ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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N.Ya. Vilenkin and Ya.A. Smorodinsky

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INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES

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1. Introduction

In the present paper we shall consider the problem of the expansion of relativistic functions in series and integrals of the eigenfunctions of the Laplace operator in the Lobachevsky's space.

The problem is to investigate the Laplace operator on the two-sheet hyperboloid (the angular part of the d'Alembert operator) in various coordinate systems. In the next paper we shall consider a similar problem for a single-sheet hyperboloid. Many points of our investigation can be illustrated in geometrical terms. We shall use this method constantly. The expansion on the light cone first obtained by Shapiro (1,2) appears to be very important in these considerations. Gelfand and Graev (3,4) have developed a general theory of such expansions using the method of integral geometry. These methods allow one to pass from the investigation of functions on the hyperboloid to that on the cone where expansion reduces to the Fourier transformation. In this case the connection with the theory of the Lorentz group representations is clearly established[x/].

The constructed systems of eigenfunctions are a generalization of the non-relativistic systems such as the spherical and cylindrical ones. However, one of the systems described below namely the cylindrical, system has no analogy in non-relativistic mechanics. The eigenfunctions in this system are not expressed in terms of the Legendre and Bessel functions but in terms of a hypergeometrical function of a general form.

We note some features of the described expansions which point to interesting trends in the further work.

1) The systems of functions have an interesting symmetry property with respect to the crossing- channel. The relativistic nature of the crossing reaction is clearly manifested in geometrical terms. In particular, the cylindrical and horospherical coordinate system generates functions giving a one-dimensional representation of the crossing operation. The group properties of such transformations will be described later.

[x/] The expansion of relativistic functions was also investigated by Dolginov/4-6/ but he has not constructed the complete system of functions.

2) Of great interest is the connection with the four-dimensional angular momentum and the complete set of commuting observables in the relativistic theory. The four-momentum operator properties are richer than in the three-dimensional case and their theory makes it possible to understand the physical nature of the obtained results. This theory will be presented in the next paper.

The investigation of functions in the Lobachevsky's space was hitherto undeservedly forgotten. This paper illustrates the efficiency of the old and new aspects of the Lobachevsky's geometry for spinless particles. In further papers we shall consider particles with spin.

2. The Scattering Amplitudes

The scattering kinematics is described by means of the kinematical graph (Fig. 1): velocity-diagram in the Lobachevsky's space (see 7,8).

Points 1,2 represent the particle velocities before scattering, points 3,4 are the velocities after scattering. Point s is the velocity of the c.m.s. As the coordinate axes we choose two orthogonal axes, one of which bisects the scattering angle. As was shown in [9], the point of intersection of 14 and 23 is the velocity of the c.m.s. of the crossing reaction. For equal masses the point of intersection of 13 and 24 represents the u - channel c.m.s. velocity. In this case the coordinate axes coincide with the directions of the lines st and su and three points s, t, u form a so-called autopolar triangle (comp. [10]). For different masses (dashed lines of Fig. 1) the connection between the coordinate axes and the u - system is not so simple.

The third axis (non-essential for the binary collisions of spinless particles) is the normal to the plane.

If the Lobashevsky's space is represented as the interior of a sphere (the Beltrami model), then it is convenient to choose the origin in the center of the sphere. The points of the sphere correspond to particles with zero mass[x].

The Lobachevsky's space may be realized also as a three-dimensional manifold - the upper sheet of a two-sheet hyperboloid $u^2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1.$

We shall assume the coordinate origin to coincide with the vertex of the hyperboloid i.e. the coordinates of the c.m.s. will be (1,0,0,0). The zero mass

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x/ The planes tangential to the sphere are isotropic (the distance from the point of contact to any point of the plane is equal to zero). This is connected with the gauge invariance of the photon.

particles are repsented by the points on the light cone.

In this coordinate system the scattering is described by two parameters. For our purpose the usual kinematic parameters s and t turn out to be inconvenient. Instead of them we shall describe the process by the coordinates of one of the points, e.g. point I. The conservation laws of energy and momentum then define the coordinates of the other points. Since the bisectrix of the scattering angle was chosen as one of the coordinate axis, then the vectros of the initial and final velocity will be symmetrical with respect to this axis.

The scattering angle is denoted by θ . The coordinates of the point I can be chosen in different ways. This leads to different systems of eigenfunctions.

3. The Coordinate System and the d'Alembert Operator

As we have pointed out already, the Lobachevsky velocity-space is realized as the three-dimensional surface of the upper sheet of the two-sheet hyperboloid $u^2 = 1$ in the Minkowski's space. We cut such a hyperboloid by a plane parallel to one of the coordinate planes. Then the cross section represents again a hyperboloid (two-dimensional surface) if the plane does not pass through the axis "0", or a two-dimensional sphere if the plane crosses this axis normally, In the second case we can construct on the sphere a spherical coordinate system. In the first one the cutting can be continued in two ways: along a circle or along a hyperbola. In this case we get two more coordinate systems. In the Minkowski's space it is possible to construct two orthogonal two-dimensional planes. One of these plane possesses an Euclidean metric, the second a hyperbolic one. This gives rise to one more coordinate system. Constructing in the first plane a polar coordinate system, and in the second plane a hyperbolic one we get a cylindrical coordinate system. These systems exhaust all the orthogonal systems whose coordinate surfaces are spheres on hyperboloids (hyperspheres in the Lobachevsky space, i.e. spheres with the center on a one-sheet hyperboloid). If the center will be placed on the light cone then the coordinate surface will be a so-called horosphere: a surface whose geometry is isomorphic to the Euclidean plane geometry. Horospherical coordinate system will also be described in this paper. The physical meaning of all the coordinate systems can be easily understood if we notice that the transition from one frame of reference to any other can be performed by three successive transformations one of which, at least, is a Lorentz transformation.

In the spherical coordinate system a point is characterised by two space rotations and one Lorentz transformation, in other systems by two or three Lorentz transformations. Note that for binary reactions without spins whose amplitude is independent of the azimuthal angle the Lobachevsky's system (three Lorentz rotations)[x] is not convenient.

The coordinate systems in the velocity-space will be specified by the connection between four homogeneous cartesian coordinates u_0, u_3, u_2, u_1 and corresponding angles. In this case a point on the hyperboloid is specified by four numbers which are the projective coordinates of a point. Along with the projective coordinates we introduce inhomogeneous coordinates obtained by dividing the projective coordinates by u_0.

As the interval of variation of the ratios u_i/u_0, $i = 1,2,3$ is between 0 and 1, then it is convenient to denote them by $th\, z_i$ where already $0 \leq z_i \leq \infty$ thus as inhomogeneous coordinates we choose z_i determined by the formula:

$$th\, x_i = \frac{u_i}{u_0} \qquad (3.1)$$

Finally, we agree to denote the spacial rotation angles by Greek letters and the hyperbolic angles by Latin ones.

Now we go over to the description of the coordinate systems.

1. The Spherical System

The S -system is given by the formulas

$$\begin{aligned} u_0 &= ch\, a \\ u_3 &= sh\, a \cos\theta \\ u_2 &= sh\, a \sin\theta \cos\phi \\ u_1 &= sh\, a \sin\theta \sin\phi \end{aligned} \qquad (3.2)$$

or

$$\begin{aligned} th\, z_3 &= th\, a \cos\theta \\ th\, z_2 &= th\, a \sin\theta \cos\phi \\ th\, z_1 &= th\, a \sin\theta \sin\phi \end{aligned} \qquad (3.3)$$

The inhomogeneous coordinates are nothing more than the three-dimensional spherical ones where $th\, a$ is the radius vector. The formulas in the

x/ The problem of the coordinate systems, in which variables in the Klein-Gordon equations are separable, has been considered by M.P.Olevsky[10] and A.I. Shoom[11].

S -system are therefore found to be the closest to the non-relativistic ones. The d'Alembert operator in the four-dimensional Minkowski's space can be written in the form:

$$\Box = \frac{1}{U^3} \frac{\partial}{\partial U} U^3 \frac{\partial}{\partial U} - \frac{1}{U^2} \Delta_L \qquad (3.4)$$

where U is the four-vector length in the Minkowski's space, and Δ_L is the Laplace operator on the hyperboloid[x], whose eigenfunctions are to be found.

In the S -system

$$\Delta_L = \frac{1}{sh^2 a} \frac{\partial}{\partial a} sh^2 a \frac{\partial}{\partial a} + \frac{1}{sh^2 a} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) \qquad (3.5)$$

II. The Lobachevsky Coordinate System

Lobachevsky was the first to consider one of the hyperbolic coordinate systems, therefore the corresponding coordinates are referred to as the Lobachevsky's coordinates (coordinates a , b and c):

$$\begin{aligned} u_0 &= ch\, a\; ch\, b\; ch\, c \\ u_3 &= ch\, a\; ch\, b\; sh\, c \\ u_2 &= ch\, a\; sh\, b \\ u_1 &= sh\, a \end{aligned} \qquad (3.6)$$

$$\begin{aligned} th\, z_3 &= th\, c \\ th\, z_2 &= th\, b \frac{1}{ch\, c} \\ th\, z_1 &= th\, a \frac{1}{ch\, b} \frac{1}{ch\, c} \end{aligned} \qquad (3.7)$$

The homogeneous coordinates u_0 , u are called the Weierstrass coordinates. In this system all the three angles are hyperbolic ones.

The Laplacian is determined by the formula

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x/ I.e. operator of the second order on the hyperboloid which commutes with the Lorentz transformations.

$$\Delta_L = \frac{1}{ch^2 a}\frac{\partial}{\partial a} ch^2 a \frac{\partial}{\partial a} + \frac{1}{ch^2 a}\left(\frac{1}{ch\, b}\frac{\partial}{\partial b} ch\, b \frac{\partial}{\partial b} + \frac{1}{ch^2 b}\frac{\partial^2}{\partial c^2}\right) \tag{3.8}$$

III. The Hyperbolic System - H

This system differs from the Lobachevsky one by that in the H -system one of the angles is spherical

$$\begin{aligned} u_0 &= ch\, a\; ch\, b \\ u_3 &= ch\, a\; sh\, b\; \cos\phi \\ u_2 &= ch\, a\; sh\, b\; \sin\phi \\ u_1 &= sh\, a \end{aligned} \tag{3.9}$$

and the inhomogeneous coordinates

$$\begin{aligned} th\, z_3 &= th\, b\; \cos\phi \\ th\, z_2 &= th\, b\; \sin\phi \\ th\, z_1 &= \frac{1}{ch\, b}\, th\, a \end{aligned} \tag{3.10}$$

The inhomogeneous coordinates are similar to the cylindrical ones with the radius-vector (in the plane) $th\, b$ .The coordinate planes z_1 = const., are orthogonal to the axis z_1 . For the Laplacian we get:

$$\Delta_L = \frac{1}{ch^2 a}\frac{\partial}{\partial a} ch^2 a \frac{\partial}{\partial a} + \frac{1}{ch^2 a}\left(\frac{1}{sh\, b}\frac{\partial}{\partial b} sh\, b \frac{\partial}{\partial b} + \frac{1}{sh^2 b}\frac{\partial^2}{\partial \phi^2}\right). \tag{3.11}$$

IY. The Cylindrical System

The C -system and H -system are very alike

$$\begin{aligned} u_0 &= ch\, b\; ch\, a \\ u_3 &= sh\, b\; \cos\phi \\ u_2 &= sh\, b\; \sin\phi \\ u_1 &= ch\, b\; sh\, a \end{aligned} \tag{3.12}$$

The correspondence between two systems is easily seen in inhomogeneous coordinates

$$\begin{aligned} th\, z_3 &= th\, b \frac{1}{ch\, a} \cos \\ th\, z_2 &= th\, b \frac{1}{ch\, a} \sin\phi \\ th\, z_1 &= th\, a \end{aligned} \tag{3.13}$$

We see that the difference comes from the third (axial) coordinate. In the C - system the distance between planes is measured along the cylinder axis, so that the coordinate surfaces are equidistant surfaces.

The Laplacian in this system reads:

$$\Delta_L = \frac{1}{ch\, b\, sh\, b} \frac{\partial}{\partial b}\, ch\, b\, sh\, b \frac{\partial}{\partial b} + \frac{1}{ch^2 b} \frac{\partial^2}{\partial a^2} + \frac{1}{sh^2 b} \frac{\partial^2}{\partial \phi^2} \tag{3.14}$$

V. The Horospherical Coordinate System (0)

Finally we consider the horospherical coordinate system

$$\begin{aligned} u_0 &= ½ [\, e^{-a} + (r^2+1)\, e^{a} \,] \\ u_3 &= ½ [\, e^{-a} + (r^2-1)\, e^{a} \,] \\ u_2 &= r\, e^{a} \cos\phi \\ u_1 &= r\, e^{a} \sin\phi \end{aligned} \tag{3.15}$$

In this system the coordinate surface $a = const$ is a horosphere in the Lobachevsky's space. In the horospherical coordinate system the Laplacian is

$$\Delta_L = e^{-2a} \left(\frac{\partial}{\partial a} e^{2a} \frac{\partial}{\partial a} + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \tag{3.16}$$

It is not difficult to see that from the system (0) we can get the system (C) by the transformation

$$\begin{aligned} r e^{a} &= sh\, b \\ th\, a_c &= \frac{e^{-a} + (r^2-1)\, e^{a}}{e^{-a} + (r^2+1)\, e^{a}} \end{aligned} \tag{3.17}$$

where by a_c we denote the coordinate a in (3.12). After the transformation (3.17) has been performed the systems will differ only by an interchange of axes.

Now let us consider the kinematical meaning of the coordinates introduced above. As long as we restrict ourselves to the binary reactions (i.e. to the plane graphs), we shall not consider the Lobachevsky's system.

In the S -system the coordinates are the angle $\theta/2$ (the half of the scattering angle in c.m.s.) and the length $(s1)$ which defines the energy of the particle I in c.m.s.

In the H -system the point I is defined by the segment (sB) the Breit system velocity with respect to c.m.s. and the length (B1) which defines the energy of the particle in the Breit system

In the C -system the particle I is specified symmetrically by the segments (sB) and (sB') i.e. by the velocities of both Breit systems (the system with vanishing sum of velocities of the particle I before and after the collision and the system in which vanishes the sum of the velocities of the particle I before the collision and particle 2 after the collision).

In the O - system the coordinate a is the distance to the horosphere with the center at the point k (1100) defined by the formula $\delta = \ln(ku)$ The scalar product of the type (ku) is often used as an independent variable in the problems of quantum electrodynamics. The two other variables u_2 and u_3 determine together with δ an invariant rectangular three-dimensional cartesian system in the Lobachevsky space.

4. The Eigenfunctions

In this section we shall obtain the eigenfunctions of the Laplacian on the hyperboloid in the five systems S, L, H, C and O. For this purpose we use the method of separation of variables. As is known (comp.[12]), the eigenvalues of the multidimensional Laplacian are $-\ell(\ell+n-2)$ where n is the dimension of the space. For $n=4$ the eigenvalue is $-\ell(\ell+2)$. The transition from the sphere to the hyperboloid leads to that ℓ ceases to be integer and may be any complex number. Since in this case the sign of Δ_L (see eq.(3.4)) is negative the equation of the eigenvalues becomes:

$$\Delta_L f = \sigma(\sigma+2) f \tag{4.1}$$

In order that $\sigma(\sigma+2)$ be real it is necessary that: either

$$\sigma = -1 + ip \qquad (p - \text{real}) \tag{4.2}$$

or

$$\sigma - \text{real} \tag{4.3}$$

From the theory of representations it is known that these values correspond to the unitary representations of the Lorentz group. Here (4.2) corresponds to the representation of the principal series and (4.3) to the additional one (see[13]).

It is convenient to put $\sigma = -1 + ip$, then the real values of p correspond to the unitary representations of the principal series. In this case the eigenvalue $\sigma(\sigma+2) = -(p^2+1)$ and we can write (4.1) in the form:

$$\Delta_L f_p = -(p^2+1) f_p \tag{4.4}$$

In the following we shall find other quantum numbers. We notice that in the case of a two-dimensional hyperboloid the eigenvalues will be $\sigma(\sigma+1)$. For $\sigma = -\frac{1}{2} + iq$ (q – real) they will be real and equal to $-\sigma(\sigma+1) = q^2 + \frac{1}{4}$.

We start with the system S. By separating the coordinates θ and ϕ by the ordinary spherical function $Y_{\ell,m}(\theta,\phi)$; $m = 0, \pm 1, \pm 2, \ldots$, we get the equation

$$\left[\frac{1}{sh^2 a}\frac{\partial}{\partial a} sh^2 a \frac{\partial}{\partial a} - \frac{\ell(\ell+1)}{sh^2 a}\right] A(a) = -(p^2+1) A(a) \tag{4.5}$$

It is easy to check that the solution of this equation will be

$$A(a) = (sh\, a)^{-\frac{1}{2}} P^{-(\ell+\frac{1}{2})}_{-\frac{1}{2}+ip}(ch\, a) \tag{4.6}$$

Thus, the eigenfunctions (unnormalized) in the S-system are of the form:

$$\langle p, \ell, m \,|\, a, \theta, \phi \rangle = (sh\, a)^{-\frac{1}{2}} P^{-(\ell+\frac{1}{2})}_{-\frac{1}{2}+ip}(ch\, a)\, Y_{\ell m}(\theta,\phi) \tag{4.7}$$

The expansion in the coordinate a is a generalization of the well-known Moller-Fock expansion.

In the Lobachevsky's system the coordinate c is separated by means of the functions e^{imc} (m is real; the expansion is a Fourier expansion). Then we obtain two equations

$$\left(\frac{d^2}{da^2} + \frac{2}{cth\, a}\frac{d}{da} - \frac{q^2+\frac{1}{4}}{ch^2 a}\right) A(a) = -(p^2+1) A(a) \tag{4.8}$$

$$\left(\frac{d^2}{db^2} + \frac{1}{cth\, b}\frac{d}{db} - \frac{m^2}{ch^2 b}\right) B(b) = -(q^2+\tfrac{1}{4}) B(b) \tag{4.9}$$

to determine the eigenfunctions in the system L. The solutions of these equations are

$$A(a) = (ch\, a)^{-1} P^{ip}_{-\frac{1}{2}+iq}(th\, a) \tag{4.10}$$

$$B(b) = (ch\, b)^{-\frac{1}{2}} P^{iq}_{-\frac{1}{2}+im}(th\, b) \tag{4.11}$$

for the eigenfunction we obtain

$$<p, q, m \mid a, b, c> = (ch\, a\sqrt{ch\, b}\,)^{-1}\, P^{ip}_{-1/2+iq}(th\, a)\, P^{iq}_{-1/2+im}(th\, b)\, e^{imc} \qquad (4.12)$$

In the next system, the H -system, the operator in brackets (3.11) has the eigenvalue $-(q^2+1/4)$ and the eigenfunctions

$$P^{m}_{-1/2+iq}(ch\, b)\, e^{im\phi} \qquad (4.13)$$

where m is an integer. Replacing the bracket by the eigenvalue we are led to the equation

$$\left(\frac{d^2}{da^2} + \frac{2}{cth\, a}\frac{d}{da} - \frac{q^2+1/4}{ch^2 a}\right) A(a) = -(p^2+1)\, A(a) \qquad (4.14)$$

The functions:

$$A(a) = (ch\, a)^{-1}\, P^{ip}_{-1/2+iq}(th\, a) \qquad (4.15)$$

represent the solution of this equation. Thus, we get for the H -system x/

$$<p, q, m \mid a, b, \phi> = (ch\, a)^{-1}\, P^{ip}_{-1/2+iq}(th\, a)\, P^{m}_{-1/2+iq}(ch\, b)\, e^{im\phi} \qquad (4.16)$$

In the horospherical coordinate system we separate the variables r and ϕ by means of the Bessel function $J_m(\kappa r)\, e^{im\phi}$. Then eq. (3.18) will be

$$\left\{ \frac{d^2}{db^2} - \frac{1}{b}\frac{d}{db} + \left(\kappa^2 + \frac{p^2+1}{b^2}\right) \right\} B(b) = 0 \qquad (4.17)$$

The solution of this equation is the Macdonald function (the Bessel function of an imaginary index and an imaginary argument)

$$B(b) = \kappa\, b\, K_{ip}(\kappa\, b) \qquad (4.18)$$

Thus the eigenfunctions of the system O are the functions

$$<p, \kappa, m \mid b, r, \phi> = (\kappa\, b)\, K_{ip}(\kappa\, b)\, J_m(\kappa r)\, e^{im\phi} \qquad (4.19)$$

Finally, we consider the cylindrical system. Separating the variables ϕ and a by $e^{im\phi}$ and e^{ira} respectively, we are led to the equation

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x/ The argument $th\, a$ can be replaced by the trigonometrical function $\cos\, a$. Formula $th\, a = \cos\, a$ is the well-known relation for parallel lines in Lobachevsky space.

$$\frac{d^2B}{db^2} + (th\, b + cth\, b) \frac{dB}{db} + [p^2 + 1 - \frac{m^2}{sh^2 b} - \frac{r^2}{ch^2 b}] B = 0 \qquad (4.20)$$

The solution of (4.20) is the product of hyperbolical functions with the hypergeometrical ones:

$$B(b) = (th\, b)^m (ch\, b)^{-1-ip} F(\frac{m+1+ip+ir}{2}, \frac{m+1+ip-ir}{2}; m+1; th^2 b) \qquad (4.21)$$

and the eigenfunctions of the system C turn out to be equal to

$$< p, r, m | a, b, \phi > = e^{i(r a + m\phi)} \frac{(sh\, b)^m}{(ch\, b)^{m+1+ip}}$$

$$F(\frac{m+1+ip+ir}{2}, \frac{m+1+ip-ir}{2}; m+1; th^2 b)$$

This finishes the construction of the orthogonal systems of functions in the five coordinate systems. It can be shown that four systems S, H, O and C just exhaust all the systems, possessing the axial symmetry property and constructed from spheres, horospheres and hyperboloids. (The system L does not have this property and is given here as very close to the others; the remaining systems include ellipsoidal coordinate surfaces). In what follows it will be shown that the restri ction to orthogonal systems is not obligatory and the expansion can be performed in non-unitary representations (the quantum number p is any complex number).

5. The Method of the Horosphere and the Expansion on the Cone

We have constructed different systems of the Laplace eigenfunctions on a two-sheet hyperboloid. Now we have to normalize these functions, or, what is the same, to find inversion formulas using these eigenfunctions. The general method of the horosphere developed by Gelfand and Graev gives a firm basis to the derivation of all these formulas.

To understand the essence of the method we consider the classical Fourier transformation in n dimensional space

$$F(k) = \int f(x) \exp(ikx)\, d^n x \qquad (5.1)$$

where (only in this and the next formula):

$$kx = k_1 x_1 + \dots + k_n x_n .$$

This can be reduced to the integration over planes

$$\Phi(k, p) = \int f(x)\, \delta(kx - p)\, d^n x \qquad (5.2)$$

and a subsequent one-dimensional Fourier transformation

$$F(k) = \int_{-\infty}^{\infty} \Phi(k,p)\, e^{ip}\, dp \tag{5.3}$$

The integral transformation (5.2) (integration over the planes) is called the Radon transformation.

Gelfand and Graev have shown that in the same way one reduces the invariant expansions on any homogeneous manifold, the role of the planes being played by so-called horospheres. For the upper sheet of the two-sheet hyperboloid $u^2 = 1$ the horospheres are the cross sections of the hyperboloid by the planes $uk = 1$ where k is a point of the cone $k^2 = 0$. These planes are parallel to the cone generating lines.

Each function $f(u)$ given on the hyperboloid generates a function on the cone

$$h(k) = \int f(u)\, \delta(uk-1)\, \frac{d^n u}{u_0} \tag{5.4}$$

where

$$d^n u = du_1 \dots du_n \tag{5.5}$$

$\frac{d^n u}{u_0}$ is the invariant measure on the hyperboloid. The transition from $f(u)$ to $h(k)$ we call the integral Gelfand-Graev transformation.

The inversion formula for this transformation for $n = 2m+1$ is of the form

$$f(u) = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \int \delta^{(n-1)}(uk-1)\, h(k)\, \frac{d^n k}{k_0} \tag{5.6}$$

and for $n = 2m$ of the form x/

$$f(u) = \frac{(-1)^{n/2}\, \Gamma(n)}{(2\pi)^n} \int (ku-1)^{-n}\, h(k)\, \frac{d^n k}{k_0} \tag{5.7}$$

Here

$$\frac{d^n k}{k_0} = \frac{dk_1 \dots dk_n}{k_0} \tag{5.8}$$

is the invariant measure on the cone.

It is convenient to consider the functions on the cone since it is posible

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x/ The divergent integral implies here a regularized one by means of the analytic continuation of the exponent.

to use homogeneous functions on this surface. The expansion of the function into the homogeneous components $\Phi(k,\sigma)$ is of the form[x/]:

$$h(k) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \Phi(k,\sigma)\, d\sigma \tag{5.9}$$

where

$$\Phi(k,\sigma) = \int h(tk)\, t^{-\sigma-1}\, dt \tag{5.10}$$

From eqs. (5.4) and (5.10) it follows that

$$\Phi(k,\sigma) = \int f(u)\,(uk)^{\sigma} \frac{d^n u}{u_0} \tag{5.11}$$

Whereas from eqs. (5.6), (5.7) and (5.9) it follows that for $n = 2m+1$

$$f(u) = \frac{(-1)^{\frac{n-1}{2}}}{2i(2\pi)^n} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)} \int_{\Gamma} \Phi(k',\sigma)(uk')^{-\sigma-n+1} d^{n-1}k'\, d\sigma \tag{5.12}$$

for $n = 2m$

$$f(u) = \frac{(-1)^{\frac{n}{2}-1}}{2i\,(2\pi)^n\,\Gamma(n)} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)}\, ctg\, \pi\sigma \int_{\Gamma} \Phi(k',\sigma)(uk')^{-\sigma-n+1} d^{n-1}k'\, d\sigma \tag{5.13}$$

The contour of integration Γ is any contour on the cone which crosses all the cone generating lines, and $d^{n-1}k'$ is the measure on this contour defined by the equality

$$d(tk') = t^{n-3}\, dt\, dk' \qquad (0<t<\infty)$$

For $\sigma = -\frac{n-1}{2} + ip$ where p is real, the integral transformation (5.11), (5.12) is unitary.

For the two-sheet hyperboloid the obtained formulas[xx/] are just an analog of the Fourier function expansion in the n -dimensional Euclidean space. The analogy becomes more obvious if we notice that

$$uk' = e^{\ell(uk')} \qquad uk' = 1$$

where $\ell(uk')$ is the distance from the point u to the horosphere $uk' = 1$

A further expansion of the function is already performed on the contour

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[x/] The value of δ is chosen so that no poles of the function are left inside the strip $0 \leq Re\,\sigma \leq \delta$.

[xx/] For $n = 3$ these formulas (5.10) and (5.12) were obtained by I. Shapiro[1] before Gelfand and Graev.

Γ. In this case to different contours there correspond different expansions. We restrict ourselves to the case $n=3$. We consider the contours:

a) Γ - section of the cone by the plane $k_0 = 1$ i.e. the sphere $k_1^2 + k_2^2 + k_3^2 = 1,\ k_0 = 1$

b) Γ - section of the cone by the planes $k_3 = 1$ and $k_3 = -1$ i.e. the upper sheets of the hyperboloids $k_0^2 - k_1^2 - k_2^2 = 1,\ k_3 = \pm 1$.

c) Γ - section of the cone by the plane $k_0 - k_3 = 2$ (paraboloid).

d) Γ - section of the cone by the cylinder $k_0^2 - k_1^2 = 1$.

To these types of the cross sections there correspond the above considered coordinate systems S, H, O and C. Thus to case a) there corresponds the S -system, to b) the H -system, to c) the O -system and to d) the C -system.

6. Derivation of the Inversion Formulas

We pass to the derivation of the inversion formulas connected with the Laplace eigenfunctions obtained in 4.

1. The Inversion Formula in the S -system

We start with the case when the cross-section Γ is a sphere $k_0 = 1$. In this case the function $\Phi(k', \sigma)$ is given on the sphere and is therefore expanded in the set of spherical harmonics

$$\Phi(k', \sigma) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\sigma)\, Y_{\ell m}(k') \tag{6.1}$$

where k' is a point of the sphere.

We insert the expansion (6.1) into eq. (5.12) of § 5. We get that

$$f(u) = -\frac{1}{2i(2\pi)^3} \sum_{\ell m} \int_{\delta - i\infty}^{\delta + i\infty} \sigma(\sigma+1)\, a_{\ell m}(\sigma) \int_{\Gamma} (uk')^{-\sigma-2}\, Y_{\ell m}(k')\, d^2k'\, d\sigma \tag{6.2}$$

where d^2k' is the ordinary Euclidean measure on the sphere Γ.

We have to calculate the integral

$$J = \int_{\Gamma} (uk')^{-\sigma-2}\, Y_{\ell m}(k')\, d^2k' \tag{6.3}$$

Let $u = (u_0, u')$ where $u_0 = \operatorname{ch} a$, $u' \equiv u$. We choose a polar axis in the direction u' and go over to the spherical coordinates, after simple calculations we obtain

$$J = \frac{(-1)^{\ell}(2\pi)^{3/2}\Gamma(-\sigma-1)}{\Gamma(-\sigma-\ell-1)\sqrt{\operatorname{sh} a}} P^{-\ell-1/2}_{-\sigma-\frac{3}{2}}(\operatorname{ch} a)\, Y_{\ell m}(\theta,\phi) \tag{6.4}$$

By substituting J into eq. (6.2) we get

$$f(u) = \frac{1}{2i(2\pi)^{3/2}} \sum_{\ell,m} (-1)^{\ell+1} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(1-\sigma)}{\Gamma(-\sigma-\ell-1)} \frac{a_{\ell m}(\sigma)}{\sqrt{\operatorname{sh} a}} P^{-(\ell-1/2)}_{-\sigma-3/2}(\operatorname{ch} a)\, Y_{\ell m}(\theta,\phi)\, d\sigma \tag{6.5}$$

Now we proceed to the calculation of the coefficients $a_{\ell m}$. From eq. (6.1) it follows that

$$a_{\ell m}(\sigma) = \int_{\Gamma} \Phi(k',\sigma)\, Y^{*}_{\ell m}(k')\, d^2 k' \tag{6.6}$$

We insert into this equation the expression (5.11) for $\Phi(k',\sigma)$

$$a_{\ell m}(\sigma) = \int f(u) \int_{\Gamma} (uk')^{\sigma}\, Y^{*}_{\ell m}(k')\, d^2k' \,\frac{d^3u}{u_0} \tag{6.7}$$

Repeating the calculations of the integral (6.4) we get

$$a_{\ell m}(\sigma) = \frac{(-1)^{\ell}(2\pi)^{3/2}\Gamma(\sigma+1)}{2\Gamma(\sigma-\ell+1)} \int f(u)(\operatorname{sh} a)^{-1/2} P^{-(\ell+1/2)}_{\sigma+1/2}(\operatorname{ch} a)\, Y^{*}_{\ell m}(\theta,\phi) \frac{d^3u}{u_0} \tag{6.8}$$

In terms of the coordinates a, θ, ϕ we have

$$\frac{d^3u}{u_0} = \operatorname{sh}^2 a \sin\theta \; da\, d\theta\, d\phi \tag{6.9}$$

Eqs. (6.5), (6.8) give the expansion of the function $f(u)$ in the eigenfunctions of the Laplacian.

Formulae become simpler if $\sigma = -1 + ip$ i.e. if we consider only the unitary case. In this case they take the form:

$$f(u) = \frac{(-1)^{\ell}}{(2\pi)^{3/2}} \sum_{\ell,m} \int_0^{\infty} \frac{\Gamma(2+ip)}{\Gamma(ip-\ell)} \frac{a_{\ell m}(p)}{(\operatorname{sh} a)^{1/2}} P^{-(\ell+1/2)}_{-1/2+ip}(\operatorname{ch} a)\, Y_{\ell m}(\theta,\phi)\, p^2\, dp \tag{6.10}$$

where

$$a_{\ell m}(p) = \frac{(-1)^{\ell}(2\pi)^{3/2}\Gamma(-ip)}{2\Gamma(-ip-\ell)} \int f(u)(\operatorname{sh} a)^{-1/2} P^{-(\ell+1/2)}_{-1/2-ip}(\operatorname{ch} a)\, Y^{*}_{\ell m}(\theta,\phi) \frac{d^3u}{u_0} \tag{6.11}$$

Expansion in the Π - System

The expansion in the coordinate system Π is performed in just the same way. We choose as the Γ contour the section of the cone by the planes $u_3 = \pm 1$. This cross section consists of two parts: the upper sheets of the two-sheet hyperboloids Γ_+ and Γ_-. We denote by $\Phi_+(k', \sigma)$ and $\Phi_-(k', \sigma)$ the values of $\Phi(k', \sigma)$ on these hyperboloids and apply to the functions $\Phi_+(k', \sigma)$ and $\Phi_-(k', \sigma)$ the expansion on the two-demensional hyperboloid (see Appendix). By substituting the obtained result into the formula (5.12) we get:

$$f(u) = \frac{1}{8} \frac{1}{(2\pi)^4} \sum_{m=-\infty}^{\infty} \int_{\delta - i\infty}^{\delta + i\infty} \sigma(\sigma+1) \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(1-\tau)}{\Gamma(m-\tau)} \tau \, ctg \pi\tau \, . \tag{6.12}$$

$$\cdot a_m^+(\tau, \sigma) \left(\int_{\Gamma_-} + \int_{\Gamma_+} \right) (uk')^{-\sigma-2} J^m_{-\tau-1}(k') d^2k' \, d\tau \, d\sigma$$

where

$$J^m_{-\tau-1}(k') = P^m_{-\tau-1}(ch\, b)\, e^{im\phi} \equiv J^m_{-\tau-1}(b, \phi)$$

$$d^2k' = sh\, b\, db\, d\phi$$

In this case we have to calculate the integrals

$$I_\pm = \int_{\Gamma_\pm} (uk')^{-\sigma-2} J^m_{-\tau-1}(k')\, d^2k' \tag{6.13}$$

They are calculated in just the same way as (6.4)

$$I\pm = \frac{\Gamma(\sigma+\tau+2)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma+2)} (ch\, a)^{-1} P^{-\sigma-1}_{-\tau-1}(\mp th\, a) J^m_{-\tau-1}(b, \phi) \tag{6.14}$$

Therefore

$$f(u) = -\frac{(ch\, a)^{-1}}{8(2\pi)^4} \sum_m \int_{\delta - i\infty}^{\delta + i\infty} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{\Gamma(\sigma+\tau+2)\Gamma(\sigma-\tau+1)}{\Gamma(\sigma)\ \Gamma(m-\tau)} \tau \, ctg\pi\tau \cdot J^m_{-\tau-1}(b, \phi). \tag{6.15}$$

$$\left\{ a_m^+(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(-th\, a) + a_m^-(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(th\, a) \right\} d\tau \, d\sigma$$

Calculating a_m^+ and a_m^- we get

$$a_m^\pm(\tau, \sigma) = \frac{\Gamma(\tau)\ \Gamma(-\sigma-\tau-1)\,\Gamma(\tau-\sigma)}{\Gamma(\tau-m+1)\quad \Gamma(-\sigma)} \int f(u) P^{\sigma+1}_\tau(\mp th\, a) J^{-m}_\tau(b, \phi)\, d^3u \tag{6.16}$$

where

$$d^3u = sh^2 a \ sh\, b\ da\, db\, d\phi \tag{6.17}$$

As was pointed out in §5 (p.14) the unitary case corresponds to the values

$\sigma=-1+ip,\ \tau=-\frac{1}{2}+iq$. Then eqs. (6.15), (6.16) take the form

$$f(u)=\frac{1}{(2\pi)^4\,\mathrm{ch}\ a}\sum_m\int_0^\infty \sigma^2\int_0^\infty \frac{\Gamma(\frac{1}{2}+iq+ip)\,\Gamma(\frac{1}{2}+ip-iq)}{\Gamma(1+ip)\ \Gamma(m+\frac{1}{2}-iq)}\ q\,\mathrm{cth}\ \pi q\ \times \tag{6.18}$$

$$J^m_{-\frac{1}{2}+iq}(b,\phi)\,\{a^+_m\,P^{ip}_{-\frac{1}{2}+iq}(-\mathrm{th}\,a)+a^-_m\,P^{-ip}_{-\frac{1}{2}+iq}(\mathrm{th}\,a)\}\,d\phi\,dq$$

$$a^{\pm}_m(p,q)=\frac{\Gamma(-\frac{1}{2}+iq)\,\Gamma(\frac{1}{2}-ip-iq)\,\Gamma(\frac{1}{2}-ip+iq)}{\Gamma(\frac{1}{2}+iq-m)\,\Gamma(1-ip)}. \tag{6.19}$$

$$\int f(u)\,P^{ip}_{-\frac{1}{2}-iq}(\mp\mathrm{th}\,a)\ J^{-m}_{-\frac{1}{2}+iq}(b,\phi)\,d^3\phi$$

The Expansion in the O System

In the horospherical coordinate system the calculation is performed in just the same way as above. As the Γ - contour we choose the section of the cone by the plane $k_0-k_3=2$.

$$k_0=(1+\rho^2)\qquad\qquad k_3=(-1+\rho^2) \tag{6.20}$$

$$k_2=2\rho\cos a\qquad\qquad k_1=2\rho\ \sin\ a$$

We put

$$\Phi(k',\sigma)=\int_0^{2\pi}\int_0^\infty \psi(\kappa,\theta,\sigma)\,e^{i\kappa\rho\cos(\theta-a)}\,\kappa\,d\kappa\,d\theta \tag{6.21}$$

(the Hanckel transformation) and insert it in eq. (5.12). After simple transformations we get that

$$f(u)=-\frac{1}{2i(2\pi)}\int_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\qquad e^{-a(\sigma+2)} \tag{6.22}$$

$$\int_0^{2\pi}\int_0^\infty \psi(\kappa,\theta,\sigma)\,e^{i\kappa r\cos(\theta-\phi)}\qquad \int_0^{2\pi}\int_0^\infty e^{i\kappa\rho\cos(\theta-a)}$$

$$\cdot(\rho^2+e^{-2a})^{-\sigma-2}\,\kappa\rho\,d\rho\,da\,d\kappa\,d\theta\,d\sigma$$

$$I=\int_0^{2\pi}\int_0^\infty e^{i\kappa\rho\cos(\theta-a)}(\rho^2+e^{-2a})^{-\sigma-2}\rho\,d\rho\,da=\frac{2\pi}{\Gamma(\sigma+2)}\left(\frac{\kappa}{2}e^{a}\right)^{\sigma+1}K_{-\sigma-1}(e^{-a}\kappa) \tag{6.23}$$

where $K_a(x)$ is the Macdonald cylindrical function. Therefore (assuming also $e^{-a} = b$)

$$f(u) = -\frac{b}{i(2\pi)^2} \int_{\delta - i\infty}^{\delta + i\infty} \frac{1}{\Gamma(\sigma)} \int_0^{2\pi} \int_0^{\infty} \psi(\kappa, \theta, \sigma) \left(\frac{\kappa}{2}\right)^{\sigma+2} e^{i\kappa\rho\cos(\theta - \phi)} \cdot K_{-\sigma-1}(b\kappa)\, d\kappa\, d\theta\, d\sigma \tag{6.24}$$

The coefficients $\psi(\kappa, \theta, \sigma)$ are calculated by the Fourier formula

$$\psi(\kappa, \theta, \phi) = \frac{1}{(2\pi)^2} \int \Phi(k', \sigma)\, e^{-i\kappa\rho\cos(\theta - a)}\, d^2k'$$

where $d^2k' = 4\rho\, d\rho\, da$. Substituting here eq. (5.11) for $\Phi(k', \sigma)$ we get

$$\psi(k, \theta, \sigma) = \frac{2}{\pi\,\Gamma(-\sigma)} \left(\frac{2}{\kappa}\right)^{\sigma+1} \int f(u)\, e^{-i\kappa r\cos(\theta - \phi)} K_{\sigma+1}(b\kappa) \frac{1}{b^2}\, d^3u \tag{6.25}$$

$$d^3u = -r\, dr\, e^{-a} da\, d\phi = r\, dr\, db\, d\phi$$

In the unitary case as above: $\sigma = -1 + i\rho$

The Expansion in the Cylindrical Coordinate System

Finally we consider the cylindrical coordinate system. It corresponds to the cross section Γ of the hyperboloid by the cylinder $k_0^2 - k_1^2 = 1$ We choose on Γ the parameters c and a :

$$k_0 = \operatorname{ch} c, \quad k_1 = \operatorname{sh} c, \quad k_2 = \sin a, \quad k_3 = \cos a \tag{6.26}$$

We perform the Fourier transformation of the function $\Phi(k', \sigma)$ with respect to the parameters c, a:

$$\Phi(k', \sigma) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} a_m(r, \sigma)\, e^{i(ma + rc)}\, dr \tag{6.27}$$

We substitute this expansion to eq. (5.12). Assuming

$$u_0 = \operatorname{ch} b \operatorname{ch} a \qquad u_3 = \operatorname{sh} b \cos\phi$$
$$u_2 = \operatorname{sh} b \sin\phi \qquad u_1 = \operatorname{ch} b \operatorname{sh} a \tag{6.28}$$

we get

$$f(u) = -\frac{1}{2i(2\pi)^3} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(m\phi + r a)} \cdot \int_{\delta - i\infty}^{\delta + i\infty} \sigma(\sigma+1)\, a_m(r, \sigma) \int_0^{2\pi} \int_{-\infty}^{\infty} (\operatorname{ch} b \operatorname{ch} c - \operatorname{sh} b \cos a)^{-\sigma-2} e^{i(ma + rc)} \cdot dc\, da\, d\sigma\, dr. \tag{6.29}$$

Now we have to calculate the integral

$$I = \int_0^{2\pi} \int_{-\infty}^{\infty} (ch b \; ch c - sh b \cos a)^{-\sigma-2} \, e^{i(ma+\tau c)} \, dc \, da \tag{6.30}$$

We separate $ch b \; ch c$ outside the brackets and expand $(1 - th b \frac{\cos c}{ch c})^{-\sigma-2}$ into the binomial series. After the integration term by term we get

$$I = \frac{2^{\sigma+2}\pi \Gamma(\frac{m+\sigma+i\tau}{2}+1)\Gamma(\frac{m+\sigma-i\tau}{2}+1)}{\Gamma(m+1)\Gamma(\sigma+2)} \tag{6.31}$$

$$\frac{sh^m b}{ch^{m+\sigma+2} b} F(A, B, m+1; th^2 b)$$

where $A = \frac{1}{2}(m+\sigma+i\tau+2)$ $\quad B = \frac{1}{2}(m+\sigma-i\tau+2)$

Therefore

$$f(u) = -\frac{1}{16\pi^2 i} \sum_m \frac{th^m b \; e^{im\phi}}{\Gamma(m+1)} \int_{-\infty}^{\infty} e^{i\tau a}$$

$$\int_{\delta-i\infty}^{\delta+i\infty} \frac{a_m(\tau,\sigma)}{\Gamma(\sigma)} \Gamma(A)\Gamma(B)\left(\frac{2}{ch\, b}\right)^{\sigma+2} F(A, B, m+1; th^2 b) \, d\sigma \, d\tau \tag{6.32}$$

The coefficients $a_m(\tau,\sigma)$ are expressed by the formula

$$a_m(\tau,\sigma) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \Phi(k',\sigma) e^{-i(m\phi+\tau a)} \, d^2k' \tag{6.33}$$

$$d^2k' = d\phi \, da$$

We insert for $\Phi(k',\sigma)$ the eq. (5.11). After simple transformations we obtain

$$a_m(\tau,\sigma) = \frac{\Gamma(A')\Gamma(B')}{4\pi \Gamma(m+1)\Gamma(-\sigma)} \, . \tag{6.34}$$

$$\cdot \int th^m b \left(\frac{2}{ch\, b}\right)^{-\sigma} F(A', B', m+1; th^2 b) \, e^{i(m\phi+\tau a)} \, f(u) \, d^3u$$

where $d^3u = sh^2 a \; da \; db \; d\phi$

$A' = \frac{1}{2}(m-\sigma+i\tau)$

$B' = \frac{1}{2}(m-\sigma-i\tau)$

The unitary case corresponds to $\sigma = -1 + i\rho$. These formulas solve the problem of the amplitude expansion in the described four systems of coordinates.

APPENDIX

In the expansion in the H system use has been made of the expansion of functions on a two-dimensional hyperboloid. Introducing the coordinates

$$u_0 = ch\, b$$
$$u_2 = sh\, b \cos\phi$$
$$u_1 = sh\, b \sin\phi$$

we have

$$f(u) = -\frac{1}{8\pi i}\sum_{m=-\infty}^{\infty}\int_{\epsilon-i\infty}^{\epsilon+i\infty} a_m(\sigma)\,\frac{\Gamma(1-\sigma)}{\Gamma(m-\sigma)}\, J^m_{-\sigma-1}(b,\phi)\,\sigma\, cth\,\pi\sigma\, d\sigma$$

where

$$a_m(\sigma) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-m+1)}\int f(u)\, \overset{*}{J}{}^{-m}_{\sigma}(b,\phi)\, d^2u$$

$$d^2u = sh\, b\; db\; d\phi$$

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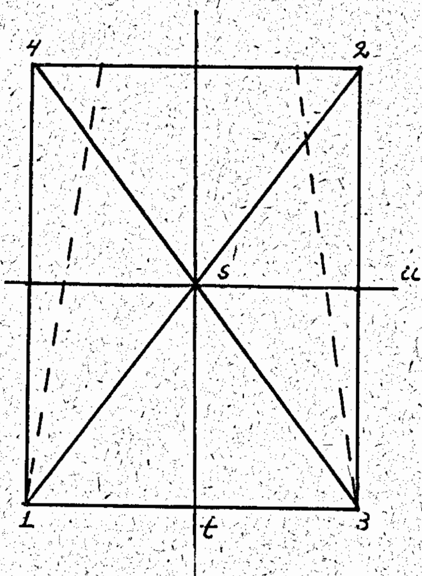

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