ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

LABORATORY OF THEORETICAL PHYSICS

G. Domokos, P. Suranyi and A. Vancura

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FERMI INTERACTIONS AT HIGH ENERGIES

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Аннотация

Исследована 4-фермионная лептонная вершина в лестничном приближении. Вершинная функция или амплитуда рассеяния лептонов определена однозначно: произвольные константы перенормировки не появляются в конечных выражениях. Фермиевская константа связи перенормируется конечным множителем $(\frac{2}{\pi})^{1/2}$. Амплитуда рассеяния является чисто мнимой при больших энергиях. Дифференциальное сечение убывает с энергией. Упругое сечение убывает как $s^{-1} \log s$. Полу-ширина пика вперед постоянна.

Abstract

The four lepton vertex function is investigated in the ladder approximation. The vertex function of lepton-lepton scattering amplitude is determined uniquely: Arbitrary renormalisation constants do not appear in the final expressions. The partial wave amplitude is a meromorphic function of the four dimensional angular momentum in the whole complex plane. The Fermi coupling constant is renormalized by the finite factor $(\frac{2}{\pi})^{1/2}$. The scattering amplitude is purely imaginary at high energies. The correction factor to the angular distribution is a universal function of the variable $x = -\log s\,(\log(-t))^{-1}$.
The differential cross section decreases with energy as $s^{-1}(\log s)^{-2}$.
The elastic cross section decreases as $s^{-1} \log s$ The half-width of the forward peak is constant. The contribution of the direct channel to the cross section decreases as $s^{-1/3}$.

1. Introduction

Ever since Heisenberg divided the local interactions in field theory into two kinds[1-4], the theory of interactions of the second kind have been the "enfant terrible" of quantum field theory. It seems on the one hand that they do exist in nature (in particular weak interactions belong to this kind), on the other hand with the development of renormalization theory it turned out that interactions of the second kind* are "nonrenormalizable" in the sense that the infinities (more correctly, the arbitrariness) arising in a perturbation expansion cannot be removed by renormalizing the constants of the theory (i.e. masses and coupling constants).

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* i.e. those where the coupling constant has the dimensions of a positive power of length. (Throughout we work in a system of units where $\hbar = c = 1$).

Actually, in a perturbative expansion of S matrix elements or Green functions, there arises an infinite number of arbitrary subtraction constants (See e.g.[5]). A more physical aspect of the problem is that interactions of the second kind are expected to "become strong" at high energies. In fact, the dimensionless expansion parameter in a perturbative expansion is not the coupling constant itself, but the coupling constant times some power of a characteristic momentum of the system (total C.M. momentum, momentum transfer, etc.). Focussing our attention to Fermi interactions, the coupling constant f has a dimension (length)[2], so one expects a dimensionless expansion parameter to be fE^2 where E is the total C.M. momentum. This shows that $fE^2 \sim 1$ (i.e. "the weak interaction becomes strong") at $|E| \sim 300$ GeV. This circumstance has given rise to many speculations about high energy lepton physics and higher order corrections to weak interactions; a survey of them with a practically complete bibliography can be found in the recent work of Markov[6].

This situation is to be confronted with the following observation of Pais [7]. If - as follows from the previous considerations - higher order corrections to weak interactions were large, we should get a large $K_1^\circ - K_2^\circ$ mass difference in striking contradiction with experimental results.

In the framework of the W -meson theory, Feinberg and Pais[8-9], by applying a special procedure to remove the dangerous divergences, succeeded in summing ladder diagrams their result is that higher order corrections do play a non - negligible role; in particular, they obtained that the coupling constant is renormalized by a factor of 3/4.

The aim of the present work is to investigate the behaviour of the four-lepton vertex (or equivalently, the two-lepton propagator) at high energies. Instead of assuming an intermediate boson theory, we assume the existence of an elementary Fermi interaction of the V-A type. To calculate the vertex, we apply the ladder approximation: we consider two fermions propagating with the successive exchange of a fermion loop. The Bethe-Salpeter equation with singular interaction kernel arising thereby is treated by a method, proposed in ref. |10|. The essence of that method consisted in transforming the *BS* equation into a non-Fushsian differential equation, which can be investigated by known methods |11|.

In Sec. 2 after a brief general discussion of the properties of the four-fermion vertex, we derive the differential equation mentioned above. Sec. 3 is devoted to the description of the approximation method used to solve the equation. Sec. 4 contains the main result of the paper: We show that subtraction terms do not contribute to the amplitude and we obtain an asymptotic expression for the scattering amplitude of two leptons; we find that the leading term is purely imaginary and decreases with energy. The fifth section deals with the calculation of the asymptotic behaviour of the scattering amplitude in the direct channel by the application of the *WKB* method. In the last sixth section we discuss some physical consequences of the results obtained; further we point out the probable limitations of our approximation and possible improvements.

2. Derivation of the differential equation

Consider a four-fermion interaction of the $V-A$ type with charged currents only. As an interaction kernel for the *BS* equation, we choose the diagram drawn on Fig. 1. If the diagram on Fig. 1 is to describe the interaction between two charged leptons (μ, e), the loop in the intermediate state contains two neutrinos while if we consider the scattering of a charged lepton on a neutrino, there is a charged lepton and a neutrino exchanged. However, in what follows, we neglect all the lepton masses, thus, obtaining an "asymptotic equation" in the sense, explained in ref. |10|; in this approximation all the lepton-lepton ($\ell\ell$) and antilepton-antilepton (aa) amplitudes equal each other, while the lepton-antilepton (ℓa) amplitudes differ from them by sign. Instead of going through the usual formal argument to show this, we prefer the following elementary argument. The two-fermion propagator can be conceived as the infinite iteration of the interaction kernel of Fig. 1. A glance at Fig. 2 shows that if we change the orientation of an internal fermion loop, we obtain the same expression*; going from $\ell\ell$ to ℓa amplitude means reversing the orientation of the "last", open fermion line, which obviously results in a change of the sign of the amplitude; for an ($a\,a$) amplitude we have to change the orientation of two open lines.

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* Like in quantum electrodynamics for a loop with four external photon lines.

By the same argument we see that instead of treating a multichannel problem, it is sufficient to calculate the ladder diagram with one definite orientation of the lines only.

Denoting the two-lepton propagator by G, the free one by G_0 the interaction kernel by K, the BS equation in operator form reads:

$$G = G_0 + G_0 K G \tag{2.1}$$

where the quantities involved have the following expressions in momentum representation:

$$\begin{aligned} G_0(p,q;E) &= \delta(p-q)\,[\tfrac{1}{2}\hat{E}^{(1)} - \hat{p}^{(1)}]^{-1}\,[\tfrac{1}{2}\hat{E}^{(2)} - \hat{p}^{(2)}]^{-1}, \\ K(p,q) &= \gamma^{(1)}_{\rho}(1 + i\gamma^{(1)}_{5})\,\gamma^{\rho(2)}(1 + i\gamma^{(2)}_{5}).\,K, \\ G(p,q;E) &= \gamma^{(1)}_{\rho}(1 - i\gamma^{(1)}_{5})\,\gamma^{\rho(2)}(1 - i\gamma^{(2)}_{5})\,G. \end{aligned} \tag{2.2}$$

In eq. (2.2) E is the total CM four momentum of the leptons, p, q are their relative momenta; the superscripts (1) and (2) refer to the "first" and "second" lepton, respectively. The scalar function K is given by the following expressions:

$$K = K_1(t) + K_2(t) = \frac{f^2 t^2}{(2\pi)^4 i}\,(2\pi)^3 \int_0^\infty \frac{dt'}{t'(t'-t)} + g_1 + g_2 t\,. \tag{2.3}$$

Here $t=(p-q)^2$, f stands for the Fermi coupling constant, g_1 and g_2 are subtraction constants. The symbols $K_1(t)$ and $K_2(t)$ stand for the spectral integral and the subtraction polynomial, respectively. (Notice that we have only one invariant function in the expression of G; this is the consequence of the $V-A$ character of the four-lepton interaction and the vanishing of the lepton masses). The $B.S.$ equation for the invariant function G can be derived in a straightforward way. The result is:

$$(p^2 - \tfrac{1}{4}E^2)\,G(p,q;E) = \delta(p-q) + \frac{16 f^2}{(2\pi)^4 i}\int dk\,K(p,k)\,G(k,q;E)\,. \tag{2.4}$$

Following the procedure described in ref. |10| we now separate the "analytic part" of G, called G_1 (i.e. that obeying eq. (2.4) with K_1 only), go over to Euclidean metric and expand G_1 according to four-dimensional spherical harmonics. (See, however, Sec.6), Finally we perform a Hankel transformation on the resulting radial equation. After these operations we are left with the following differential equation for the radial Green function $G_l(z,z')$:

$$[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \frac{n^2-1}{r^2} + k^2]\ G_l(r,r') =$$

$$= \lambda^2 r^{-6}\ G_l(r,r') + (rr')^{-3/2}\delta(r-r'), \tag{2.5}$$

where $k^2 = \frac{1}{4}E^2$, $\lambda^2 = 32\pi^{-1} f^2$; n^2 is the square of the four dimensional orbital momentum, defined as in ref. 10 (i.e. its physical values being 1,2,3 ...). Alternatively, we can write the equation for the wave function, $\psi(r) = r^{-3/2}u(r)$:

$$[\frac{d^2}{dr^2} + k^2 - \frac{n^2-\frac{1}{4}}{r^2} - \frac{\lambda^2}{r^6}]\cdot u(r) = 0 \tag{2.6}$$

Eq. (2.6) has exactly the form of a radial Schrödinger equation with a repulsive potential $\lambda^2 r^{-6}$. Therefore, in order to find the scattering amplitude, we can apply the procedure, familiar in non relativistic quantum mechanics. Eq. (2.6) cannot be integrated in terms of known transcendental functions, so we are going to develop an approximation procedure, valid for small values of k^2 and potentials more singular than $O(r^{-2})$ at the origin (A usual effective range expansion would diverge, cf. Landau-Lifschiz/12/).

3. Approximate Solution of the Differential Equation

We start from the observation that if we drop either the kinetic energy term or the potential from eq. (2.6), it can be integrated exactly in terms of Bessel functions.

In fact, the equation:

$$(\frac{d^2}{dr^2} + k^2 - \frac{n^2-\frac{1}{4}}{r^2})\ v_0(r) = 0 \tag{3.1}$$

has the following two independent solutions:

$$v_0^1(r) = r^{1/2} H_n^{(1)}(kr), \tag{3.2}$$

$$v_0^2(r) = r^{1/2} H_n^{(2)}(kr),$$

with the boundary conditions:

$$\left.\begin{aligned} v_0^1(r) &\sim \exp\{i[kr - \frac{\pi}{2}n - \frac{\pi}{4}]\}, \\ v_0^2(r) &\sim \exp\{-i[kr - \frac{\pi}{2}n - \frac{\pi}{4}]\}, \end{aligned}\right\} r\to\infty \tag{3.3}$$

Putting $k^2 = 0$ we arrive at the equation:

$$[\frac{d^2}{dr^2} - \frac{n^2-\frac{1}{4}}{r^2} - \frac{\lambda^2}{r^6}]\ v_1(r) = 0, \tag{3.4}$$

with the independent solutions:

$$v_1^1(r) = r^{1/2} K_{\frac{n}{2}}\left(\frac{\lambda}{2 r^2}\right),$$
$$v_1^2(r) = r^{1/2} I_{\frac{n}{2}}\left(\frac{\lambda}{2 r^2}\right), \tag{3.5}$$

and the boundary conditions:

$$v_1^1(r) = O\left(r^{3/2}\exp\left\{-\frac{\lambda}{2r^2}\right\}\right),$$
$$v_1^2(r) = O\left(r^{3/2}\exp\left\{\frac{\lambda}{2r^2}\right\}\right), \qquad r \to 0 . \tag{3.6}$$

We construct the approximate solution of eq. (2.6) from the functions (3.2) and (3.5), by neglecting the kinetic energy term if r is less than some conveniently chosen value, r_0 and neglecting the potential for $r > r_0$. It is intuitively clear that such an approximation procedure should work well if the potential is very large at small distances while decreases sufficiently rapidly as $r \to \infty$.

We choose for r_0 the distance at which the kinetic and potential energy terms are equal in magnitude, i.e. $r_0 = (\lambda / k)^{1/3}$. (If $k^2 < 0$, which is the domain we are interested in, when going over to the crossed channel, the coefficients of eq. (2.6) are even continuous at $r = r_0$ with $r_0 = (i\lambda / k)^{1/3}$.

We look for a solution of eq. (2.6), vanishing at $r=0$ and being a combination of ingoing and outgoing waves at infinity.

therefore we choose

$$u(r) = \begin{cases} v_1^1(r) & (r < r_0) \\ F_1\, v_0^1(r) + F_2\, v_0^2(r) & (r > r_0) \end{cases} \tag{3.7}$$

Matching the logarithmic derivatives at $r = r_0$, we obtain for the scattering matrix element:

$$S(n, k^2) = e^{2i\delta_n} = \frac{F_1}{F_2} = -\frac{H_n^{(2)}(\epsilon)\, K'_{n/2}\left(\frac{\epsilon}{2}\right) + H_n^{(2)\prime}(\epsilon)\, K_{n/2}\left(\frac{\epsilon}{2}\right)}{H_n^{(1)}(\epsilon)\, K'_{n/2}\left(\frac{\epsilon}{2}\right) + H_n^{(1)\prime}(\epsilon)\, K_{n/2}\left(\frac{\epsilon}{2}\right)} . \tag{3.8}$$

In eq. (3.8) primes mean derivatives with respect to the arguments of the cylindrical functions, and $\epsilon = (k^2 \lambda)^{1/3}$

The approximation procedure described above can be considered as a zeroth order term of a perturbation series to the differential equation:

$$\left[\frac{d^2}{dr^2} + \theta(r - r_0)\, k^2 - \theta(r_0 - r)\frac{\lambda^2}{r^6} - \frac{n^2 - 1/4}{r^2}\right] u(r) = W(r)\, u(r), \tag{3.9}$$

$\theta(x)$ being the unit step function and the perturbation operator $W(r)$ has the following expression:

$$W(r) = -k^2\,\theta(r_0 - r) + \frac{\lambda^2}{r^4}\,\theta(r - r_0) \qquad (3.10)$$

Let us sketch the proof of the convergence of the perturbation expansion for $k^2 < 0$. In order to simplify matters, let us split off the factor $r^{1/2}$ from v in eqs. (3.2) and (3.6): $v(r) = r^{1/2} w(r)$ and multiply the equation for $w(r)$ by r^2. Thus we obtain with $k^2 = -\kappa^2$ $(\kappa^2 > 0)$:

$$\left[r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - \theta(r - r_0)\kappa^2 r^2 - \frac{\lambda^2}{r^2}\,\theta(r_0 - r) - n^2 \right] w(r) = \qquad (3.11)$$

$$= v(r)\, w(r),$$

with

$$V(r) = \kappa\, r\, \theta(r - r) + \frac{\lambda}{r}\,\theta(r - r)$$

Considering eq. (3.11) as an eigenvalue problem for n with the boundary condition $w(0) = w(\infty) = 0$, the unperturbed solution is:

$$w_\nu^{(0)}(r) = \theta(r - r_0)\, K_{\frac{\nu}{2}}\left(\frac{\lambda}{2r_0^2}\right) K_\nu(\kappa r) + \theta(r_0 - r)\, K_{\frac{\nu}{2}}\left(\frac{\lambda}{2r^2}\right) K_\nu(\kappa r), \qquad (3.12)$$

while the unperturbed eigenvalue ν satisfies the usual determinantal equation

$$f(\nu) = K_\nu(\kappa r_0)\frac{d}{dr_0} K_{\frac{\nu}{2}}\left(\frac{\lambda}{2r_0^2}\right) - K_{\frac{\nu}{2}}\left(\frac{\lambda}{2r_0^2}\right)\frac{d}{dr_0} K_\nu(\kappa r_0) = 0. \qquad (3.13)$$

In the standard way [13] we find that the solutions of eq. (3.13) are pure imaginary so that the functions $w_\nu^{(0)}$ are real. The functions $w_\nu^{(0)}$ are orthogonal:

$$\int_0^\infty \frac{dr}{r}\, w_\nu^{(0)}(r)\, w_{\nu'}^{(0)}(r) = N_\nu\, \delta_{\nu\nu'},$$

(N_ν being a normalization coefficient). The perturbed eigenvalue, n, is of course given by:

$$n = \nu + V_{\nu\nu} + \sum_{\nu'} \frac{V_{\nu\nu'}\, V_{\nu'\nu}}{\nu^2 - \nu'^2} + \dots, \qquad (3.14)$$

where $V_{\nu\nu'}$ are the matrix elements of V taken between the normalized functions (3.12).

It is easy to see that

$$|V_{\nu\nu'}| < (\kappa r_0)^2 = (\kappa^2\lambda)^{2/3}$$

(which is really small for small values of κ^2, as we have expected).

Making use of the familiar expression for the number of roots $N(R)$ inside a circle with radius $|n| = R$

$$\int_0^R \frac{N(x)}{x}\, dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi - \log|f(0)|,$$

we obtain

$$N(R) < C\, R \log R$$

where C is some constant.

Hence, for the ℓ eigenvalue ν_ℓ we have the lower bound:

$$\nu_\ell^2 > C' \frac{\ell^2}{(\log \ell)^2}$$

where C' is again some constant. So we obtain that the series (3.14) converges. In a similar way one can demonstrate the convergence of the series for the perturbed eigenfunctions as well.

We can check by inspection that our S-matrix element is a meromorphic function of n in the whole complex n-plane; it satisfies the symmetry relation, characteristic of potentials with a hard core [14]:

$$S(-n, k^2) = e^{-2\pi i n}\, S(n, k^2)\,. \tag{3.15}$$

Finally we quote the expression for the transition matrix element, defined by the relation:

$$T(n, k^2) = \frac{1}{i \pi k^2}\left[S(n, k^2) - 1 \right],$$

$$T(n, k^2) = \frac{-1}{\pi k^2}\; \frac{J_n(\epsilon) K'_{\frac{n}{2}}\left(\frac{\epsilon}{2}\right) + J'_n(\epsilon)\, K_{\frac{n}{2}}\left(\frac{\epsilon}{2}\right)}{H_n^{(1)}(\epsilon)\, K'_{\frac{n}{2}}(\epsilon) + H_n^{(1)\prime}(\epsilon)\, K_{\frac{n}{2}}\left(\frac{\epsilon}{2}\right)}\,. \tag{3.16}$$

One can check by direct calculation that the expression (3.16) gives the correct continuation of the transition matrix element for $k^2 < 0$, $|k^2 \lambda| \ll 1$.

4. Subtraction Terms and Asymptotic Behaviour of the Scattering Amplitude

All the expressions derived up to now are originally valid for $\mathrm{Re}\, n > 2$ (as we had two subtractions in the kernel (2.3)). However, as explained in ref. [10], we can continue them even for $\mathrm{Re}\, n < 2$, if only we take into account that at the subtraction points, $n = 1, 2$, the expression for the transition amplitude is not given by (3.16) but by a suitably modified expression, derivable from eq. (2.5) of ref. [10]. To formulate the content of that equation in an intuitive way, the subtraction polynomial of the kernel can be conceived as a set of new local interaction Lagrangians. The transition amplitude at the subtraction points is given by the infinite chain diagrams formed with these additional interactions, taking into account the vertex - and propagator corrections caused by the analytic part of the transition amplitude, T.

In fact, at the subtraction points, according to eq. (2.5) of ref. [10] the Green function G is given by:

$$G = G_1 + G_1 K_2 G . \tag{4.1}$$

Taking into account the relation between the Green function and the transition amplitude off the mass shell:

$$G_1 = G_0 + G_0 T \, G_0 \tag{4.2}$$

and defining at the subtraction points

$$G = G_0 + G_0 U \, G_0$$

with the help of eq. (4.1) we obtain the correction to the analytic part of the amplitude:

$$W(n,k^2) = [\, 1 - (1 + T G_0)\, K_2(n)\, G_0 \,]^{-1} \times \tag{4.3}$$

$$\times (1 + T G_0)\, K_2(n) (1 + G_0 T)$$

where $W(n,k^2) \equiv T(n,k^2) - U(n,k^2)$

It is clear that $W(n,k^2)$ can differ from zero for $K_2(n)$ only, because of the properties of $n = 1$ (cf. eq. (2.3)). If we write out the operator equation (4.3) in momentum representation, we arrive at an integral equation with degenerate kernel; its solution reads e.g. for

$$< p \mid W(1,k^2) \mid p' > = g_1 \frac{F(k^2,p)\, \tilde{F}(k^2,p')}{1 - g_1 \phi(k)} , \tag{4.4}$$

where

$$F(k^2,p) = 1 + \int_0^\infty \frac{q'^3 dq' < p \mid T(1,k^2) \mid q' >}{q'^2 - k^2} ,$$

$$\tilde{F}(k^2,p) = 1 + \int_0^\infty \frac{q'^3 dq' < q' \mid T(1,k^2) \mid p >}{q'^2 - k^2} \tag{4.5}$$

$$\phi(k) = \int_0^\infty \frac{q'^3 dq'}{q'^2 - k^2} + \int_0^\infty q'^3 dq' \int_0^\infty q''^3 dq'' \frac{< q' \mid T(1,k^2) \mid q'' >}{(q'^2 - k^2)(q''^2 - k^2)}$$

(The first integral in the expression of $\phi(k)$ is divergent as it stands; it could be defined in the usual way by means of subtractions; however, we shall see in what follows that we can operate formally with such divergent integrals).

Let us now observe that the expressions (4.5) can be expressed with the help of the Green functions in coordinate space. In fact, remembering that

$$\lim_{r\to 0} \frac{J_1(pr)}{pr} = ½$$

one immediately recognizes (cf. eq. (4.2)) that

$$\phi(k) = \lim_{\substack{r_1 \to 0 \\ r_2 \to 0}} G_1(r_1, r_2) = G_1(0,0)$$

For small values of r one can immediately write down the expression of the Green function of eq. (2.5).

$$G_1(r, r') = ½ (rr')^{-1} [\, \theta(r-r') I_{½}(\frac{\lambda}{2r^2}) K_{½}(\frac{\lambda}{2r'^2}) +$$

$$+ \theta(r'-r) I_{½}(\frac{\lambda}{2r'^2}) K_{½}(\frac{\lambda}{2r^2}) \,]$$

Now, the value of $G_1(0,0)$ depends on the order of the limits: if $r'>r$ or $r>r'$ then $G_1(0,0)=0$ if the limit is taken along the line $r'=r$, $G_1(0,0)$ is a finite constant. However, as one can easily check, $F(k^2,p) = \tilde{F}(k^2,p) = 0$. Thus we see that $W(1,k^2) = 0$ and the same result is obtained for $W(2,k^2)$ Hence, <u>the transition amplitude is determined by its analytic part everywhere</u>. Let us immediately remark that this remarkable property is a consequence of the strong singularity of the Green function at the origin. Had we to do with a "regular" theory (the interaction at small distances is weaker than the centrifugal term), the Green function at small distances would behave as r^{n-1}, thus allowing a nonvanishing contribution from the subtraction terms at $n=1$. We can now turn to the determination of the total scattering amplitude from the partial amplitudes (3.16). Remembering the addition theorem for four dimensional spherical harmonics (ref. |10|, Appendix) and the well-known identity:

$$C^{1}_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}$$

(C^{λ}_{ν} being a Gegenbauer function), we obtain the scattering amplitude in the form of a Fourier series:

$$T(z,k^2) = (\frac{\pi}{2})^{-½} (\sin\theta)^{-1} \sum_{n=1}^{\infty} \sin n\theta \;\; T(n,\kappa^2) \qquad (4.\;)$$

Here $z = \cos\theta$ or expressed in Mandelstam's variables, $z = 1 + \frac{2s}{t}$, $t = 4^{-1}\kappa^2$

It is convenient to transform the series (4.6) into a contour integral:

$$T(z,\kappa^2) = (\frac{2}{\pi})^{½} (\sin\theta)^{-L} \int_C dn \, ctg\, \pi n \; \times \qquad (4.6)$$

$$\times \sin n\theta \;\; T(n,\kappa^2),$$

the contour C being shown on Fig. 3a. We have already remarked that $T(n,\kappa^2)$ is a meromorphic function of n in the whole complex n - plane, its poles lying for $k^2<0$ on the imaginary axis.

A simple investigation shows that if we put $n = R \exp\{i\phi\}$ then for $R \to \infty$

$$T(n, ik^2) = O(\exp(-R \log R \cos\phi))$$

Thus for $k^2 < 0$ the contour C can be deformed almost to the imaginary axis (C') (an infinitesimal sector being excluded) as shown on Fig. 3b.

Let us now obtain an approximate expression for small values of k^2 and nonintegral values of n. Making use of the familiar expansions of the cylindrical function, after some rearrangements we arrive at the expression:

$$T(n,t) \sim \frac{e^{in\pi} \sin n\pi}{Y - 1}, \tag{4.7}$$

where

$$Y = \frac{\Gamma(1+n)\,\Gamma(1+\frac{n}{2})}{\Gamma(1-n)\,\Gamma(1-\frac{n}{2})} \left(\frac{te^{-i\pi}}{64}\right)^{-n} \tag{4.8}$$

To find the poles of the amplitude we put $n = ir$ then

$$\frac{\Gamma(1+n)\;\Gamma(1+\frac{n}{2})}{\Gamma(1-n)\,\Gamma(1-\frac{n}{2})} = \exp i\psi ,$$

$$\psi = -3cr + O(r^2)$$

where C is the Euler-Mascheroni constant.

$T(n,t)$ obviously has poles in the n-plane determined by the equation:

$$\psi - r \log \frac{te^{-i\pi}}{64} = 2\pi N , \tag{4.9}$$

$$N = 0, \pm 1, \pm 2, \ldots$$

or, approximately,

$$n = \frac{2\pi i N}{3C + \log \frac{te^{-i\pi}}{64}} \tag{4.10}$$

Thus, we have an infinite number of poles; all of them have a common trajectory in the n-plane, as sketched on Fig. 4. For $t = 0$ the poles fill the whole imaginary axis, giving a continuous spectrum, as can be directly checked e.g. from eq. (2.6).

These considerations show that the concept of Regge poles to determine the behaviour of the amplitude in the crossed channel is not a useful one in our case), because we have no "leading pole".

Nevertheless, the contour integral representation of the scattering amplitude (4.6) can be used to determine the asymptotic behaviour for large positive values of s and small negative values of t. The integral in (4.6) can be evaluated by means of the method of steepest descents.

The calculation is a bit tedious but quite elementary; we simply quote the resulting asymptotic expression of the scattering amplitude, as a function of s and t:

$$T(s,t) \sim \left(\frac{2}{\pi}\right)^{1/2} \frac{i \cdot 16 f}{s \log s} M(x) \qquad (4.11)$$

where

$$x = \frac{-\log s}{\log(-t)}, \qquad M(x) = x^{3/2} \exp\left(-\frac{i}{x} \log x\right).$$

The function $(M(x))^2$ is plotted on Fig. 5. The quantities s and t are measured in units of f^{-1}. The formula (4.11) is valid for $s \gg 1$ and $|t| \ll 1$.

We can immediately indicate the "rule of thumb" 2 for the use of the expression $T(s,t)$. As the infinite iteration of our massless $V-A$ bubble gave again an expression with a $V-A$ kinematic structure, the expression for any transition amplitude, corrected with the higher order contributions is obtained, if in the lowest order expression for the amplitude in question, the coupling constant f is replaced by our $T(s,t)$

Closing this section, we mention a low energy theorem for our amplitude T.

One expects that for $k^2 \to +0$, the partial amplitude $T(1,k^2)$ tends to a multiple of the Fermi coupling constant, f, while $T(n,k^2)$, $n > 1$, tends to zero; thus at the physical threshold the original $V-A$ interaction would be reproduced (perhaps with a renormalized coupling constant).

In fact, a glance at eq. (3.16) shows that

$$\lim_{k^2 \to 0} T(1,k^2) = \left(\frac{2}{\pi}\right)^{1/2} f, \qquad (4.12)$$

$$\lim_{k^2 \to 0} T(n,k^2) = 0, \quad (n > 1).$$

Thus the original Fermi constant is renormalized by a factor $(2/\pi)^{1/2}$. It is perhaps amusing to remark that the numerical value of our renormalization factor $((2/\pi)^{1/2} \approx 0.8)$ is rather close to that obtained by Feinberg and Pais (loc. cit.) for the renormalization factor of the W-meson coupling constant $3/4$).

5. Scattering Amplitude in Direct Channel

In the preceding chapters we described the "weak diffraction scattering" of two leptons in the region of $s > 0$, $t < 0$.

Nevertheless, the same diagram, considered in the physical domain in another channel, namely $s < 0$, $t > 0$, gives a non diffractional contribution to the scattering amplitude, which decreases slower than the expression (4.11). To calculate the transition matrix element for large positive values of k^2, we apply a WKB approximation to eq. (2.6).

Thus the leading term in k^{-1} for the phase shift reads:

$$\delta(b,k) = -\frac{\lambda^2}{2k}\int_b^\infty dr\; r^{-5}(r^2-b^2)^{-1/2} + O(k^{-2}) \tag{5.1}$$

where we have introduced the "relativistio impact parameter" $b = n\,k^{-1}$. After an elementary caloulation we find from (5.1) with an accuracy up to $O(k^{-2})$:

$$\delta(b,k) \simeq -3\,f^2 b^{-5} k^{-1} \tag{5.2}$$

The partial wave amplitude oan be easily found if we remark [15] that to the same order in k^{-1}, $\delta(b,k) \approx \operatorname{tg}\delta(b,k)$ so that the partial wave amplitude is

$$t(b,k) \sim \frac{\pi}{k^2}\;\frac{-3\,f^2 k^{-1}\, b^{-5}}{1 + 3if\,k^{-1} b^{-5}}\,. \tag{5.3}$$

The soattering amplitude is given in terms of the partial wave amplitudes by eq. (4.6). Going over to integrating over b instead of summation over n we arrive at the following quasiclassical approximation to eq. (4.6):

$$T(x,k) \sim \left(\frac{2}{\pi}\right)^{1/2} k^2 x^{-1}\int_0^\infty db\,\sin bx\; t(b,k) \tag{5.4}$$

where $x = (-s)^{1/2} \sim k\,\theta$ Let us insert eq. (5.3) into (5.4) and introduce the variable $bx = y$; then we get:

$$T(x,k) = \left(\frac{2}{\pi}\right)^{1/2} 3\pi f^2 x^3 k^{-1}\int_0^\infty dy\,\frac{\sin y}{y^5 + i\gamma x^5}\,, \tag{5.5}$$

where the notation $3f^2k^{-1} = \gamma$ has been introduced. To **evaluate** (5.5) we split the integration interval into two parts: $0 \le y \le 1$ and $1 \le y < \infty$. In the integral taken from 0 to 1, $\sin y$ can be expanded into a Taylor series, the leading contribution for small values of x being obtained from the linear term. In the second integral the denominator can be expanded in powers of $\gamma x^5 y^{-5}$ the leading contribution being again given by the zero order term. The integrals arising after this oan be evaluated in a familiar way. We quote the resulting expression for the invariatnt amplitude, inserting numerical values for the coefficients:

$$T(x,t) = \frac{1{,}05\,f}{t^{1/5}}\left[\left(3{,}65 - 4{,}97\,\frac{x^3}{t^{3/10}}\right) + i\left(0{,}62 - 4{,}59\,\frac{x^3}{t^{3/10}}\right)\right] + O(x^5), \tag{5.6}$$

$$x = (-s)^{1/2}\,.$$

In eq. (5.6) the quantities s, t are measured in units of f^{-1}.

By using our recipe, formulated at the end of chapter IY and of the optical theorem, we see that eq. (5.6) gives a total cross section, decreasing as $s^{-1/3}$. The cross section as calculated from eq. (5.6) is equal to the total cross section, calculated from the low energy approximation of the preceding chapter (which is almost equal to the lowest order contribution in f) at approximately the "critical energy" $s = 1$.

Thus, assuming that this approximation is at least qualitatively correct, we find that the elastic scattering cross section of leptons rises practically as s till $s = 1$ afterwards decreases roughly as $s^{-1/3}$. The total cross section contains inelastic contributions as well; however, they seem to lie out rapidly at high energies.

6. Discussion

In our opinion, the foregoing calculations yielded two results which are in a rather striking contrast with common belief. The first one is that we got a completely well- determined expression for the four-lepton vertex, without any arbitrary renormalization constants. As we have already pointed out, this is a consequence of the singular nature of the Fermi interaction. If the present result would hold generally (independently of our approximation scheme) this would mean that "unrenormalizable" interactions are even "better" than renormalizable ones.

In this respect the low-energy theorem (4.12) is far from being trivial, because, as one can see, the expression for the four fermion vertex is in general singular in the coupling constant f at $f = 0$.

The second surprise is that weak interactions do not "become strong" at high energies. In fact, the scattering amplitude becomes imaginary at high energies and decreases with energy. As to the correction to the angular distribution, given by the function $(M(x))^2$ one sees that the forward peak does not become narrower with energy. (In order to avoid the "infrared" infinity at $t = 0$, one should put $-t + m^2$ in the final formulas instead of $-t$ where m is some lepton mass). According to our "rule of thumb" the differential cross section would be roughly given by

$$\frac{d\sigma}{d\Omega} \approx \frac{(M(x))^2}{S \log^2 s}$$

for unpolarized leptons, near forward direction $|\frac{t}{s}| \ll 1$ If we tentatively define the elastic cross section to be proportional to the differential cross section, integrated over the width of the forward peak, this quantity would show an energy dependence

$\sigma \sim s^{-1} \log s$,   for $s \to \infty$ The contribution from the non-diffractional scattering is also decreasing with energy, namely, like $s^{-1/8}$ as shown in section Y. At a first glance the experimental verification of the above statements seems rather hopeless in the near future, as most of our formulas are asymptotic expressions. Nevertheless, there are two circumstances which allow one to be more optimistic in this respect. First of all, apart from neglecting an infinite number of diagrams usually believed to be unimportant at high energies and or low momentum transfers, there is actually only one asymptotic approximation in our calculation. We took namely the leading term of the saddle-point expression for the scattering amplitude. However, for a great number of functions the leading term of the saddle-point expression is known to give very reasonable estimates even for moderate values of the argument.

Secondly, one can be encouraged by the- at least partial -success of the one- Regge-pole expressions in interpreting the results of scattering experiments in strong interaction physics. The one- pole formulae are just as asymptotic expressions, as our ones.

In view of this it would be perhaps not quite unreasonable to test the predictions of the present theory at lepton energies of the order of some tens of GeV-s (In this respect one thinks of course in the first line of $\nu e - e$ scattering, because of the absence of Coulomb scattering).

We want to make a remark which may be of some interest. Put in a Lagrangian language, we investigate here the two-body Green function starting from the following Lagrangian:

$$L = \bar{\psi} \hat{\partial} \psi + f (\bar{\psi} \gamma_\mu (1 + i \gamma_5) \psi)^2 + h.c.$$

This Lagrangian is identical in its appearence to Heisenberg's one in his unified field theory. However, we quantized the theory in a completely "conservative" way and still arrived at finite and uniquely determined results. One could speculate whether would it be really impossible to construct a finite theory without introducing indefinite metric in Hilbert space, non-canonical quantization etc.

One has to make an important remark at this place concerning the transition to Euclidean metric.

One can prove that in the case of a renormalizable theory, the BS equations written either in Euclidean or Minkowskian metric are completely equivalent to each other in the sense that in eq. (2.4) the integration over k^0 can be performed both along the imaginary or real axes. In our case, however, the situation is completely different. The Green function has an essential singularity in k^0 at $|k^0| \to \infty$ (This can be immediately seen from the behaviour of the solution in coordinate representation at $r \to 0$).

Thus the solution constructed by means of our procedure does not satisfy eq. (2.4) in Minkowskian metric. (It is even possible that the Minkowskian equation has no physically reasonable solutions at all).

Nevertheless we believe that our procedure may serve as a reasonable definition of nonrenormalizable Green functions.

First, as it is well known [15] the existence of a Euclidean theory is a necessary condition for the existence of a Minkowskian one. Second, all the physically interesting quantities (scattering amplitudes etc.) constructed in the framework of the Euclidean theory, when continued in the kinematic invariants to the domain, corresponding to Minkowskian theory, give functions possessing familiar analytic properties.

One could, of course, object that the ladder approximation is not a justified one, one must not neglect masses etc. As to the first objection, at the moment we can give a practical answer only: although the ladder approximation has been blamed several times - and with good reason- it is at present practically the only approximation for Green functions of field theory, where one can push the calculations till numerical estimations. Still more, we feel that the vanishing of the renormalization terms is a consequence of the singular nature of the interaction and not of the ladder approximation and thus is more general than it would be seen from the present calculation. Concerning the second objection, we can say a bit more. According to our preliminary calculations, the inclusion of lepton masses induced other invariant functions (S, T, P) but the latter are probably small (they are proportional to the product of the masses of propagating leptons).

The resulting equations for the invariant functions are somewhat less singular than that we dealt with in the present paper, but still with an essential singularity at $r = 0$. The results of the calculations concerning the role of lepton masses and other physical consequences of the present approach to the theory of Fermi interactions will be published in subsequent papers.

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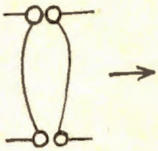

Fig. 1. Diagram of the interactions kernel. Particles meeting in one small circle belong to the same current. The arrow indicates the direction of summation.

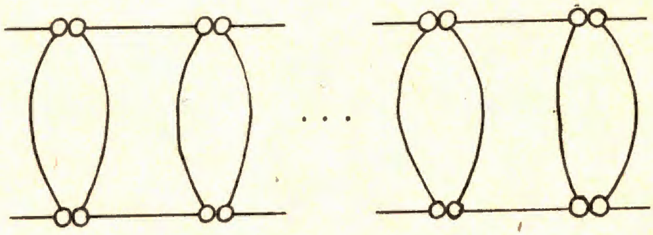

Fig. 2. Scheme of iterated diagrams.

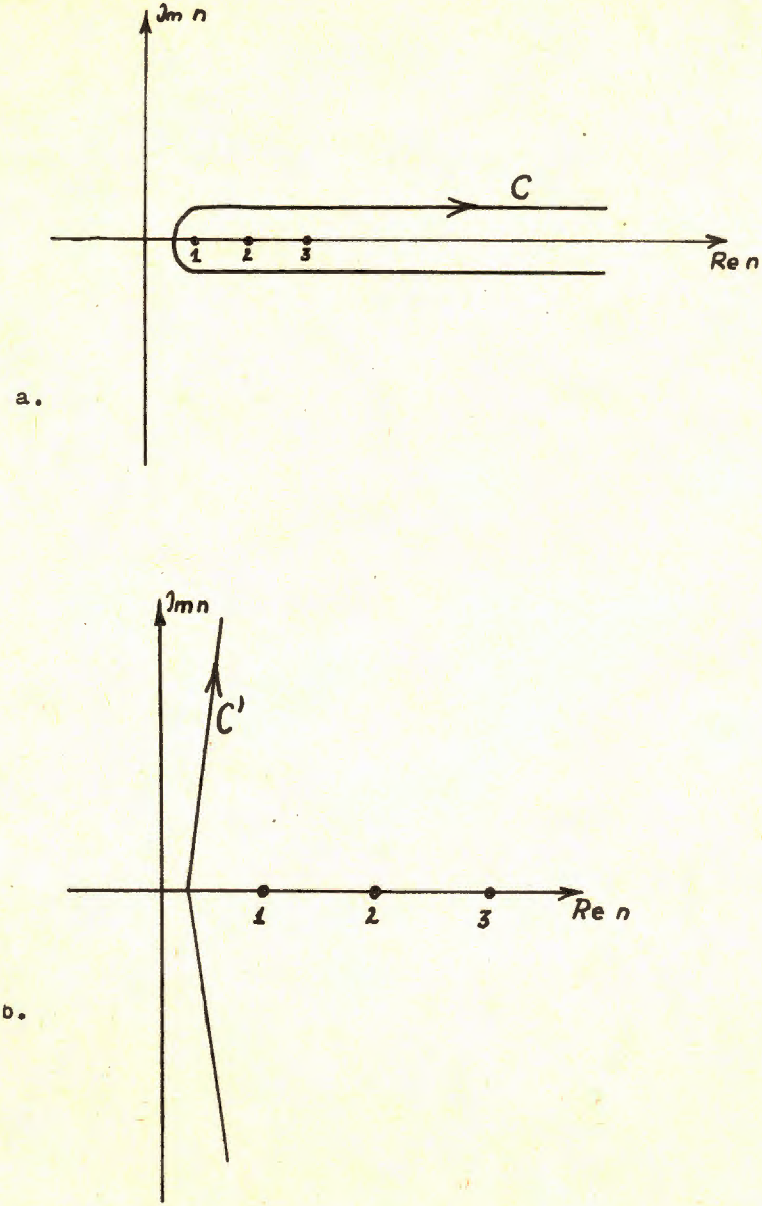

Fig. 3. a. Contour of integration in the n-plane
b. Deformed contour of integration in the n -plane.

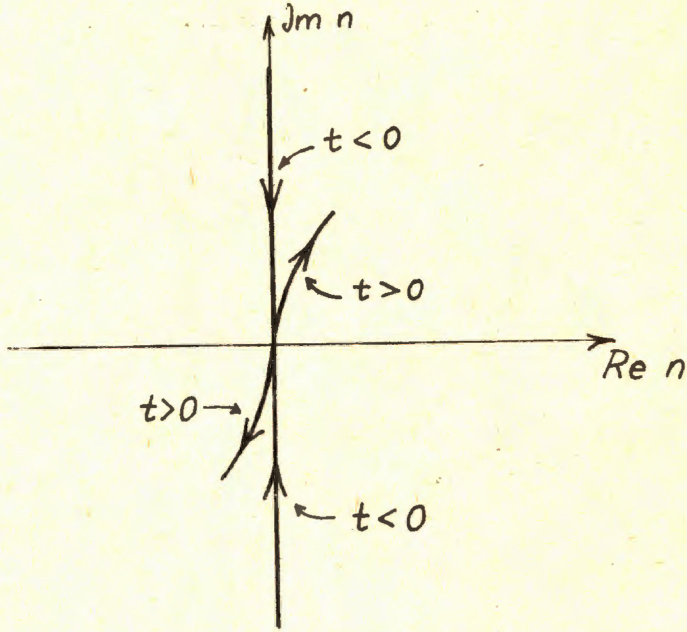

Fig. 4. Sketch of the trajectory of poles in the n -plane.

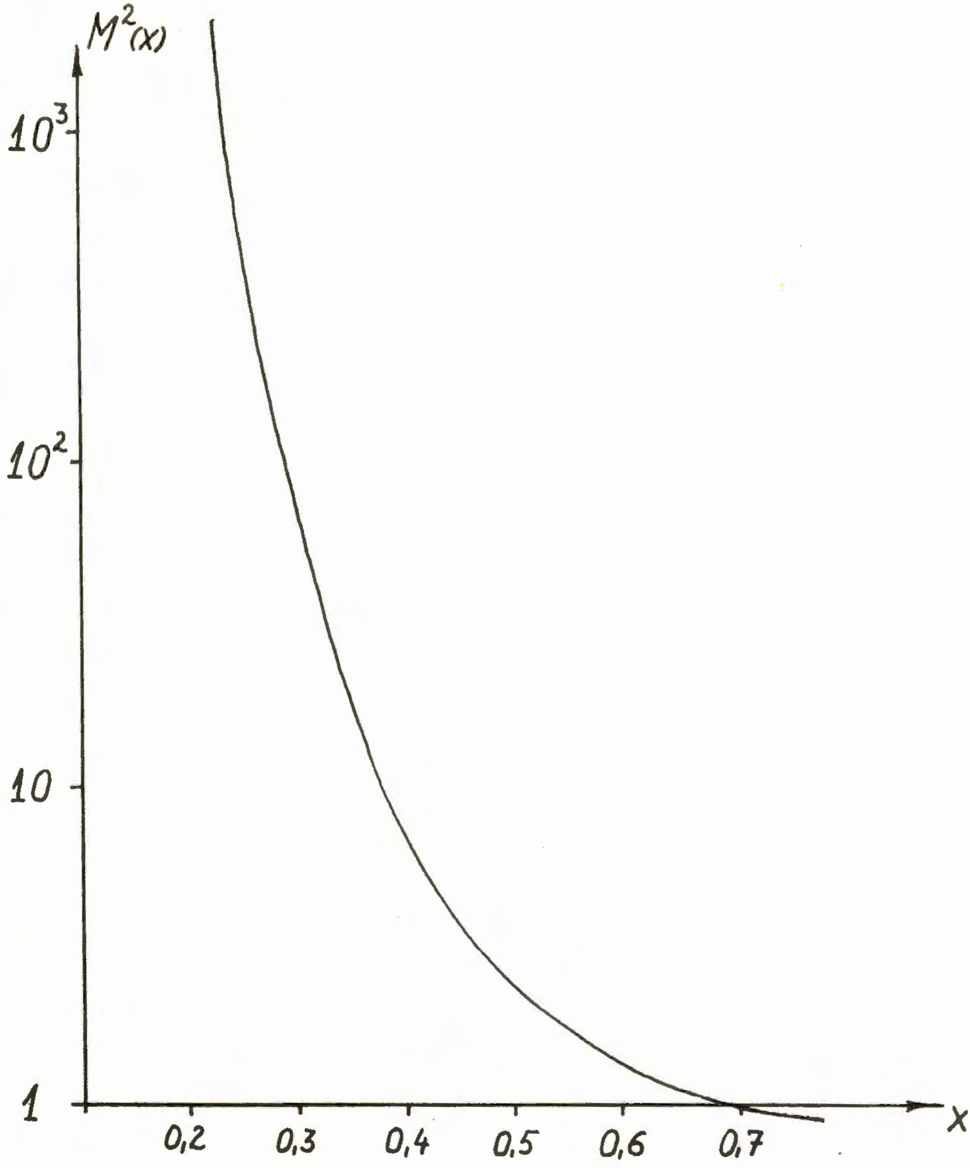

Fig. 5. Plot of the correction factor to angular distribution in lepton-lepton scattering.

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ФЕРМИ-ВЗАИМОДЕЙСТВИЕ ПРИ ВЫСОКИХ ЭНЕРГИЯХ

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