ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ
ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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E-1496

ON THE PROBLEM OF BOUND STATES IN PION-PION INTERACTION

Nucl. Phys., 1964, v 55, n 2, p 300-304.

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1. Introduction

In a previous work[1] we have shown that the Bethe-Salpeter equation for the two-particle Green function can be solved even if the kernel in momentum representation does not belong to the Fredholm class.

The method proposed in ref.[1] contained the following essential steps:

1) Put E the total energy equal to zero.

2) Go over to Euclidean metric in the space of relative momenta.

3) Expand the Green function according to four dimensional spherical harmonics.

4) Perform a Hankel transform on the radial Green function.

As a result of these operations we arrived at a local differential equation for the radial part of the Green function, provided the BS kernel was a generalized local one, as specified in ref.[1].

We have shown that in this manner the energy-independent singularities of the scattering amplitude can be found in a comparatively simple way, the latter being determined by the most singular part of the "potential" (the Hankel transform of the kernel) at $r \to 0$, where r was the corresponding radial coordinate.

In particular, we have found that the two-pion Green function possesses an infinite series of energy-independent cuts as a function of the angular momentum ℓ, provided, we take the lowest, nonvanishing approximation to the kernel (Example 3 of ref.[1]). In the present work we want to investigate the problem of bound states in the case of singular interactions, confining ourselves to the pion-pion interaction, in the approximation mentioned above.

Our method consists in the following. Taking the solutions of the radial BS equation for the singular part of the potential, we expand the total Green function according to these functions rather than the free ones.

In this way we arrive at an integral equation for the Green function, which is already of the Fredholm type, and so familiar methods can be applied for finding the bound state poles.

Sec. 2 is devoted to the derivation of the integral equation just mentioned, while in Sec. 3 we investigate the positions of the roots of the Fredholm determinant in the weak coupling approximation. We find that for $E = 0$ and weak coupling there are no poles on the physical sheet of the angular momentum plane, while on the first unphysical sheet we find a pair of poles.

The possible physical implications of this situation are discussed in sec. 4.

Throughout the present work extensive use will be made of the notations and results of ref. /1/.

2. Derivation of the Integral Equation for the Green Function

In ref. /1/ we have found that the radial part of the two-pion Green-function at $E = 0$ satisfies the following equation:

$$\left[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - m^2 - \frac{n^2-1}{r^2} \right]^2 G(r, r') = \frac{\delta(r-r')}{r^3} + V(r)\, G(r, r') \qquad (2.1)$$

where the "potential" $V(r)$ has been given by the following expression:

$$V(r) = \lambda^2 \int_{2m}^{\infty} d\mu\, \mu^3 \left(\frac{\mu^2 - 4m^2}{\mu^2} \right)^{1/2} \frac{K_1(\mu r)}{\mu r} . \qquad (2.2)$$

The potential can be split into a "singular" and a "regular" part: $V = V_s + V_r$, where

$$V_s = \lambda^2 \int_0^{\infty} d\mu\, \mu^3 \frac{K_1(\mu r)}{\mu r} = \frac{2\lambda^2}{r^4} \qquad (2.3)$$

and

$$V_r = \lambda^2 \int_0^{\infty} d\mu \left[\Theta(\mu - 2m) \left(\frac{\mu^2 - 4m^2}{\mu^2} \right)^{1/2} - 1 \right] \times \\ \times \mu^3 \frac{K_1(\mu r)}{\mu} \qquad (2.4)$$

Equation (2.1) with V_s only can be solved exactly. We found for the wave function $\psi(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - m^2 - \frac{n^2-1}{r^2} \right]^2 \psi(r) = V_s(r)\, \psi(r) \qquad (2.5)$$

with the four independent solutions:

$$\psi^1_{\pm}(r) = I_{p_\pm}(mr)$$

$$\psi^2_{\pm}(r) = K_{p_\pm}(mr) \qquad (2.6)$$

$$(p_\pm)^2 = 1 + n^2 \pm (4n^2 + 2\lambda^2)^{1/2}$$

The Green function with the singular part of the potential, can be constructed in the usual way; we obtain:

$$G^{*}_{n}(r,r') = \frac{-1}{2\sqrt{4n^2+2\lambda^2}} \{ \Theta(r-r')[K_{p_+}(mr)I_{p_+}(mr') -$$

$$-K_{p_-}(mr)I_{p_-}(mr')] + \tag{2.7}$$

$$+\Theta(r'-r)[K_{p_+}(mr')I_{p_+}(mr) -$$

$$-K_{p_-}(mr')I_{p_-}(mr)] \}$$

Now we make use of the operator identity, already used in ref./1/ in a different context.

Assuming that we have to solve the linear operator equation:

$$G = G_0 + G_0(K_s + K_r)G$$

and the operator satisfying the equation

$$G^{*} = G_0 + G_0 K_s G^{*}$$

is known, we find that G satisfies the equation:

$$G = G^{*} + G^{*} K_r G \tag{2.8}$$

In the present case, if G^{*} is the solution vanishing at $r \to 0$ and $r' \to 0$ the equation (2.8) is an integral equation with a L kernel in momentum representation.

Transforming back to momentum representation, we find:

$$\langle p | G^{*}_{n} | p' \rangle = \int_0^\infty r^3 dr \int_0^\infty r'^3 dr' \times \tag{2.9}$$

$$\times \frac{{}_n(pr)}{pr} G^{*}_{n}(r,r') \frac{{}_n(p'r')}{p'r'}$$

and

$$\langle p | G^{*} K_r | p' \rangle = \int_0^\infty r^3 dr \int_0^\infty r'^3 dr' \times \tag{2.10}$$

$$\times \frac{J_n(pr)}{pr} V_r(r) G_n^{(0)}(r,r') \frac{J_n(p'r')}{p'r'} \tag{2.10}$$

where the expressions of V_r and G_n^0 are to be taken from eqs. (2.4) and (2.7), respectively.

Thus $\langle p|G|p'\rangle$ satisfies the integral equation:

$$\langle p|G_n|p'\rangle = \langle p|G_n^0|p'\rangle + \int_0^\infty q^2 dq \langle p|G_n^0 K|q\rangle\langle q|G_n|p'\rangle \tag{2.11}$$

In order to find the positions of the bound state poles of G_n we have to find the roots of the Fredholm determinant:

$$D_n = \exp \int_0^\infty dp\, p^2 \langle p|\log(1-G_n^0 V_r)|p\rangle = 1 - \int_0^\infty dp p^2 \langle p|G^0 V_r|p\rangle + \ldots, \tag{2.12}$$

3. Poles of the Green Function

The Fredholm determinant (2.12) being independent of the representation, we need not go through the steps outlined in eqs. (2.9) - (2.11) but can insert the matrix elements in the r -representation directly.

Thus the Fredholm determinant (2.10) takes the form:

$$D_n = 1 - \int_0^\infty r^2 dr\, V_r(r)\, G_n^0(r,r') + O(\lambda^4) \tag{3.1}$$

Inserting the expressions (2.4) and (2.7), we find to an accuracy up to λ^2 :

$$D_n = 1 - \tfrac{1}{2}\lambda^2 (4n^2 + 2\lambda^2)^{-1/2} \int_0^\infty r^2 dr\, F(r) \tag{3.2}$$

where

$$F(r) = \int_0^\infty \mu^2 d\mu \; [\Theta(\mu - 2m)\left(\frac{\mu^2 - 4m^2}{\mu^2}\right)^{1/2} - 1] \times \tag{3.3}$$

$$\times K_1(\mu r)\,[K_{p_-}(mr)\,I_{p_-}(mr) - $$

$$- K_{p+}(mr)\,I_{p+}(mr)\,]. \qquad (3.3)$$

One has to bear in mind that the Fredholm parameter of eq. (2.11) is the λ^2 standing before V_1 ; therefore one must not expand the expressions $p_\pm$ and $(4n^2 + 2\lambda^2)^{-1/2}$ in powers of λ^2. (By expanding these expressions, one would just loose we were attempting at, namely, to build in explicity the "dangerous" singularities into the solution).

It is clear that for small λ^2 , the right hand side of eq. (3.2) can vanish in the neighbourhood of a pole of $\int_0^\infty r^2 dr\; F(r)$ in the n-plane. As the Bessel functions in eq. (3.3) are integral functions of their indices, a pole can arise if the integral over r begins to diverge for some value of n .

Taking into account the well-known asymptotic expressions of the Bessel functions, we find that the integral over r at the upper limit converges independently of the value of n :

$$F(r) = O(r^{-5}) \quad (r \to \infty)$$

Therefore, poles can arise from the behaviour of $F(r)$ at $r \to 0$ only. Here we find:

$$F = O(r^{-a})$$

where

$$a = \max\{1,\; q-2p_-,\; 1-2p_+\}$$

Thus the integral in (3.2) begins to diverge at $r=0$ if $a \geq 3$.

Thus a pole of $\int_0^\infty r^2 F(r)\,dr$ will arise at $\mathrm{Re}\,p_\pm = -1$

If we choose the positive sign of the square root in the expression of $p_\pm$ from eq. (2.6) the latter condition can never be satisfied. Hence on the sheet of the n - plane where $p_\pm$ is defined with the positive sign, in the weak coupling limit there are no poles at all (at least for $\mathrm{Re}\,n > 0$). On the other sheet we have poles, situated at the points:

$$n = \pm 2^{1/2}\left[1 \pm \sqrt{1 - \frac{\lambda^2}{2}}\,\right]^{1/2} \qquad (3.4)$$

We are thus faced with the problem of telling which is the "physical sheet" of the

n -plane. We know that for sufficiently large values of n , the iteration solution of eq. (2.11) must exist. This happens only in the case if we choose the positive sign in $p_{\pm}$. (This result can be made plausible by the following argument as well. p_{-} plays the role of a "modified four dimensional angular momentum" just as the quantity $\bar{\ell} = -\frac{1}{2} + \sqrt{(\ell + \frac{1}{2})^2 + g}$ in the nonrelativistic Schrödinger equation with a potential $g r^{-2}$. Thus, if the coupling constant vanishes, p- should go over to $n-1$ and not $1-n$).

Hence we conclude that in the weak coupling limit there are no poles on the physical sheet of the n -plane, for $E=0$.

4. Discussion

In a superrenormalizable field theory, where the lowest order kernel of the BS equation is square integrable/2/, one finds a result/3/, familiar from non-relativistic quantum mechanics/4/: the scattering amplitude as a function of the angular momentum has no other singularities but poles- and in view of the classical theorem of Schmidt, it has at least one pole. In our case, which is an example of a renormalizable (but not super-renormalizable) interaction, we came to a qualitatively different conclusion. The angular momentum plane has several sheets, and we have found that the physical sheet does not contain poles at all. Nevertheless, we have to bear in mind that our result has been found in the weak coupling limit and for $E=0$. At present we cannot tell anything about the behaviour of the poles as the coupling constant increases or the energy is varied.

One can, however, imagine that with increasing λ^2 or some values of the energy the poles would move towards the cut and cross it, thus appearing on the physical sheet. If this happened, we would encounter the strange situation that a composite state (in the general sense) could be formed above a minimal value of the coupling constant or energy only.

An example for a similar behaviour of bound states has been indeed given in a recent paper of Bastai et al./5/.

The authors would like to express their sincere thanks to Prof. N.N.Bogolubov, for his constant interest in the present research.

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Received by Publishing Department
on December 17, 1963.