ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ
ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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ON RELATIVISTIC ANGULAR MOMENTUM THEORY

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I. Introduction

In paper [1] invariant expansions of the scattering amplitude are introduced in terms of the eigenfunctions of the Laplace operator in the Lobachevsky space of relativistic velocities. It has been shown [2] that 34 three-orthogonal coordinate systems exist in which the variables of the Laplacian in a three-dimensional space of constant negative curvature are separable. Four of these systems, specially convenient for investigating binary collisions [3], i.e. those which have one center and are axially symmetrical have been studied in detail in [1].

In this paper we shall consider some of the properties of these four systems from the point of view of group theory and their connection with the four-dimensional angular momentum. Complete systems of commuting operators, corresponding to definite subgroups of the homogeneous Lorentz group, are introduced in each of these systems. The classical quantities corresponding to these operators are given explicitly and electromagnetic fields, in which they are conserved, are calculated.

II. Infinitesimal Operators of the Homogeneous Lorentz Group

We shall work in the space of functions determined on the upper sheet of a two-sheet three-dimensional hyperboloid, which is an invariant hyper-surface in the four-dimensional velocity space (this is a realization of the Lobachevsky space).

Let us consider representations of the homogeneous Lorentz group which have eigenfunctions of the Laplacian on the hyperboloid as their basis vectors and write down explicitly the infinitesimal operators. In the following A_i are the infinitesimal operators of space rotations, B_i of hyperbolic ones. Other notations are the same as in [1]. The infinitesimal operators can be obtained by solving the Killing equations [4], or directly as they are obtained in [5] for the rotation group.

Spherical system S.

$$A_1 = \cos\phi \frac{\partial}{\partial \theta} - \operatorname{cotg}\theta \sin\phi \frac{\partial}{\partial \phi} \tag{1}$$

$$A_2 = -\sin\phi \frac{\partial}{\partial \theta} - \operatorname{cotg}\theta \cos\phi \frac{\partial}{\partial \phi}$$

$$A_3 = \frac{\partial}{\partial \phi}$$

$$B_1 = \sin\theta\ \sin\phi \frac{\partial}{\partial a} + \mathrm{cth}\, a\ \cos\theta\ \sin\phi \frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\ \mathrm{cth}\, a\ \cos\phi \frac{\partial}{\partial\phi}$$

$$B_2 = \sin\theta\ \cos\phi \frac{\partial}{\partial a} + \mathrm{cth}\, a\ \cos\theta\ \cos\phi \frac{\partial}{\partial\theta} - \frac{1}{\sin\theta}\ \mathrm{cth}\, a \sin\phi \frac{\partial}{\partial\phi}$$

$$B_3 = \cos\theta \frac{\partial}{\partial a} - \sin\theta\ \mathrm{cth}\, a \frac{\partial}{\partial\theta}$$

Hyperbolical system H

$$A_1 = \frac{\partial}{\partial\phi}$$

$$A_2 = -\mathrm{sh}\, b \cos\phi \frac{\partial}{\partial a} + \mathrm{th}\, a\ \mathrm{ch}\, b\ \cos\phi \frac{\partial}{\partial b} - \mathrm{th}\, a \frac{1}{\mathrm{sh}\, b} \sin\phi \frac{\partial}{\partial\phi}$$

$$A_3 = \mathrm{sh}\, b\ \sin\phi \frac{\partial}{\partial a} - \mathrm{th}\, a\ \mathrm{ch}\, b\ \sin\phi \frac{\partial}{\partial b} - \mathrm{th}\, a \frac{1}{\mathrm{sh}\, b} \cos\phi \frac{\partial}{\partial\phi} \qquad (2)$$

$$B_1 = \mathrm{ch}\, b \frac{\partial}{\partial a} - \mathrm{th}\, a\ \mathrm{sh}\, b \frac{\partial}{\partial b}$$

$$B_2 = \sin\phi \frac{\partial}{\partial b} + \mathrm{cth}\, b\ \cos\phi \frac{\partial}{\partial\phi}$$

$$B_3 = \cos\phi \frac{\partial}{\partial b} - \mathrm{cth}\, b\ \sin\phi \frac{\partial}{\partial\phi}$$

Cylindrical system C:

$$A_1 = \frac{\partial}{\partial\phi}$$

$$A_2 = -\mathrm{ch}\, a\ \mathrm{th}\, b\ \cos\phi \frac{\partial}{\partial a} + \mathrm{sh}\, a\ \cos\phi \frac{\partial}{\partial b} - \mathrm{sh}\, a\ \mathrm{cth}\, b\ \sin\phi \frac{\partial}{\partial\phi} \qquad (3)$$

$$A_3 = \mathrm{ch}\, a\ \mathrm{th}\, b\ \sin\phi \frac{\partial}{\partial a} - \mathrm{sh}\, a\ \sin\phi \frac{\partial}{\partial b} - \mathrm{sh}\, a\ \mathrm{cth}\, b\ \cos\phi \frac{\partial}{\partial\phi}$$

$$B_1 = \frac{\partial}{\partial a}$$

$$B_2 = -\mathrm{sh}\, a\ \mathrm{th}\, b\ \sin\phi \frac{\partial}{\partial a} + \mathrm{ch}\, a\ \sin\phi \frac{\partial}{\partial b} + \mathrm{ch}\, a\ \mathrm{cth}\, b\ \cos\phi \frac{\partial}{\partial\phi}$$

$$B_3 = -\mathrm{sh}\, a\ \mathrm{th}\, b\ \cos\phi \frac{\partial}{\partial a} + \mathrm{ch}\, a\ \cos\phi \frac{\partial}{\partial b} - \mathrm{ch}\, a\ \mathrm{cth}\, b\ \sin\phi \frac{\partial}{\partial\phi}$$

Horospherical system O:

$$A_1 = r\cos\phi \frac{\partial}{\partial a} - \frac{e^{-a}}{2}[-e^{-a} + (r^2+1)e^{a}]\cos\phi \frac{\partial}{\partial r} + \frac{e^{-a}}{2r}[-e^{-a} + (-r^2+1)e^{a}]\sin\phi \frac{\partial}{\partial\phi}$$

$$A_2 = -r\ \sin\phi \frac{\partial}{\partial a} + \frac{e^{-a}}{2}[-e^{-a} + (r^2+1)e^{a}]\sin\phi \frac{\partial}{\partial r} + \frac{e^{-a}}{2r}[-e^{-a} + (-r^2+1)e^{a}]\cos\phi \frac{\partial}{\partial\phi} \qquad (4)$$

$$A_3 = \frac{\partial}{\partial\phi}$$

$$B_1 = r\ \sin\phi \frac{\partial}{\partial a} - \frac{e^{-a}}{2}[-e^{-a} + (r^2-1)e^{a}]\sin\phi \frac{\partial}{\partial r} + \frac{e^{-a}}{2r}[e^{-a} + (r^2+1)e^{a}]\cos\phi \frac{\partial}{\partial\phi}$$

$$B_2 = r\cos\phi \frac{\partial}{\partial a} - \frac{e^{-a}}{2}[-e^{-a} + (r^2-1)e^{a}]\cos\phi \frac{\partial}{\partial r} - \frac{e^{-a}}{2r}[e^{-a} + (r^2+1)e^{a}]\sin\phi \frac{\partial}{\partial\phi}$$

$$B_3 = -\frac{\partial}{\partial a} + r\frac{\partial}{\partial r}$$

Note that the commutation relations for these operators are the same in all systems,

$$[A_i, A_k] = \epsilon_{ik\ell} A_\ell$$

$$[A_i, B_k] = \epsilon_{ik\ell} B_\ell \qquad (5)$$

$$[B_i, B_k] = -\epsilon_{ik\ell} A_\ell$$

However, the matrix "canonical" form of these operators will differ in each system and will coincide with that given in /5,6/ only in the S-system.

The connection with the four-dimensional angular momentum is given in all four systems by the formulas (using the metric $ds^2 = du_0^2 - du_1^2 - du_2^2 - du_3^2$).

$$M_{k\ell} = -i\left(u_k \frac{\partial}{\partial u_\ell} - u_\ell \frac{\partial}{\partial u_k}\right) = i\epsilon_{k\ell m} A_m$$

$$M_{0k} = u_0 \frac{\partial}{\partial u_k} + u_k \frac{\partial}{\partial u_0} = B_k \qquad (6)$$

$$k, \ell, m = 1, 2, 3$$

III. Invariants of the Lorentz Group and its Subgroups

It is well-known /6/ that the homogeneous Lorentz-group has two invariants

$$\Delta_L = \sum_{k=1}^{3} (B_k^2 - A_k^2)$$

$$\Delta' = \sum_{k=1}^{3} A_k B_k \qquad (7)$$

In our case the first is equal to the Laplacian on the hyperboloid, the second is related to the intrinsic spin and is equal to zero identically.

The difference between the systems S, H, C and O is in invariants corresponding not to the whole Lorentz group, but to definite subgroups, i.e. commuting only with all of the operators of the subgroup. The separation of variables in the Laplace equation

$$\Delta_L f(u) = -(p^2+1) f(u) \qquad (8)$$

is directly connected with these subgroup invariants and the function $f(u)$ will in each system be an eigenfunction of the complete set of such (commuting) invariants. The eigenvalues, originating as separation constants in (8), will play the role of quantum numbers. The quantum number p always corresponds to the invariant Δ_L . Note that all the eigenfunctions, corresponding to a certain p form the basis of an irreducible (in general infinite) representation of the Lorentz group.

Let us consider all four coordinate systems from this point of view.

The S system. The invariants are

$$L^2 = A_1^2 + A_2^2 + A_3^2 = \frac{\partial^2}{\partial \theta^2} + \cot g\, \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \tag{9}$$

$$L_z = A_3 = \frac{\partial}{\partial \phi}$$

and they correspond to the three-parametrical rotation group and to its one-parametrical subgroup. Here we naturally obtain the quantum numbers ℓ and m (usual three-dimensional angular momentum and its projection).

H - system. The invariants are

$$H^2 = B_2^2 + B_3^2 - A_1^2 = \frac{\partial}{\partial b^2} + \operatorname{cth} b \frac{\partial}{\partial b} + \frac{1}{\operatorname{sh}^2 b} \frac{\partial^2}{\partial \phi^2} \tag{10}$$

$$L_z = A_1 = \frac{\partial}{\partial \phi}$$

and correspond to the three-dimensional Lorentz group with the infinitesimal operators B_2 , B_3 and A_1 and to its subgroup - space rotations around the axis 1 . This gives rise to the quantum numbers a and m where $a = -\frac{1}{2} + iq$.

C - system. This system is symmetrical with respect to space and hyperbolic rotations. The invariants are

$$L_z = A_1 = \frac{\partial}{\partial \phi}$$
$$K_z = B_1 = \frac{\partial}{\partial a} \tag{11}$$

and correspond simply to the one-parametrical groups of rotations in the (23) plane or (01) plane respectively. The corresponding quantum numbers are r and m.

0 - system. The invariant operators are

$$0^2 = (B_1 + A_2)^2 + (B_2 - A_1)^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \tag{12}$$

$$L_z = A_3 = \frac{\partial}{\partial \phi}$$

and correspond to a three-parametrical group with the infinitesimal operators $A_3, B_1 + A_2, B_2 - A_1$. Their commutation relations are

$$[A_3, B_1 + A_2] = B_2 - A_1, \quad [A_3, B_2 - A_1] = -(B_1 + A_2)$$
$$[B_1 + A_2, B_2 - A_1] = 0 \tag{13}$$

and we see that infinitesimally they determine the group of motions of an Euclidean plane (one rotation, two translations). In our case the finite group is the group of motions on an horosphere. The operator 0^2 is the Laplacian on a plane in cylindrical coordinates, which explains why the corresponding eigenfunctions are Bessel functions. As usual L_z just corresponds to rotations around one axis.

IV. Related Topics in Classical Dynamics

The connection between the infinitesimal operators and relativistic angular momentum has already been stressed. To illustrate the physical meaning of the subgroup invariants, let us consider the coordinate space. All the coordinate systems introduced in velocity space, can naturally also be introduced in the x - space, e.g. the S - system can be written as

$$\begin{aligned} x_0 &= x \operatorname{ch} a \\ x_3 &= x \operatorname{sh} a \cos\theta \\ x_2 &= x \operatorname{sh} a \sin\theta \cos\phi \\ x_1 &= x \operatorname{sh} a \sin\theta \sin\phi \end{aligned} \tag{14}$$

(x and a can also be complex). Formulas (1) - (4) hold, if the right hand sides are multiplied by x.

Let x^i be arbitrary curvilinear coordinatss. The infinitesimal operators of the homogeneous Lorentz group can be written as /7/:

$$X_a = \xi^i_{(a)} \frac{\partial}{\partial x^i} \qquad i = 1, \ldots, 4 \qquad a = 1, \ldots, 6 \tag{15}$$

where ξ^i_a are solutions of the Killing equations.

A linear first integral

$$C^{(a)} = \xi_{i(a)} \frac{dx^i}{ds} \tag{16}$$

of the geodesic equation corresponds to each such operator.

We shall not write down the values of all such integrals in our systems, but only consider combinations of them corresponding to the subgroup invariants.

The group invariant Δ_L corresponds to the classical quantity

$$D = \sum_{a=1}^{3} C^{(a)2} - \sum_{a=4}^{6} C^{(a)2}$$

In Cartesian coordinates this gives

$$D = m^2 \{ x_\mu x^\mu - (x_\mu u^\mu)^2 \} \tag{17}$$

and we see that D is just the square of the four-dimensional angular momentum.

Further let us consider the quantities, corresponding to the invariants of the subgroups (m is the particle rest mass):

S -system

$$L_z \to \mathfrak{L}_z = m x^2 sh^2 a \sin^2\theta \frac{d\phi}{ds}$$

$$L^2 \to \mathfrak{L}^2 = m^2 x^4 sh^4 a \left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 \right]. \tag{18}$$

H -system

$$L_z \to \mathfrak{L}_z = m x^2 ch^2 a\, sh^2 b \frac{d\phi}{ds}$$

$$H^2 \to \mathfrak{H}^2 = m^2 x^4 ch^4 a \left[\left(\frac{db}{ds}\right)^2 + sh^2 b \left(\frac{d\phi}{ds}\right)^2 \right]. \tag{19}$$

C -system

$$L_z \to \mathfrak{L}_z = m x^2 sh^2 b \frac{d\phi}{ds}$$

$$K_z \to \mathfrak{K}_z = m x^2 ch^2 b \frac{da}{ds} \tag{20}$$

O - system

$$L_z \to \mathfrak{L}_z = m x^2 r^2 e^{2a} \frac{d\phi}{ds}$$

$$O^2 \to \mathfrak{O}^2 = m^2 x^4 e^{4a} \left\{ \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\phi}{ds}\right)^2 \right\} \tag{21}$$

We see that in the S -system we just obtain the square of the three-dimensional angular momentum. In the other systems new integrals appear and it is necessary to find the fields in which they are conserved.

Let us consider the equations of motion in curvilinear coordinates

$$m\left\{ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} \right\} = F^\mu \tag{22}$$

F^μ is the Minkowski force; for a charged particle in an electromagnetic field we have

$$F^\mu = e\left(\frac{\partial A^\sigma}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\sigma} \right) u^\sigma \tag{23}$$

where A^μ is the four-dimensional potential, e - the charge of the particle.

Writting equations (22) in the S, H, C and O systems and demanding that $\mathfrak{L}_z$ and one of the quantities $\mathfrak{L}^2, \mathfrak{H}^2, \mathfrak{K}_z$ or $\mathfrak{O}^2$ should be constant along the trajectory, we obtain conditions on F^μ and hence also on A^μ. Dropping the details we shall only give the final results.

a) The quantities, describing the S system are conserved if $A^\phi = A^\theta = 0$

and A^x and A^a do not depend on θ and ϕ. In cartesian coordinates this gives

$$A_i = x_i \Phi_1(x_1^2 + x_2^2 + x_3^2, x_0) \qquad i = 1,2,3$$

$$A_0 = \Phi_2(x_1^2 + x_2^2 + x_3^2, x_0) \tag{24}$$

where Φ_1, Φ_2 (here and in the following formulas) are arbitrary functions (of the given variables). This formula and the following ones can of course be simplified by a gauge transformation. Equation (24) is a well-known result - the usual angular momentum is conserved in a spherically symmetrical field.

b) The quantities $\mathfrak{H}^2$ and $\mathfrak{L}_z$ are conserved in the H -system if $A^b = A^\phi = 0$, A^x and A^a do not depend on b and ϕ. In cartesian coordinates this means

$$A_1 = \Phi_1(x_1, x_2^2 + x_3^2 - x_0^2)$$

$$A_\lambda = x_\lambda \phi_2(x_1, x_2^2 + x_3^2 - x_0^2) \qquad \lambda = 2,3,0 \tag{25}$$

c) The quantities $\mathfrak{L}_z$ and $\mathfrak{K}_z$ of the C -system are conserved if $A^a = A^\phi = 0$, A^x and A^b independent on a and ϕ i.e.

$$A_0 = x_0 \Phi_1(x_0^2 - x_1^2, x_2^2 + x_3^2)$$

$$A_1 = x_1 \Phi_1(x_0^2 - x_1^2, x_2^2 + x_3^2)$$

$$A_2 = x_2 \Phi_2(x_0^2 - x_1^2, x_2^2 + x_3^2) \tag{26}$$

$$A_3 = x_3 \Phi_2(x_0^2 - x_1^2, x_2^2 + x_3^2)$$

d) The quantities $\mathfrak{O}^2$ and $\mathfrak{L}_z$ in the O -system are conserved if $A^\phi = A^r = 0$ A^x and A^a independent on r and ϕ i.e.

$$A_1 = x_1 \Phi_1(x_0^2 - x_1^2 - x_2^2 - x_3^2, x_0 - x_3)$$

$$A_2 = x_2 \Phi_1(x_0^2 - x_1^2 - x_2^2 - x_3^2, x_0 - x_3)$$

$$A_3 = \tfrac{1}{2} \frac{\sqrt{x_\mu x^\mu}}{x_0 - x_3} \Phi_1(x_\nu x^\nu, x_0 - x_3) + \tfrac{1}{2}\left(\frac{x_1^2 + x_2^2}{(x_0 - x_3)^2} - 1 \right) \frac{x_0 - x_3}{\sqrt{x_\mu x^\mu}}, \Phi_2(x_\nu x^\nu, x_0 - x_3) \tag{27}$$

$$A_0 = \tfrac{1}{2} \frac{\sqrt{x_\mu x^\mu}}{x_0 - x_3} \Phi_1(x_\nu x^\nu, x_0 - x_3) + \tfrac{1}{2}\left(\frac{x_1^2 + x_2^2}{(x_0 - x_3)^2} + 1 \right) \frac{x_0 - x_3}{\sqrt{x_\mu x^\mu}} \Phi_2(x_\nu x^\nu, x_0 - x_3)$$

The results of this chapter can be used to write down directly the first integrals of the equations of motion of a particle in fields of the type (24)-(27).

V. Conclusion

We have already mentioned that a number of other coordinate systems, mostly if the elliptic type, exists, in which the variables in the Laplacian separate. These systems are connected with each other by transformations belonging to the 15-parametrical group of conformal transformations. This question and also the relation between the 15-parametrical group and relativistic angular momentum theory will be investigated in a future paper.

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