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ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

G.Domokos and P.Suranyi

E - 1400

# BOUND STATES AND ANALYTIC PROPERTIES IN ANGULAR MOMENTUM

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# BOUND STATES AND ANALYTIC PROPERTIES IN ANGULAR MOMENTUM

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## 1. Introduction

The investigation of the problem of the relativistic theory of composite states and its interrelation to analytic properties of Green functions in angular momentum in quantum field theory has produced several interesting results in last years.

Bound states appear as singular surfaces of the S - matrix elements or Green functions, considered as functions of the total angular momentum and energy of the system. The actual calculation of singular surfaces in question, however, meets with several serious difficulties. Strictly speaking, the Green functions satisfy an infinite system of coupled integral equations, or - in an alternative but equivalent formulation-equations in functional derivatives.

Even the question as to the existence of solutions of such equations or finding practical methods to handle them in relatistic field theories is almost completely unsolved.

Practically tractable has proved to be so far the Bethe-Salpeter (BS) equation only. The essence of the BS method consists in finding a linear integral equation containing one Green function only, all the higher and lower order Green functions being lumped together into the kernel of the equation, and considered as known ones. In practice one approximates the kernel by the contribution of a finite number of Feynman diagrams and tries to solve the resulting integral equation. It turns out, however, that in realistic field theories, at least in the simplest approximations for the BS kernel, the integral equations do not belong to the Fredholm class, because the fundamental domain of the integration variables is an infinite one and the kernel does not decrease rapidly enough at infinity. ( The simplest examples for such a phenomenon are the pion-pion Green function in the $g\phi^4$ theory with a simple "bubble-kernel" and the scattering of spinless particles with the exchange of vector mesons in the ladder approximation /1/ ).

The cases mentioned above have been treated either by summing the most singular terms of the iterated diagrams /1/ or by solving the "asymptotic integral equation /1,2/, i.e. the BS equation with a kernel, reproducing the asymptotic behaviour of the original one for large relative momenta.

As a result it turned out that the two particle Green function contains singularities not corresponding to bound states. ( For the cases mentioned above, these were energy independent cuts, in the angular momentum plane).

In the present work we develop a method for handling such "singular" Green functions, for a certain class of kernels, to be specified later. ( The class of the kernels considered, although rather special, comprises all the kernels practically treated so far). The method to be presented consists in converting the original BS equation into an ordinary differential equation for the covariant radial wave function. The energy-independent singularities can then be immediately found by examining the behaviour of the wave function for small distances.

The solutions of the equation are then classified according to the nature of the singularity of the kernel at the origin. It turns out, that for any power-like singularity of the kernel there exist "no collapse" solutions of the differential equation. The positions of the energy independent singularities of the Green function in the angular momentum are then found exactly for the cases mentioned above in a surprisingly simple manner.

Further it is shown that the bound states described by Regge trajectories, can be classified on a simple group theoretical basis, independently of the singularity of the wave function at the origin. Throughout the paper we treat the interaction of spinless particles. All our considerations, however, can be extended to the case of particles with spin. Examples of this kind are defferred to a future publication.

While this research was in progress, we learnt that several authors have found results, which either partly coincide with, or are special cases of our ones. ( See e.g. /3,4/ ).

Despite of possible overlaps we should like to present this paper in this self-contained form.

## 2. Separation of renormalization terms and classification of bound states

In order to simplify kinematics, we consider the interaction of particles of equal mass, m. We define the normalized two-particle Green function by the following relation:

$$\tilde{\tilde{G}}(x_1, \dots, x_4) = \langle 0 | T(\phi(x_1) \dots \phi(x_4) S) S^+ | 0 \rangle ,$$

where $\phi(x_i)$ are asymptotic fields and $S$ is the adiabatic $S$-operator

$$S = T \exp i \int g(x) L(x) dx,$$

as defined by Bogolubov and Shirkov/5/.

Because of translation invariance, $\tilde{\tilde{G}}$ depends on three coordinate differences only; we define its Fourier transform by

$$\tilde{\tilde{G}}(x_1, \dots, x_4) = \frac{1}{(2\pi)^6} \int dp \; dq \; dE \; G(p, q | E) \times$$

$$\times \exp[-ip(x_1 - x_2) + iq(x_3 - x_4) + i\frac{E}{2}(x_3 + x_4 - x_1 - x_2)].$$

The Fourier transform $G$ satisfies the BS equation/6/:

$$[(p + \tfrac{1}{2}E)^2 - m^2][(p - \tfrac{1}{2}E)^2 - m^2] \; G(p, q | E) =$$
$$= \delta(p - q) + \frac{g^2}{(2\pi)^4 i} \int dp' K(p, p' | E) \, G(p', q | E) \qquad /2.1/$$

Here $K(p, p'|E)$ is the BS kernel, consisting of the contribution of the Feynman diagrams not possesing two particle cuts in $E$ .

We treat kernels which possess a simple spectral representation in the nomentum transfer, $(p - p')^2$, of the form:

$$K(p,p'|E)=\frac{(p-p')^{2N}}{\pi}\int_{\mu_0^2}^{\infty}\frac{d\mu^2\sigma(\mu^2,E)}{\mu^{2N}(\mu^2-(p-p')^2)}$$

$$+P_{N-1}((p-p')^2;E). \tag{2.2}$$

$P_{N-1}$ is a polynomial of $(N-1)$ th degree in $(p-p')^2$ ; its coefficients being-finite-renormalization constants of $K$ . The spectral density $\sigma(\mu^2,E)$ may contain $\delta$ – functions.

We now split the kernel $K$ into two parts:

$$K=K_1+K_2$$

where $K_1$ means the spectral integral, $K_2$ the polynomial. First of all, we show that there is a corresponding splitting of the Green function G . In fact, the BS equation written in operator form, reads:

$$G=G_0+G_0(K_1+K_2)G \tag{2.3}$$

( G and $G_0$ are the Green operators with and without interaction, respectively ).

A simple algebra shows that if $G_1$ is the solution of the equation:

$$G_1=G_0+G_0K_1G_1 \tag{2.4}$$

then (2.3) can be rewritten with the help of $G_1$ as follows :

$$G=G_1+G_1K_2G \tag{2.5}$$

Equations (2.4) and (2.5) tell us that first one can solve the BS equation with $K_1$ - the spectral integral only - as if the substraction polynomial were not there at all, afterwards the complete Green function can be found by solving another BS equation with $K_2$ only, but with another "free" Green function.

The point in such a separation is that by expanding G into partial waves, the contribution of $K_2$ differs from zero for the first $N-1$ physical partial waves only, while $G_1$ is an analytic function of the angular momenta. G coincides with $G_1$ except for the few discrete points, where $K_2$ differs from zero. Therefore, in order to study analytic properties of G in the angular momentum, one can ignore the contribution of the renormalization terms at all.

A second point connected with the separation of renormalization terms is a natural classification of bound ( or quasistationary) states into two classes.

I. We call regular bound states those corresponding to the poles of $G_1$ . They are characterized by the fact that there is an analytic function connecting their masses and spins. ( The "Regge trajectory" or its inverse function) .

II. We call singular bound states those which correspond to poles of $G$ but not of $G_1$. Their spin can take on discrete values only, corresponding to the subtracted partial waves.

Remarks. 1. In order to simplify the terminology we call "regular" bound states" the whole singular surface of the Green function, corresponding to complex energies and angular momenta or one Regge trajectory, corresponding to real energies, or sometimes one point on the Regge trajectory, corresponding to a physical angular momentum value. The sense of the term will be always clear from the context.

2. The appearence of singular bound states seems to be characteristic of relativistic field theory.

Whether a singular bound state appears or not, depends on the theory considered, while in a nontrivial theory there is at least one regular bound state ( not necessarily a physical one). We demonstrate at present this latter assertion with an example. Consider the $g\,\phi^3$ theory in the ladder approximation. The BS kernel needs no substruction, it belongs to the Fredholm class/7/. One can symmetrize the kernel by introducing $g^2 \; G_0^{1/2} K \; G_0^{1/2}$ instead of $g^2 G_0 K$. Then by Schmidt's theorem there exists at least one eigenvalue $g^2(\ell, s)$ giving an inplicit relation for the Regge trajectory. Because of the absence of subtraction terms, certainly no singular bound states appear.

In what follows, we study the analytic part of $G$ ( i.e. $G_1$ ) ignoring the subtraction terms in $K$. We can therefore omit the index "1" from $K_1$ and $G_1$ without the danger of confusion.

## 3. Expansion into partial waves.

Following Wick/8/, in eq. (2.1) we restrict the value of $E^2$ to $0 < E^2 < 4m^2$ and go over to Euclidean variables by turning the integration path in $p^{0\prime}$ tothe imaginary axis. For a kernel given by eq. (2.2) no singularities are crossed thereby.

The symmetry group of eq. (2.1) becomes then $R_4$ instead of $L$. Actually, for $E_\mu \neq 0$ fixed, eq. (2.1) is invariant under the subgroup of rotations around the direction of $E$ only ( which is isomorphic to $R_3$ ), nevertheless, we find it convenient to expand $G$ according to a basis of $R_4$. Choosing the coordinate system such as to have $E_0 = E$, $\underline{E} = 0$ we have the usual representation of $R_3$.

Thus we expand the equation accodring to four dimensional spherical harmonics:

$$Z^m_{n\ell}\,(\psi,\theta,\phi) = \frac{1}{N_{n\ell}}\,(\operatorname{Sin}\psi)^\ell \times \times C^{\ell+1}_{n-\ell-1}\,(\cos\psi)\, Y^m_\ell\,(\theta;\phi) \equiv p_{n\ell}\,(\psi)\, Y^m_\ell\,(\theta,\phi) \qquad (3.1)$$

( of. Appendix ).

where $\psi, \theta, \phi$ mean the polar coordinates of a unit vector:

$$
\begin{aligned}
e^0 &= \cos\psi\,, \\
e^3 &= \sin\psi\,\cos\theta\,, \\
e^1 &= \sin\psi\,\sin\theta\,\cos\phi\,, \\
e^2 &= \sin\psi\,\sin\theta\,\sin\phi\,,
\end{aligned}
\qquad (3.2)
$$

$C^{\ell+1}_{n-\ell-1}$ is a Gegenbauer polynomial, $Y^m_\ell$ the three dimensional spherical harmonic, normalized to unity on the surface of the three dimensional unit sphere. The function $p_{n\ell}(\psi)$ is normalized to unity:

$$\int_0^\pi (\sin\psi)^2\, p_{n\ell}(\psi)\, p_{n'\ell}(\psi)\, d\psi = \delta_{nn'},$$

$$N_{n\ell} = \frac{2^{-(\ell+1/2)}}{\Gamma(\ell+1)} \sqrt{\frac{\pi\,\Gamma(n+\ell+1)}{n\,\Gamma(n-\ell)}}$$

( Because of the compactness of $K_4$, both $n$ and $\ell$ are discrete indices).

The four dimensional volume element is given by

$$dV = r^3\, dr\, \sin^2\psi\, d\psi\, \sin\theta\, d\theta\, d\phi\,,$$

so we have the representation:

$$\delta(p-q) = \frac{\delta(p-q)}{(p\,q)^{3/2}} \sum_{n\ell m} Z^m_{n\ell}(\hat{p})\, Z^{m*}_{n\ell}(\hat{q}) \qquad (3.3)$$

( Note that on the left hand side of eq. (3.3) we have a four dimensional – on the r.h.s. – a one dimensional $\delta$ function. Denoting a four vector and it's radial coordinate by the same letter may cause no confusion. $\hat{x}$ stands for the unit vector in the direction of the four-vector $\hat{x}$ ).

In what follows, we choose the 0-axis of the coordinate system in the direction of $E$ ( c.m. system) and characterize $G$ by the independent variables: $p^2$, $q^2$, $E^2$, $\hat{p}$, $\hat{q}$.

Thus we have the following expansion:

$$G(p,q\,|\,E) = \sum_{nn'\ell m} Z^{m*}_{n\ell}(\hat{p}) \langle n'q^2|\, G(E^2,\ell)\, |np^2\rangle\, Z^m_{n\ell}(\hat{q}) \qquad (3.4)$$

and similarly for $K$ and the operator $F \equiv [(1/2\,E+p)^2 - m^2][1/2\,E-p)^2 - m^2]$.

(Notice that $G$ is not diagonal in $n!$ ). In order to find the matrix elements of $K$ ( eq. (2.2) ) in the $n\,\ell$ representation, we first choose $n > N$, and expand in the angle between $p$ and $p'$. Denoting this angle by $\omega$, we expand the denominator $[\mu^2 - (p-p')^2]^{-1}$. In Euclidean metric we have:

$$\frac{1}{\mu^2+p^2} = \int_0^\infty r^3 dr\, \sin^2\psi\, d\psi\, \sin\theta\, d\theta\, d\phi\, e^{ipx}\, G(r)\,.$$

Choosing $p = (p, 0)$ we have $px = pr\cos\psi$ and we can integrate over the three dimensional solid angle to find:

$$\frac{1}{\mu^2 + p^2} = 4\pi \int_0^\infty r^3 dr \; G(r) \; F(pr),$$

where

$$F(pr) = \int_0^\pi d\psi \; \sin^2\psi \; e^{ipr\cos\psi} = \frac{\pi}{pr} J(pr)$$

and $J_1(x)$ is the Bessel function of the first kind of index one. Hence, by a well-known integral formula ( ref./9/, p. 686):

$$G(r) = \frac{\mu}{4\pi^2 r} K_1(\mu r) \tag{3.5}$$

$K_1(x)$ being MacDonalds's function of index one.

In a similar way, noting that $(p-q)^2 = p^2 + q^2 - 2pq\cos\omega$ and making use of the addition theorem of Bessel functions: ( GR, p. 993)

$$\frac{J_1(r|p-q|)}{r|p-q|} = 2\sum_{n=1}^\infty n \frac{J_n(pr)\,J_n(qr)}{pr \cdot qr} C^1_{n-1}(\cos\omega) =$$

$$= 2^{-1/2}\sum_{n=1}^\infty \frac{J_n(pr)}{pr} \frac{J_n(qr)}{qr} \, n \, Z^0_{n0}(\cos\omega)$$

Expressing $\cos\omega$ in terms of the unit vectors $\hat{p}$ and $\hat{q}$ by means of the four dimensional cosine theorem and making use of the addition theorem of four dimensional spherical functions, we arrive at the expansion:

$$K(p, p'|E) = \sum_{n\ell m} Z^{m*}_{n\ell}(\hat{p}) \times$$

$$\times K_n(p^2, p'^2|E^2) \, Z^m_{n\ell}(\hat{p}') \tag{3.6}$$

with

$$K_n(p^2, p'^2|E^2) = \int_0^\infty r^3 dr \frac{J_n(pr)}{pr} V(r) \frac{J_n(qr)}{qr},$$

$$V(r) = \int_{\mu_0}^\infty d\mu \, \mu^3 \sigma(\mu) \frac{K_1(\mu r)}{\mu r} \tag{3.7}$$

( We omitted the variable $E^2$ ).

Strictly speaking we have established the formula (3.7) for $n > N$ only, where in the expansion

$$K(p, p' | E) = \sum_n K_n C^1_{n-1} (\cos\omega)$$

the coefficients $K_n$ can be expressed in terms of Legendre functions of the second kind.

However, the radial equation to be obtained can be continued in $n$ beyond the line $Re\, n = N$ ( For the analogous case in three dimensions see. e.g. Domokos/10/) Thus, in what follows, we relax the restriction

$$Re\, n > N$$

The matrix elements of the operator

$$F = [(\tfrac{i}{2} E + p)^2 + m^2][(\tfrac{i}{2} E - p)^2 + m^2]$$

can be easily found. We have:

$$F = (-p^2 + \tfrac{1}{4} E^2 - m^2)^2 + p^2 E^2 \cos(p, E)$$

So the matrix elements in the $n, \ell$ representation take on the form:

$$< n' \ell' m | F | n \ell m > = \delta_{\ell\ell'} \delta_{mm'} \times$$

$$\times [\delta_{nn'} (-p^2 + \tfrac{1}{4} E^2 - m^2)^2 + p^2 E^2 f^{\ell}_{nn'}] \tag{3.8}$$

where

$$f^{\ell}_{nn'} = f^{\ell}_{n'n} = \frac{1}{N_{n\ell} N_{n'\ell}} \int_{-1}^{1} dx\, C^{\ell+1}_{n'-\ell-1}(x)\, C^{\ell+1}_{n-\ell-1}(x)\, x^2 (1 - x^2)^{\ell+1/2} \tag{3.9}$$

As $x^2$ is a-reducible-tensor of second rank, the matrix elements $f^{\ell}_{nn'}$ will be different from zero for $n' = n$ , $n' = n \pm 2$ only. In order to find the explicit expression of $f^{\ell}_{nn'}$ , one has to apply the recursion formula ( GR, p. 1044):

$$(n+2) C^{\lambda}_{n+2}(x) = 2(\lambda + n + 1) C^{\lambda}_{n+1}(x) - (2\lambda + n) C^{\lambda}_{n}(x)$$

twice. As a result, we find the non vanishing matrix elements:

$$f^{\ell}_{nn} = -\frac{1}{2} \frac{\ell(\ell+1)}{(n-1)(n+1)} + \tfrac{1}{2} = \frac{(n-1)(n+1) - \ell(\ell+1)}{2(n-1)(n+1)} ,$$

?

$$f^{\ell}_{n+2,n} = \frac{1}{4(n+1)} \sqrt{\frac{(n+\ell+2)(n+\ell+1)(n-\ell+1)(n-\ell)}{n(n+2)}} , \tag{3.10}$$

$$f^{\ell}_{n-2,n} = \frac{1}{4(n-1)} \sqrt{\frac{(n+\ell)(n+\ell-1)(n-\ell-1)(n-\ell-2)}{n(n-2)}}$$

Inserting the expansions (3.4), (3.6) and the corresponding one for $F$ into eq. (2.1) ( with the substitution $p^{\circ} \to i p'^{\circ}$ ), we find the BS equation in the BS - representation:

$$[(-p^2+\tfrac{1}{4}E^2-m^2)^2 + p^2 E^2 f^{\ell}_{nn}] < n, p^2 | G(\ell m E^2) | n' q^2> -$$
$$- p^2 E^2 f^{\ell}_{n,n-2} < n-2, p^2 | G(\ell, E^2) | n', q^2>$$
$$- p^2 E^2 f^{\ell}_{n,n+2} <n+2, p^2 | G(\ell, E^2) | n', q^2>$$
$$= \delta_{nn'} \frac{\delta(p-q)}{(pq)^{3/2}} - \frac{g^2}{(2\pi)^4} \int_0^\infty p'^3 dp' K_n(p^2, p'^2, E) \times \tag{3.11}$$
$$\times < n \; p'^2 | G(\ell, E^2) | n' q^2> .$$

Formula (3.11) gives an infinite system of coupled equations for the matrix elemens $<n p^2 | G(p, E^2) | n q^2>$ We note immediately that for $E^2 \to 0$ ( which in our case means $E_\mu \to 0$ , as the direction of $E$ is kept fixed), the system is decoupled and $G$ becomes diagonal in $n$ as well. This corresponds to the fact that the little group of the Poincare group, corresponding to a null-momentum, is isomorphic to $L$ , not to $R_3$ .

4. The differential equation for the wave function and its solutions for $E = 0$

The representation (3.7) for the kernel suggests to go over to a new representation by the substitution:

$$<p^2 n | G(\ell, E^2) | q^2 n'> =$$
$$= \int_0^\infty r^3 dr \; r'^3 dr' \frac{J_n(pr)}{pr} G^{\ell}_{nn'}(r, r', E^2) \frac{J_n(qr')}{qr'} \tag{4.1}$$

In fact, this transformation converts our equations (3.11) into a system of local differential equations. With the help of the Bessel differential equation one verifies immediately that $p^2$ corresponds to the differential operator:

$$p^2 \to -[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} + \frac{1-n^2}{r^2}] \tag{4.2}$$

in the new representation. We write out the resulting differential equation for $E^2 = 0$ :

$$[\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - (\frac{n^2-1}{r} + m^2)]^2 G_{n\ell}(r, r') =$$
$$= \frac{\delta(r-r')}{(r,r')^{3/2}} + \frac{g^2}{(2\pi)^4} V(r) G(r,r') . \tag{4.3}$$

We go over to the equation for the wave function instead, which reads

$$\left[\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}-\left(\frac{n^2-1}{r^2}+m^2\right)\right]^2\psi_n(r)=\frac{g^2}{(2\pi)^4}V(r)\,\psi_n(r) \qquad (4.4)$$

This equation is by definition sufficient to investigate the energy-independent singularities of the Green functions. It is obvious from (4.4) that the solutions with the correct behaviour at infinity are those which behave as:

$$\psi(r)=\phi(r)\,e^{-mr} \qquad (4.5)$$

where $\phi(r)$ has at most a pole at infinity.

In order to study the behaviour of the solutions at the origin, we assume that the spectral density in (3.7) has the following asymptotic behaviour for $\mu\to\infty$

$$\sigma(\mu)\sim\sigma_0\,\mu^a(\log\mu)^\beta, \qquad (4.6)$$

which is the behaviour to be expected at least if $K$ is approximated by the contribution of a finite number of diagrams. Correspondingly, the "potential" $V(r)$ behaves at $r\to 0$ as follows:

$$V(r)\approx V_0\,\frac{(\log r)^\beta}{r^{4+a}}, \qquad (4.7)$$

with

$$V_0=-\sigma_0\int_0^\infty dx\;x^{2+a}\,K_1(x)$$

Introducing a new independent variable:

$$u=\log r^{-1} \qquad (4.8)$$

and keeping in (4.4) the leading terms only, we arrive at the following asymptotic equation for $\phi$:

$$\phi^{IV}(u)-4n^2\phi''(u)+\left[(n^2-1)^2-G^2e^{au}(-u)^\beta\right]\phi(u)=0 \qquad (4.9)$$
$$u\to\infty$$

where

$$G^2=\frac{g^2V_0}{(2\pi)^4}$$

The solutions of eq. (4.8) can be classified according to the signs of $a$ and $\beta$. Correspondingly, we find the following classes:

I. $a<0$, or $a=0$, $\beta<0$,

II. $a=0$, $\beta=0$,

III. $a=0$, $\beta>0$,

IV. $a>0$.

The leading term in the asymptotic expansion of can be found comparatively easily for all the classes.

Class I. The "potential" term vanishes with respect to the "centrifugal" one, so the solution for $r \to 0$ behaves as the free one.

Class II. The potential term equals in magnitude to the centrifugal one, so the solution behaves essentially as a free one, with a modified centrifugal term. (This class will be treated in detail in connection with the examples).

For the remaining two classes the potential term predominates over the centrifugal one at small distances.

Class III. Eq. (4.9) can be satisfied asymptotically by the Ansatz:

$$\phi(u) \approx \exp \gamma\, u^{\delta} . \tag{4.10}$$

Inserting into (4.9), we find:

$$\delta = 1 + \beta/4 , \tag{4.11}$$

while $\gamma$ is given by the solution of a quartic equation:

$$\gamma = \frac{\sqrt{G}}{1+\beta/4} \exp \frac{i\pi}{2} \left( N_1 + \beta . N_2 + \frac{\beta}{2} \right) \tag{4.12}$$

( $N_1$ , $N_2$ -integers).

For integral $\beta$ , one finds four different solutions. If $\beta$ is non integral, then there are four different solutions for each branch of the power $u^{\beta}$ . Going back to the variable $r$ , we find of course:

$$\phi(r) \sim \exp \gamma \left( -\log r \right)^{1+\beta/4} .$$

Class IV. We make the Ausatz

$$\phi(u) \sim u^{\rho} \exp \sigma\, e^{hu} \tag{4.13}$$

Inserting into (4.9), we find:

$$\rho = \beta , \quad h = a/4 , \tag{4.14}$$

while $\sigma$ is given once again as the solution of a quartic equation:

$$\sigma = \frac{4\sqrt{G}}{a} \exp \frac{i\pi}{2} \left( N_1 + \beta N_2 + \frac{\beta}{2} \right) \tag{4.15}$$

so in $\sigma r$ we have

$$\phi(r) \sim (\log r)^{\beta} \exp \sigma r^{-\frac{a}{4}}$$

There is an important feature of the solutions belonging to singular kernels. Unlike the nonrelativistic case, according to (4.12) and (4.15), there can be found always at least one solution such that the wave function vanishes sufficiently rapidly at the origin, irrespective of the sign of the interaction. Obviously, the condition for such a solution is $Re\ \gamma < 0$ or $Re\ \sigma < 0$, respectively. This is fulfilled, if $1 < ( N_1 + \beta/2 + N_2\ \beta ) < 3$. If $\beta/2 + N_2\ \beta$ is an integer, there is one solution, otherwise two. This means that a relativistic system does not "collapse", even for singular interactions.

The physical meaning of this phenomenon can be understood by considering a one-particle wave equation in an external field. Eg. a Klein-Gordon particle in a scalar field is described by the equation

$$[\ E^2 + \Delta + ( m - V(r)^2 )\ ]\ \phi\ ( r\ E\ ) = 0$$

Going over to the nonrelativistic limit, one can see that the effective potential contains a term proportional to $( V(r) )^2$. This term is always repulsive and more singular then $V(r)$; thus the system is prevented from collapsing by the relativistic effects.

Let us now link up our results just found with the analytic properties of the scattering amplitude in the angular momentum plane. To this end we first of all remind that with the quantum number $n$, the square of the four dimensional angular momentum in the basis chosen, can be expressed as follows:

$$\tfrac{1}{4}\ L_{\mu\nu}\ \ L^{\mu\nu}\ \ Z^{m}_{n\ell} = \tfrac{1}{2}\ ( n - 1 )\ ( n + 1 )\ \ Z^{m}_{n\ell}\ ,$$

with the restriction:

$$\ell \leq n - 1 \qquad\qquad ( \ell \text{ integral})$$

For a given, still integral, $\ell$, one can write:

$$n = \ell + 1 + n_r \qquad (4.16)$$

with

$$n_r = 0, 1, 2, \ldots$$

Formula (4.16) already shows, how we should continue our expressions in $\ell$. ( The circumstance that our wave function depends on $n$ only and not on $n$ and $\ell$ separately, is the consequence of the local character of our kernel (2.2) and is analogous to the "hydrogen-like" degeneracy in nonrelativistic quantum mechanics/11,12/. We have consciously borrowed our notation from atomic spectroscopy). Now, the scattering matrix element is given by the usual formula:

$$T_\ell = \int_0^\infty r^3 dr\ \ \psi_0\,(r)\ V(r)\ \psi\,(r) \qquad (4.17)$$

where $\psi_0\,(r)$ is the wave function in the absence of interaction. The possibility of analytic continuation of $T_\ell$ towards the left half of the $\ell$ -plane is governed by the behaviour of the integral for $r \to 0$ . Wave functions $\psi(r)$ belonging to classes I and II do not give anything essentially new. The-regular-free solutions behave as $r^{\ell+1/2}$ for $r \to 0$ , so, if the potential is less singular then the centrifugal term (class I ) the integrand in (4.17) behaves as $r^{2\ell-a}(\log r)^\beta$ for $r \to 0$ . So , $T_\ell$ can be continued analytically for $Re\ \ell > 1/2\,(a-1)$. Class II – as already mentioned will be considered separately; here we anticipate the result, namely that there appear energy independent branch cuts in the $\ell$ - plane.

Turn now to class III. As has been mentioned, there exists-at least – one solution, for which the interacting wave function vanishes at the origin, and the behaviour of $\psi(r)$ is independent of $n$ . Let $\gamma_0$ correspond to such a "good" solution, i.e.

$$Re\ \gamma_0 < 0$$

The integrand of $T_\ell$ in formula (4.17) behaves like

$$r^{\ell+7/2}\,(\log r)^\beta \exp \gamma_0 (\log 1/r)^{1+\beta/4}$$

This function vanishes for $r \to 0$ independently of the value of $\ell$ . Hence, for $E^2 = 0$ , $T_\ell$ is analytic in the whole $\ell$ - plane. One verifies immediately that the same is true for class IV. We go a step further and show that this result holds not only for $E^2 = 0$ but for any-finite energy, for which the kernel belongs to class III or IV.

In fact, for the classes III and IV one can see at once that the terms, proportional to $\phi$ , $\phi', ..., \phi^{III}$ do not contribute to the leading term of the asymptotic expansion at $r = 0$ . A glance at eq. (3.11) shows that the operators proportional to $E^2$ contribute to the coefficients of $\phi, ..., \phi^{III}$ , but not to $\phi^{IV}$ . Thus for any energy, for which either $a > 0$ or $a = 0$ , $\beta > 0$ holds, the wave function behaves at $r \to 0$ as described by eqs. ( 4.10)– ( 4.12) or ( 4.13) – ( 4.15) respectively and $T_\ell\,(E^2)$ is analytic in the whole $\ell$ - plane.

## 5. Invariant classification of Regge trajectories

Eq. (4.4) can be used to determine the "regular bound states" at $E^2 = 0$ , i.e. the points of Regge trajectories. In the foregoing section we have proven that there exists at least one solution with a correct behaviour at the origin.

Rewrite eq. (4.4) in terms of the quantum numbers $\ell$ , $n_r$ instead of $n$ . ( Following the spectroscopic terminology, we call $n_r$ the "radial quantum number" and $n$ the "principal quantum number").
Taking eq. (4.16) into account, we have:

$$\left[ \frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \left( \frac{(n_r+\ell)(n_r+\ell+2)}{r^2} + m^2 \right) \right]^2 \psi(r) =$$

$$= \lambda^2 V(r)\,\psi(r) , \qquad \text{with } \lambda^2 \equiv \frac{g^2}{(2\pi)^4} \qquad (4.4)$$

Suppose now that we have found a solution of eq. (4.4) ( or equivalently (4.4) ). Then we can write:

$$\lambda^2 = \frac{\int_0^\infty r^3 dr \ \psi_n^*(r) \left[ \frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \left( \frac{n^2-1}{r^2} + m^2 \right) \right]^2 \psi_n(r)}{\int_0^\infty r^3 dr \ \psi_n^*(r) \ V(r) \ \psi_n(r)} \tag{5.1}$$

The integral in the numerator is positive and so is $\lambda^2$. Hence we find that

$$\int_0^\infty r^3 dr \ \psi_n^*(r) V(r) \psi_n(r) > 0 \tag{5.2}$$

for a bound state at $E^2 = 0$.

Denoting the r.h.s. of eq. (5.1) by $F(n, \lambda^2)$ we find the implicit equation:

$$\lambda^2 = F(n, \lambda^2) \tag{5.3}$$

for the points of the Regge trajectories at $E^2 = 0$. Suppose, we have found the solution in the form of a power series in $\lambda^2$:

$$\psi(r) = \psi_n^0(r) + \lambda^2 \psi_n^1(r) + \lambda^4 \psi_n^2(r) + \ldots$$

Taking into account that

$$L\psi_n^0 \equiv \left[ \frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \left( \frac{n^2-1}{r^2} + m \right) \right] \psi_n^0(r) = 0$$

We find:

$$F(n, 0) = \frac{\partial F}{\partial \lambda^2} \Big|_{\lambda^2=0} = 0$$

So that actually the equation for the determination of the trajectory looks:

$$1 = \lambda^2 F_1(n, \lambda^2) \tag{5.4}$$

where already $F_1(n, \lambda^2) \neq 0$.

Explicitly :

$$F_1 = \frac{\int_0^\infty r^3 dr \ \psi^{1*}(r) \left[ \frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} - \left( \frac{n^2-1}{r^2} + m^2 \right) \right]^2 \psi^1(r)}{\int_0^\infty r^3 dr \ \psi^{0*}(r) \ V(r) \ \psi^0(r)} + O(\lambda^4)$$

( we omitted the index $n$ ) or taking into account the self-adjointness of $L$ :

$$F_1 = \frac{\int_0^\infty r^3 dr \ L\psi^{1*}(r) \ L \ \psi^1(r)}{\int_0^\infty r^3 dr \ \psi^{0*}(r) \ V(r) \ \psi^0(r)}$$

However, from

$$L^2 (\psi^0 + \lambda^2 \psi^1 + \ldots) = \lambda^2 V(r) (\psi^0 + \lambda^2 \psi^1 + \ldots),$$

$$\psi_1 = L^{-2} ( V(r) \psi^0(r) ),$$

so that

$$F_1 = \frac{\int_0^\infty r^3 dr \, \{ L^{-1} ( V(r) \, \psi^\circ (r) \, ) \}^2}{\int_0^\infty r^3 dr \;\; \psi^\circ \, (r) \;\; V(r) \;\; \psi^\circ (r)} + O(\lambda^6) \tag{5.5}$$

The Green function of the operator $L$ can be constructed by the standard procedure.

Solutions of the equation $L\,\Phi = 0$ are cylindrical functions of imaginary argument:

$$\Phi(r) = r^{-1} \, Z_n \, (imr)$$

We choose $\Phi^{(1)}(r) = r^{-1} I_n \, (mr)$, $\Phi^{(2)}(r) = r^{-1} K_n \, (mr)$, so we write the Green function:

$$L^{-1} (r, r') = \frac{(r, r')^{-1}}{\sqrt{W(r) \, W(r')}} \; [ \, I_n \, (mr') \, K_n ( m \, r' ) \; \theta \, ( r - r' ) - \tag{5.6}$$

$$- I_n \, (mr') \;\; K_n \, (mr) \;\; \theta \, ( r - r' ) \, ]$$

where $W(r)$ is the Wronskian of the solutions $\Phi^{(1)}$, $\Phi^{(2)}$. Inserting (5.6) into (5.5) one can see that $F_1$ is regular for $n > 0$; at $n = 0$ it has a simple pole: $F_1 \approx \frac{f_1}{n}$. Hence the solution of eq. (5.4) in the weak coupling approximation reads:

$$n = f_1 \, \lambda^2 + O(\lambda^4)$$

or with (4.16):

$$\ell = - 1 - n_r + f_1 \cdot \lambda^2 + O \, (\lambda^4) \, . \tag{5.7}$$

Eq. (5.7) shows that the Regge trajectories at $\lambda^2 = 0$ differ by integer numbers. Our considerations so far apply to kernels of class I and II only, since the solutions of class III and IV are singular at $\lambda^2 = 0$. Nevertheless, if $\frac{\partial F}{\partial n} \neq 0$ in eq. (5.3), we can solve for $n$, to obtain:

$$\ell = - 1 - n_r + f(\lambda^2) \tag{5.8}$$

So our results just found remains true. The essential difference between (5.7) and (5.8) is that for classes I and II in the limit $\lambda^2 \to 0$ the trajectories retreat to negative integers: $\lim_{\lambda^2 \to 0} \ell = - 1 - n_r$ just like in the case of ordinary Yukawa potentials in nonrelativistic quantum mechanics while for classes III and IV this is not necessarily true.

To show that the classification of Regge trajectories we have found for zero energy is valid in general, we carry out a perturbation calculation in $E^2$.

Introducing a symbolic notation for the Hankel transform of eq. (3.11) for the wave function ( cf. eq. (4.1) ), we write:

$$\langle n_r \mid H_0 \mid n_r \rangle \langle n_r \mid \psi(r) \rangle + \\ + s \sum_{n_r'} \langle n_r \mid h_1 \mid n_r' \rangle \langle n_r' \mid \psi(r) \rangle \tag{5.9}$$

$$+ s^2 \sum_{n'_s} < n_r \mid h_2 \mid n'_r > < n'_r \mid \psi (r) >$$

$$= \lambda^2 (s) V(r) < n_r \mid \psi (r) > \tag{5.9}$$

We have introduced the notation $E^2 = s$, otherwise the notation is self-explaining.

The "unperturbed" equation, we were investigating so far, can be written as

$$< n_r \mid H_0 \mid n_r > < n_r \mid \psi_0 (r) = $$
$$= \lambda_0^2 V(r) < n_r \mid \psi_0 (r) > \tag{5.10}$$

We look for the solution of (5.9) in the form of a power series.

$$\lambda^2 (s) = \lambda_0^2 + \frac{1}{1!} s \lambda_1^2 + \frac{1}{2!} s^2 \lambda_2^2 + \ldots$$
$$< n_r \mid \psi (r) > = < n_r \mid \psi_0 (r) > + \frac{1}{1!} s < n_r \mid \psi_1 (r) > + \tag{5.11}$$
$$+ \frac{s^2}{2!} < n_r \mid \psi_2 (r) > + \ldots$$

Inserting (5.11) into (5.9) and comparing powers of S on both sides of the equation, one finds the famƀliary expressions, e.g.:

$$\lambda_1^2 = \frac{\sum_{n_r, n'_r} \int_0^\infty r^3 dr < \psi_0 (r) \mid n_r > < n_r \mid h_1 \mid n'_r > < n'_r \mid \psi_0 (r) >}{\sum_{n_r} \int_0^\infty r^3 dr < \psi_0 (r) \mid n_r > V(r) < n_r \mid \psi_0 (r) >} \tag{5.12}$$

etc.

( All the quantities depend on $\ell$ of course).

Equating $\lambda^2 (s)$ to $(2\pi)^{-4} g^2$ one finds the implicit equation for Regge trajectories. The trajectory can again be found in the form of a power series:

$$\ell (s) = \ell_0 + \frac{1}{1!} \ell_1 s + \frac{1}{2!} \ell_2 s^2 + \ldots \tag{5.14}$$

Making repeatedly use of the theorem on the differentiation of implicit functions, one finds:

$$\frac{g^2}{(2\pi)^4} = \lambda^2 \qquad (0)$$

$$\ell_1 = - \frac{\lambda_s^2}{\lambda_\ell^2} \qquad (1) \tag{5.15}$$

$$\ell_2 = - \frac{\lambda_{ss}^2}{\lambda_\ell^2} + \frac{\lambda_s^2}{(\lambda_\ell^2)^2} \lambda_{\ell s}^2 + $$
$$+ \frac{\lambda_s^2 (\lambda_{s\ell}^2 \lambda_\ell^2 - \lambda_{\ell\ell}^2 \lambda_s)}{\ldots (\lambda_\ell^2)^3} \qquad (2)$$

where we have used the notation:

$$\lambda^2 \equiv \lambda^2 ( \ell , s ) \mid_{\ell = \ell , s=0}$$

$$\lambda^2_\ell \equiv \frac{\partial \lambda^2}{\partial \ell} \Big|_{\ell=\ell_0,\, s=0}$$

$$\lambda^2_s \equiv \frac{\partial \lambda^2}{\partial s} \Big|_{\ell=\ell_0,\, s=0} \qquad \text{etc.}$$

Comparing (5.15) with (5.11), one finds that

$$\lambda^2_s = \lambda^2_1$$

$$\lambda^2_{\ell s} = \frac{\partial \lambda^2_1}{\partial \ell} \Big|_{\ell=\ell_0}$$

and $\lambda^2_1$ , $\lambda^2_2$ ... are given by the perturbation expressions (4.12) and alike.

On noting that (5.13) (0) is identical with our previous eq. (5.3), one sees that the classification found for zero energy labels indeed the whole trajectory, if only the formal power series defines an analytic function. This is however true in virtue of a theorem on analytic perturbations/4/. Hence our statement on the classification of Regge trajectories by the radial quantum number $n_r$ is proved.

The question as to the completeness of the above classification remains still open. We have seen e.g. that if the asymptotic expression of the potential contains $(\log r)^\beta$ and we are on a branch, where $\beta(N_2 + \frac{1}{2})$ is not equal to an integer, then there are two "good" bound state solutions in classes III and IV for each $n_r$. Hence, for the complete classification of the trajectories there is at least one more quantum number necessary, e.g., the value of $N_1$ in eq. (4.12) or (4.15), respectively. In classes I and II the trajectories seem to be nondegenerate with respect to $n_r$, although we would not prove this so far.

## 6. Examples

We begin this short section by listing some simple BS kernels and stating the class to which they belong. The reader can verify for himself these statements. The results are summarized in Table I. For the simplest kernels there is a direct connection between the renormalizability of the theory and the class of the kernel: renormalizable theories give kernels of class I or II in the lowest approximation. Iteration of the lowest order kernel may shift it to another class - as shown by the last example - but how far is it justified to iterate a given kernel in one direction, without taking into account other diagrams of the same order in unclear.

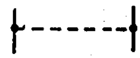
We are going now to investigate the kernel N2 of Table 1 in some more detail. The spectral function $\sigma(\mu)$ is given by

$$\sigma(\mu) = \left(\frac{\mu^2 - 4m^2}{\mu^2}\right)^{1/2},$$

it tends to unity for $\mu \to \infty$ i.e. $a = \beta = 0$. In what follows, we put $4m^2 = 1$.

The asymptotic equation, describing the BS wave function as $r \to 0,\ E^2 = 0$ can be solved exactly. In fact, by eq. (4.4), it reads:

Table 1

| NN | Description of the kernel | Diagram | Class |
|---|---|---|---|
| 1. | Scalar theory, direct coupling ladder approx |  | I |
| 2. | Scalar theory, vector meson exchange ladder approx. |  | II |
| 3. | PS pion theory, direct coupling, bubble approx. |  | III |
| 4. | PS pion theory derivative coupling, bubble approx. |  | IV |
| 5. | PS pions interacting with fermions via V-Λ coupling, bubble approx. |  | IV |
| 6. | PS pion theory, direct coupling, iterated bubble approx. |  | I |

$$\left[\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}-\left(\frac{n^2-1}{r^2}+m^2\right)\right]^2\phi(r)= \\ =\gamma^2 V_0\, r^{-4}\phi(r) \tag{6.1}$$

where

$$V_0=\int_0^\infty dx\;\; x^2K_1(x)\;,\quad \gamma^2=(2\pi)^{-4}\; g^2$$

Setting $\phi(r)=r^2 f(r)$ equation (6.1) can be written as follows:

$$\left[\frac{d^2}{dr^2}+\frac{5}{r}\frac{d}{dr}+\frac{3-n^2\pm(4n^2+\gamma^2V_0)^{1/2}}{r^2}-m^2\right]\times$$
$$\times\left[\frac{d^2}{dr^2}+\frac{5}{r}\frac{d}{dr}+\frac{3-n^2\mp(4n^2+\gamma^2V_0)^{1/2}}{r^2}-m^2\right]f(r)=0$$

The four solutions are:

$$f_1^{\pm}=I_{p_\pm}(mr)$$
$$f_2^{\pm}=K_{p_\pm}(mr)$$

where

$$p_\pm=\left[1+n^2\pm\sqrt{4n^2+\gamma^2V_0}\;\right]^{1/2}$$

The singularity in the angular momentum plane, found by several authors [1,2,3] is given by $p_{\pm} = 0$. The branch points lie at $n = \pm(1 \pm \gamma V_0^{1/2})^{1/2}$ or expressed in terms of $\ell$ :

$$\ell = -1 - n_r \pm \sqrt{1 \pm \gamma V_0^{1/2}}$$

Besides that, there are branch points at copmlex angular momentum values, given by

$$n = \pm \tfrac{1}{2} i \gamma V_0^{1/2}$$

or

$$\ell = -1 - n_r + \tfrac{1}{2} \gamma V_0^{1/2}$$

The interesting point is that with the help of our method, the branch points can be found exactly, without making a weak coupling approximation or alike.

## 7. Conclusions

We believe that the main results we have found will hold true even by relaxing the restriction imposed by eq. (2.2) on the kernel. In our opinion these are:

a) the classification of bound states into "regular" and "singular" ones ,

b) the qualitative classification of theories into classes I–IV, giving rise to qualitatively different analytic properties in angular momentum, and

c) the labeling of Regge trajectories according to the representations of the four dimensional rotation group.

As to the first point, this distinction may have a deep physical meaning. Singular bound states may correspond to what we used to call now "elementary particles", regular bound states to "composite" ones. It is of course another question, whether singular bound states really exist in nature; we do not know the answer by now.

Point b) seems to be interesting because it shows that nonrenormalizable interactions, such as weak interactions, could influence the qualitative properties of familiar strong interactions even at moderate energies. Unfortunately, we know about the behaviour at small distances of such nonrenormalizable interactions even less than about strong ones, and the authors do not feel bold enough to draw far-reaching conclusions concerning the possible role of weak interactions in manufacturing "maximal analyticity" in the angular momentum or alike, although similar speculations would be very tempting.

We want to mention only that such a possibility has been mentioned about a year ago by Predazzi and Regge [14] in the framework of nonrelativistic quantum mechanics by assuming a potential, strongly singular at the origin.

We would like to believe that the classification of regular bound states mentioned in c) is in fact a general one. Could one prove for a general kernel that the integration path in $p^0$ can be turned round to the imaginary axis at least in some interval of the total energy, this property would turn out a consequence of the assumptions of a local field theory. At any rate, to classify discrete energy levels (or, which is much the same, Regge trajectories) according to the representation of a compact group is a very attractive idea – and not new at all. The four dimensional rotation group seems to be a natural candidate in relativistic field theories, because of it connection with the Lorentz group. The interplay of analytic properties of Green functions and compact groups to produce a labeling of bound states by quantum numbers, which are not conserved for the system as a whole – would then be a beautiful property of local field theories.

As a last comment, let us mention that in our opinion the application of an $x$ - representation, although so far practical for a special class of kernels only ( eq. (2.2) ), may prove useful at least in finding out some properties of Green functions, which afterwards could be proved by more sophisticated methods.

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## APPENDIX

We list here some elementary properties of four-dimensional spherical functions used in the main text. Standard references are: the first two volumes of HTF[15], the booklet of Kratzer and Franz[16] and GR. In a canonical basis, the components of the four dimensional angular momentum $L_{\mu\nu}$ act as differential operators:

$$L_{\mu\nu} = i^{-1} ( x_\mu \partial_\nu - x_\nu \partial_\mu )$$

We are looking for the simultaneous eigenfunctions of

$$\tfrac{1}{4} L_{\mu\nu} L_{\mu\nu} \quad \text{and} \quad \underline{L}^2$$

where $\underline{L}^2$ is the three dimensional angular momentum. On separating the four dimensional Laplacean in polar coordinates:

$$\begin{aligned} x^0 &= r \cos\psi \\ x^3 &= r \sin\psi \cos\theta \\ x^1 &= r \sin\psi \sin\theta \cos\phi \\ x^2 &= r \sin\psi \sin\theta \sin\phi \end{aligned}$$

the angular part of the operator reads:

$$L^2 \Phi \equiv \{ \frac{\partial^2}{\partial\psi^2} + 2\, ctg\,\psi \frac{\partial}{\partial\psi} + \frac{1}{\sin^2\psi} [ \frac{\partial^2}{\partial\theta^2} + ctg\,\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} ] \} \Phi = -( n^2 - 1 ) \Phi \tag{A.1}$$

($n = 1,2,3 \ldots$ ; in introducing $n$ as starting from 1 instead of zero, we follow Frock[14/] ). The Ansatz

$$\Phi = Y_\ell^m(\theta, \phi)\, y(\psi)$$

where $Y_\ell^m(\theta, \phi)$ are three dimensional spherical functions, normalized to unity on the surface of the three dimensional unit sphere, transforms (A.1) into:

$$\frac{d^2 y}{d\psi^2} + 2\operatorname{ctg}\psi \frac{dy}{d\psi} - \frac{\ell(\ell+1)}{\sin^2\psi} y + (n^2 - 1)\, y = 0 \qquad (A.2)$$

Introducing the new variable: $x = \cos\psi$ and a new function by $y = (1 - x^2)^{-1/4}\, u(x)$ we get a Legendre equation:

$$(1 - x^2)\, u'' - 2xu' + [n^2 - 1/4 - \frac{(\ell + 1/2)^2}{1 - x^2}]\, u = 0 \qquad (A.3)$$

The regular solution of (A.3) is

$$u(x) = P_{n+1/2}^{-(\ell+1/2)}(x)$$

or expressed with the help of the original variable:

$$y(\psi) = (\sin\psi)^{-1/2}\, P_{n+1/2}^{-(\ell+1/2)}(\cos\psi) \qquad (A.4)$$

Another, convenient form of (A.4) is in terms of Gegenbauer functions:

$$y(\psi) = (\sin\psi)^{\ell}\, C_{n-\ell-1}^{\ell+1}(\cos\psi) \qquad (A.5)$$

The normalized functions are:

$$\begin{aligned} Z_{n\ell}^{m}(\psi\,\theta\,\phi) &= 2^{\ell+1/2} \left(\frac{n\,\Gamma(n-\ell)}{\pi\,\Gamma(n+\ell+1)}\right)^{1/2} \times \\ &\times \Gamma(\ell+1)\,(\sin\psi)^{\ell}\, C_{n-\ell-1}^{\ell+1}(\cos\psi)\, Y_\ell^m(\theta, \phi) \equiv \\ &\equiv P_{n\ell}(\psi)\, Y_\ell^m(\theta, \phi) \end{aligned} \qquad (A.6)$$

They satisfy:

$$\int_0^\pi \sin^2\psi\, d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\; Z_{n\ell}^{m}\, Z_{n'\ell'}^{m'*} = \delta_{nn'}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

i.e. the $\{Z_{n\ell}^{m}\}$ form an orthonormal system on the surface of the four dimensional unit sphere.

The system of $Z$-s is complete. The addition theorem follows immediately from those of the Gegenbauer functions and of the three dimensional spherical functions:

$$Z_{n0}^{0}(\gamma, \ldots) = \frac{2\pi}{n\sqrt{2}} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} Z_{n\ell}^{m}(\psi_1\,\theta_1\,\phi_1)\, Z_{n\ell}^{m}(\psi_2\,\theta_2\,\phi_2) \qquad (A.7)$$

if

$$\cos\gamma = \cos\psi_1 \cos\psi_2 - \sin\psi_1 \sin\psi_2 \cos\omega_1$$

$$\cos\omega = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)$$

( The dots in the argument of $Z^0_{n0}$ mean that the corresponding angles are arbitrary, as the function is independent of them )

Finally, we notice the obvious, but important property:

$$Z^m_{n\ell} = 0 \qquad \text{for} \qquad \ell > n - 1$$

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