}$$

and ξ must change within the interval $[-(2m_2 - a)\ ,\ -\infty]$ Therefore, in order to satisfy the causality conditions it is necessary that the integrands would vanish outside the given intervals. Thus, the spectrality conditions are written down in the form

$$\iint \phi^{\pm}(\alpha^2, \beta)\, e^{i\beta\xi} d\beta\ J_1(\alpha\sqrt{(Q_0-\xi)^2 + \vec{Q}^2})\,\alpha^2 d\alpha$$

$$= \begin{cases} R\, f^{\pm}(R, \xi) & \text{for} \quad R \le R_0^{\pm}(\xi) \\ 0 & \text{for} \quad R > R_0^{\pm}(\xi) \end{cases} \tag{36}$$

where ξ changes within the above-mentioned intervals. It follows from theorem /11/ that if

$$\int \alpha\,\phi^{\pm}(\alpha^2, \beta)\, e^{i\beta\xi} d\beta = \int_0^{R_0^{\pm}(\xi)} R' f^{\pm}(R', \xi)\, J\,(\alpha R')\, R' dR' \tag{37}$$

and $f^{\pm}(R';\xi)$ is a holomorphic function of R' on the segment from 0 up to $R^{\pm}(\xi)$, then the spectrality conditions (36) will be fulfilled x/. Substituting (37) into (32) we get:

$$\tilde{\phi}(Q) = \int_{2m_1-a}^{\infty} d\xi \, \frac{\mathcal{P}}{\pi i} \int_0^{R_0^+(\xi)} \frac{f^+(z,\xi)\,dz^2}{z^2-[(Q_0-\xi)^2+\vec{Q}^2]} - \tag{38}$$

$$- \int_{-\infty}^{-(2m_2-a)} d\xi \, \frac{\mathcal{P}}{\pi i} \int_0^{R_0^-(\xi)} \frac{f^-(z,\xi)\,dz^2}{z^2-[(Q_0-\xi)^2+\vec{Q}^2]} \, ,$$

where $f^{\pm}(z,\xi)$ are the holomorphic functions of the variable z on the half-axis from 0 to ∞. As to the analytic properties of ϕ^{ret} in the continuation in ω to the upper half-plane or in the continuation of ϕ^{adv} to the lower half-plane, they are determined by the properties $f^{ret}_{adv}(z,\xi)$ which is a Bessel transform of index 1 in the first argument and the Fourier transform in the second argument of the function $\phi^{ret}_{adv}(\alpha,\beta)$. One can see by examples that the above-formulated causality condition (31) allows a wide class of analyticity violations involving the appearance of poles, cuts, and singularities.

4. Dispersion Relations

At first we consider the case A) when the scattering amplitude may be represented in the form

$$\mathfrak{M}(p',k';p,k) = N(p',k';p\,k)\,\tilde{\rho}(p',k';p,k) \tag{39}$$

or $N(p',k';p\,k) = \mathfrak{M}(p',k';p,k)\,\tilde{\rho}^{-1}(p',k';p,k)$, where $N(p',k';p,k)$ is the scattering amplitude which possesses all the usual analytic properties of the causal scattering amplitude, and $\tilde{\rho}(p',k';p,k)$ is the real function determined in §3B. To go on with the construction of dispersion relations we choose a special coordinate system - the Breit system in which the expression (3) will be rewritten as

$$N(\omega,\lambda\vec{e}) = \mathfrak{M}(\omega,\lambda\vec{e})\,\tilde{\rho}^{-1}(\omega,\lambda\vec{e}), \tag{40}$$

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x/ This theorem states: If the real part exceeds 1 and if

$$f(\lambda) = \int_p^q \phi(p)\,J_n(\lambda\rho)\,\rho\,d\rho \qquad 0 \le p < q \le \infty$$

then

$$\int_0^{\infty} f(\lambda)\,J_n(\lambda r)\,\lambda\,d\lambda = \begin{cases} \phi(r), & p<r<q \\ 0, & 0<r<p,\; q<r<\infty. \end{cases}$$

where ω is the meson energy, e is the unit ort $\perp$p and $\lambda = \sqrt{\omega^2 - p^2 - m^2}$. The concrete form of dispersion relations will depend on the order of the growth of $\tilde{\rho}^{-1}$. Indeed, the dispersion relations in energy for N in case of forward scattering ($\vec{p} = 0$) without subtractions, provided that $\mathfrak{M}^*(\omega) = \mathfrak{M}(-\omega)$ x/ are as follows

$$\mathrm{Re}\,\mathfrak{M}(\omega)\tilde{\rho}^{-1}(\omega) = \frac{2\omega_f\,\mathrm{Re}[\,\mathrm{Res}\,\mathfrak{M}(\omega_f)\,]}{(\omega^2 - \omega_f^2)\tilde{\rho}(\omega_f)} + \tag{41}$$

$$+ \frac{2}{\pi}\mathcal{P}\int_m^\infty \frac{\mathrm{Im}\,\mathfrak{M}(\omega')\omega' d\omega'}{(\omega'^2 - \omega^2)\tilde{\rho}(\omega')}$$

or

$$\mathrm{Re}\,\mathfrak{M}(\omega) = \frac{2\omega_f\,\mathrm{Re}[\,\mathrm{Res}\,\mathfrak{M}(\omega_f)}{(\omega^2 - \omega_f^2)}\,\frac{\tilde{\rho}(\omega)}{\tilde{\rho}(\omega_f)} + \tag{41'}$$

$$+ \frac{2}{\pi}\tilde{\rho}(\omega)\,\mathcal{P}\int_m^\infty \frac{\mathrm{Im}\,\mathfrak{M}(\omega')\omega' d\omega'}{(\omega'^{\cdot} - \omega\ )\tilde{\rho}(\omega')} .$$

In the frequency region where $\rho(\omega), \rho(\omega_f) \approx 1$ and if the factor $(\omega'^2 - \omega^2)$ cuts off the integrand stronger than $\tilde{\rho}^{-1}(\omega)$ grows, one obtains ordinary dispersion relations. For the real dispersion relations $\tilde{\rho}^{-1}(\omega')$ must not grow faster than ω'. If $\tilde{\rho}^{-1}(\omega')$ grows faster than ω' then it is neccesary to increase the number of subtractions, and the ordinary dispersion relations will no longer hold.

If the growth of $\mathfrak{M}(\omega)$ remains bounded $\sim \omega$ what corresponds at present to the experimental data, then the acausal dispersion relations may be written down for $\mathfrak{M}(\omega)$ directly. They have the form

$$\mathrm{Re}\,\mathfrak{M}(\omega) = \frac{2\omega_f\,\mathrm{Re}[\,\mathrm{Res}\,\mathfrak{M}(\omega_f)]}{(\omega^2 - \omega_f^2)} + \frac{2}{\pi}\mathcal{P}\int_m^\infty \frac{\mathrm{Im}\,\mathfrak{M}(\omega')\omega' d\omega'}{(\omega'^2 - \omega^2)} + \Psi(\omega), \tag{42}$$

where

$$\Psi(\omega) = \mathrm{Re}\sum_f \frac{1}{2\pi i}\oint_{c_f} \frac{\mathfrak{M}(\nu)\,d\nu}{\nu - \omega} \tag{43}$$

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x/ The condition for the field being real is $\phi(x)$.

means the integration over the contours C_i which rule out the singularities of the function $\tilde{\rho}(\nu)$. In particular, if $\tilde{\rho}(\nu)$ has only the poles then for a pair of the conjugated poles we get (cf.(28))

$$\Psi(\omega) = \frac{A + B\omega}{(a-\omega)^2 + b^2} \tag{44}$$

i.e., a relation different from the ordinary dispersion relations not only in the high energy region, but also at low energies (if $A \neq 0$).

In the case B) one cannot write so generally the dispersion relations as it is done in the case A). However, it is possible to apply the following recipe. We divide the total scattering amplitude $\mathfrak{M}(\omega)$ into two parts

$$\mathfrak{M}(\omega) = \mathfrak{M}_o(\omega) + \mathfrak{M}_a(\omega)\,. \tag{45}$$

where $\mathfrak{M}_o(\omega)$ is the scattering amplitude satisfying the 'normal' dispersion relations $\mathfrak{M}(\omega)$ is the acausal part of the amplitude appearing as a result of the causality violation in the vicinity of the vertex of the light cone. In this case the dispersion relations may be written down for the difference

$$\mathfrak{M}_o(\omega) = \mathfrak{M}(\omega) - \mathfrak{M}_a(\omega)\,. \tag{46}$$

We get

$$\operatorname{Re}\mathfrak{M}(\omega) = \frac{2\omega_f \operatorname{Re}[\operatorname{Res}\mathfrak{M}(\omega_f)]}{(\omega^2 - \omega_f^2)} + \tag{47}$$

$$-\frac{2}{\pi}\mathcal{P}\int_m^\infty \frac{\operatorname{Im}\mathfrak{M}(\omega')\omega' d\omega'}{(\omega'^2 - \omega^2)} + \Psi(\omega),$$

where

$$\Psi(\omega) = \operatorname{Re}\mathfrak{M}_a(\omega) - \frac{2\omega_f \operatorname{Re}[\operatorname{Res}\mathfrak{M}_a(\omega_f)]}{(\omega^2 - \omega_f^2)} - \tag{48}$$

$$-\frac{2}{\pi}\mathcal{P}\int_m^\infty \frac{\operatorname{Im}\mathfrak{M}_a(\omega')\omega' d\omega'}{\omega'^2 - \omega^2}$$

Since the functions $\phi^{\text{ret}}_{\text{adv}}(R^2, xn)$ are concentrated near the vertex of the light cone, the function $\mathfrak{M}_a(\omega)$ which is the Fourier transform of $\phi^{\text{ret}}_{\text{adv}}(R^2, xn)$ vanishes at $\omega \to \infty$. If it is not equal to zero everywhere, it is different from zero also at small frequences. Therefore the function $\Psi(\omega)$ is different from zero over the whole frequency interval. In virtue of this, the ordinary dispersion relations will not be fulfilled both at high energies and at low ones.

In conclusion we write down the dispersion relations for $\pi - N$ scattering with two subtractions under the assumption that the scattering amplitude $\mathfrak{M}(\omega)$ is at infinity $\sim \omega$ and has singularities on the imaginary axis:

$$\tilde{\rho}(\omega) = \frac{\Omega^2}{\Omega^2 + \omega^2} \qquad \text{where} \quad \Omega = \frac{1}{a}$$

and a is a universal length. Thus, the amplitude $\mathfrak{M}(\omega)$ has additional poles at the points $\omega = \pm i\Omega$. For charged pions we obtain in this case:

$$\overset{(0)}{D}_+(\omega) + \overset{(0)}{D}_-(\omega) - \overset{(0)}{D}_+(\omega_0) - \overset{(0)}{D}_-(\omega_0) =$$

$$\frac{2}{\pi}(\omega^2 - \omega_0^2)\,\mathcal{P}\int_m^\infty \frac{[\overset{(0)}{A}_+(\omega') + \overset{(0)}{A}_-(\omega')]\,\omega' d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} + \tag{49}$$

$$\frac{2g^2}{M}\left(\frac{m^2}{2M}\right)^2 \frac{\omega^2 - \omega_0^2}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2 - (\frac{m^2}{2M})^2]} + \overset{(0)}{\Psi}_+(\omega),$$

$$\overset{(0)}{D}_+(\omega) - \overset{(0)}{D}_-(\omega) - \frac{\omega}{\omega_0}\{\overset{(0)}{D}_+(\omega_0) - \overset{(0)}{D}_-(\omega_0)\} =$$

$$\frac{2}{\pi}\,\omega(\omega^2 - \omega_0^2)\,\mathcal{P}\int_m^\infty \frac{[\overset{(0)}{A}_+(\omega') - \overset{(0)}{A}_-(\omega')]\,d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} + \tag{49'}$$

$$\frac{g^2 m^2}{M^2} \frac{\omega(\omega^2 - \omega_0^2)}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2 - (\frac{m^2}{2M})^2]} + \overset{(0)}{\Psi}_-(\omega), \tag{49''}$$

$$\overset{(1)}{D}_+(\omega) + \overset{(1)}{D}_-(\omega) - \frac{\omega}{\omega_0}\{\overset{(1)}{D}_+(\omega) + \overset{(1)}{D}_+(\omega_0)\} =$$

$$= \frac{2}{\pi}\,\omega(\omega^2 - \omega_0^2)\,\mathcal{P}\int_m^\infty \frac{[A_+^{(1)}(\omega') + \overset{(1)}{A}_-(\omega')]\,d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} + \tag{49''}$$

$$+ \frac{2g^2}{M^2} \frac{\omega(\omega^2 - \omega_0^2)}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2 - (\frac{m^2}{2M})^2]} + \overset{(1)}{\Psi}_+(\omega),$$

$$\overset{(1)}{D}_+(\omega) - \overset{(1)}{D}_-(\omega) - \overset{(1)}{D}_+(\omega_0) + \overset{(1)}{D}_-(\omega_0)$$

$$= \frac{2}{\pi}(\omega^2 - \omega_0^2)\,\mathcal{P}\int_m^\infty \frac{\overset{(1)}{A}_+(\omega') - A_-^{(1)}(\omega')]\,\omega' d\omega'}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_0^2)} + \tag{49'''}$$

$$\frac{2g^2}{M}\left(\frac{m^2}{2M}\right)^2 \frac{\omega^2 - \omega_0^2}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2 - (\frac{m^2}{2M})^2]} + \overset{(1)}{\Psi}_-(\omega) \;;$$

for neutral
$$D_0^{(0)}(\omega) - D_0^{(0)}(\omega_0) =$$

$$= \frac{2}{\pi}(\omega^2 - \omega_0^2)\mathcal{P}\int_m^\infty \frac{A_0^{(0)}(\omega')d\omega'}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)} + \tag{50}$$

$$\frac{g^2}{M}\left(\frac{m^2}{2M}\right)^2 \frac{\omega^2-\omega_0^2}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2 - (\frac{m^2}{2M})^2]} + \Psi_0^{(0)}(\omega),$$

$$D_0^{(1)}(\omega) - \frac{\omega}{\omega_0} D_0^{(1)}(\omega_0) =$$

$$\frac{2}{\pi}\omega(\omega^2-\omega_0^2)\mathcal{P}\int_m^\infty \frac{A_0^{(1)}(\omega')d\omega'}{(\omega'^2-\omega^2)(\omega'^2-\omega_0^2)} + \tag{50'}$$

$$\frac{g^2}{M^2}\frac{\omega(\omega^2-\omega_0^2)}{[\omega^2 - (\frac{m^2}{2M})^2][\omega_0^2-(\frac{m^2}{2M})^2]} + \Psi_0^{(1)}(\omega).$$

Additional terms

$$\Psi_+^{(0)}(\omega),\ \Psi_-^{(0)}(\omega),\ \Psi_+^{(1)}(\omega),\ \Psi_-^{(1)}(\omega),\ \Psi_0^{(0)}(\omega),\ \Psi_0^{(1)}(\omega),$$

may be written out in the form

$$\Psi_+^{(0)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega_0^2}\tilde{\rho}(\omega)[d_+^{(0)}(i\Omega)+d_-^{(0)}(i\Omega)], \tag{51}$$

$$\Psi_-^{(0)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega_0^2}\tilde{\rho}(\omega)\frac{\omega}{\Omega}[a_+^{(0)}(i\Omega)-a_-^{(0)}(i\Omega)], \tag{51'}$$

$$\Psi_+^{(1)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega_0^2}\tilde{\rho}(\omega)\frac{\omega}{\Omega}[a_+^{(1)}(i\Omega)+a_-^{(1)}(i\Omega)], \tag{51''}$$

$$\Psi_-^{(1)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega_0^2}\tilde{\rho}(\omega)[d_+^{(1)}(i\Omega)-d_-^{(1)}(i\Omega)]. \tag{51'''}$$

$$\Psi_0^{(0)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega^2}\tilde{\rho}(\omega)d_0^{(0)}(i\Omega), \tag{52}$$

$$\Psi_0^{(1)}(\omega) = \frac{\omega^2-\omega_0^2}{\Omega^2+\omega_0^2}\frac{\omega}{\Omega}\tilde{\rho}(\omega)d_0^{(1)}(i\Omega), \tag{52'}$$

where
$$d(z) = \mathrm{Re}\ N(z),\quad a(z) = \mathrm{Im}\ N(z).$$

Suppose that

$$d(i\Omega) \underset{\Omega\to\infty}{\approx} a\,\Omega^m, \qquad \alpha(i\Omega) \underset{\Omega\to\infty}{\approx} \beta\,\Omega^n,$$

where $m \le 0 \qquad n \le 1$,

then the additional terms $\Psi(\omega)$ will be of the order

$$\frac{\omega^2-\omega_0^2}{\Omega+\omega}\;\frac{\Omega^2}{\Omega^2+\omega^2}\;a\,\Omega^m,$$
$$\frac{\omega^2-\omega_0^2}{\Omega^2+\omega^2}\;\frac{\Omega^2}{\Omega^2+\omega^2}\;\frac{\omega}{\Omega}\,\beta\,\Omega^n. \qquad (53)$$

It is seen from here that at $\omega \ll \Omega$ the additional terms are small. But they become essential at $\omega \gtrsim \Omega$ If the length $a = \frac{h}{MC} \approx 10^{-14}$ cm, then already in the region ω of several GeV essential deviations from the normal dispersion relations will take place. The analysis made in /12, 13, 14/ shows that with the presently available accuracy the dispersion relations for πN scattering are fulfilled with an accuracy of 5-10% in the region of 0.1 – 0.5 GeV and in the region of 10-20 GeV – with an accuracy of 10-20%. This points out that the universal length is probably less than 10^{14} cm.

5. Conclusion

We have considered two types of acausality: the acausality concentrated near the surface of the light cone (the case A) and the acausality concentrated near its vertex (the case B).

A measure of concentration of acausality is a certain universal length a . As such we can take, for example, the Compton nucleon length $a_M = \frac{h}{MC} = 2.10^{-14}$ cm . or a characteristic length of weak interaction $a_F = \sqrt{\frac{g_F}{hc}} = 6.\,10^{-17}$ cm. Both these possibilities do not contradict the presently available experimental data.

In the cases A) and B) the conditions of microscopic causality and spectrality were fulfilled.

It turned out that the appearance, due to the acausality of the interaction, of the singularities at infinity in the complex plane ω is rather an exception than a rule: for this it is necessary to bound sufficiently sharply the space-time region in which the usual causality is violated.

Besides, one should borne in mind that the appearance of the factor $e^{ia\omega}$ in the scattering amplitude will lead, in virtue of the optical theorem, to the oscillations of the total cross sections, while the appearance of the factor $e^{-a^2\omega^2}$ to an essential decrease of the total cross section with the increasing ω

Both these possibilities are likely to be in contradiction with the well-known experimental facts. One can draw a conlclusion that the space-time region of acausality must have a diffuse boundary (the decrease is not faster than the exponential one).

In this case no singularities appear at infinity in the complex plane ω . However, there appear other additional singularities coinciding with those of the Fourier transforms of the functions $\rho(x,)$ (cf. (23)) or $\phi^{ret}(x,n)$ (cf. (30)). These function do not vanish in the spatial region of the variable $x \sim a$

and therefore the singularities of their Fourier transforms differ from the usual singularities characteristic of causal theory.

In view of these new singularities the dispersion relations for the scattering amplitude suffer this or that change, what depends on the nature of the singularities of the function $\tilde{\rho}(Q, n)$ or $\tilde{\phi}(Q, n)$

This change is displayed in the appearance in the dispersion relation of additional terms of the type $\Psi(\omega)$, (51) and (52) in the general case are essential not only in the high energy region ($\omega \gg \frac{1}{a}$) but also over the whole energy interval involving low energies.

The example given in the previous Section shows that the universal length a is probably less than 10^{14}cm. If this is so, then in order to find acausality it is necessary to make the verification of dispersion relations more precise. In particular, when $a \approx 10^{16}$ cm, for the pions of 10 GeV energy, the accuracy should be higher than 3 %, for 20 GeV pions it should be more than 10%.

Therefore, the experimental verification of the dispersion relations for πN scattering (in this case, the non-physical region ω is known to play no role) seems to be extrimely important and apparently quite a real problem of today's experiments.

Although we carried out the calculations in the explicit form for the case when the vector n is an internal vector of a system of interacting particles, all our conclusions hold true for the case when this vector is external, i.e. when the homogeneity of space-time is violated. Here it seems to be more important to check up a possible violation of this homogeneity rather than to verify dispersion relations. This can be accomplished by comparing the results of scattering experiments (of electrons) in the laboratory system and in the centre-of-mass system. When the vector n is external both these systems are equivalent: the system in which space inhomogeneties are at rest is singled out if compared with the others. The validity of this singling out will be treated in another publication,

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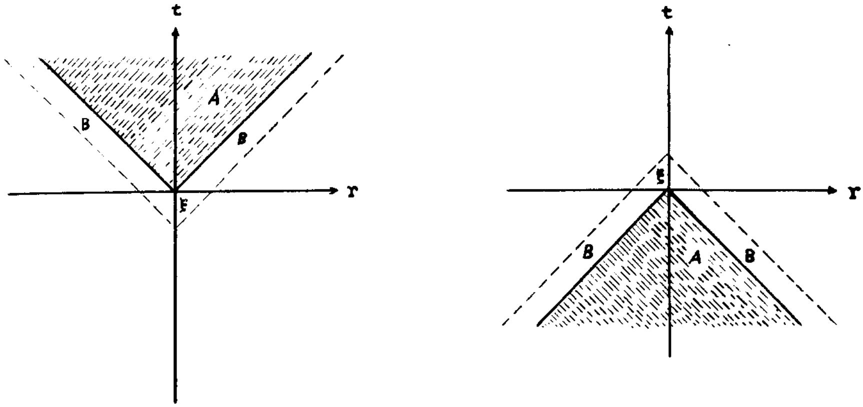

Fig 1. (a) retarded, (b) advanced interaction.
A is the region of usual causality. B is the region of acausality.

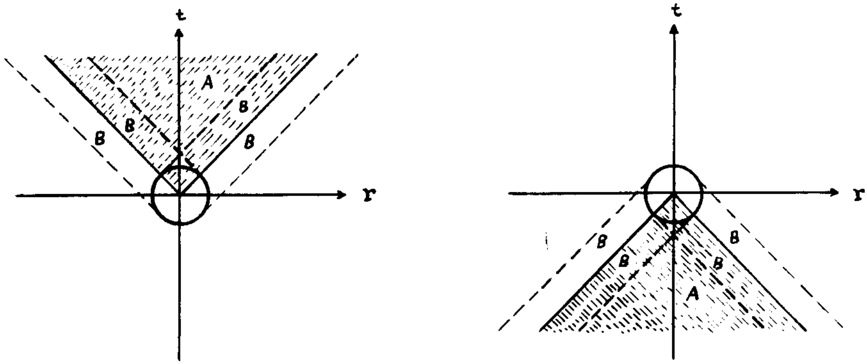

Fig. 2 (a) retarded (b) advanced interaction. A is the region of usual causality. B is the region of acausality.
The circle at the centre is the region where $\rho(R) \neq 0$.