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So far it has not been possible to find the general nature of the leading singularities in field theories (in the sense the first from the right). In the so-called two-particle approximation it is possible to prove[1] the meromorphy of the amplitude on the right half plane. Logunov et al[2] proved that the amplitude satisfies an integral equation, with an unknown kernel V, so-called potential. If this kernel is of the Fredholm type, the amplitude is a meromorphic function of ℓ on the right half-plane, otherwise we have to subtract some terms from the potential to make it belong to the Fredholm class. However, by this method some strip of the ℓ plane is excluded from the investigations of the analyticity properties of the amplitude.

Some authors[3,4] tried to attack the problem from the side of perturbation theory. Lee and Sawyer[3] proved that in the $g\phi^3$ theory in ladder approximation we have only one Regge pole on the right-hand side of the ℓ plane.

Sawyer[4] summed up the most singular parts of ladder diagrams in the $f\phi^4$ theory and in a $G\phi \frac{\partial \phi}{\partial x_\mu} \psi_\mu$ type theory (here ψ_μ is a vector boson field). He proved, that this sum of $\Sigma\, a_n \ell^{-n}$ type leads to a cut in the angular momentum plane: $A_\ell \approx \sqrt{\ell^2 - g}$ However it is not clear that the sum of less singular terms do not give a singularity which lies at higher values of $Re\, \ell$. Therefore we prefer to solve exactly the integral equation which generates the sum of ladder diagrams.

The integral equation for the ladder diagrams has the form:

$$T_\ell(s,p,\omega,p'',\omega'') = B_\ell(s,p,\omega,p'',\omega'') + \int_0^\infty dp' \int_{-\infty}^\infty d\omega'\, K_\ell(s,p,\omega,p',\omega')\, T_\ell(s,p',\omega',p'',\omega'') \tag{1}$$

or formally writing:

$$T_\ell = B_\ell + K_\ell T_\ell$$

Our abbreviations are the same as in the paper of Lee and Sawyer[3].

The kernels of integral equations (1) are the following in $g\phi^3$ (2), in vector meson exchange (3) and in $f\phi^4$ (4) theories[3,4]:

$$K_\ell^s = \frac{g^2}{i(2\pi)^3 F(p,\omega,s)}\, Q_\ell\left(\frac{p^2+p'^2+\mu^2-i\epsilon-(\omega-\omega')^2}{2pp'}\right) \tag{2}$$

$$K_\ell^v = \frac{G^2(s+2p^2+2p'^2-2\omega^2-2\omega'^2+\mu^2)}{i(2\pi)^3 F(p,\omega,s)} \times Q_\ell\left(\frac{p^2+p'^2+\mu^2-i\epsilon-(\omega-\omega')^2}{2pp'}\right) \tag{3}$$

$$K_\ell^p = \frac{f^2}{8(2\pi)^5 F(p,\omega,s)} \int_{4\mu^2}^\infty dy\, \rho(y)\, Q_\ell\left(\frac{p^2+p'^2+y-i\epsilon-(\omega-\omega')^2}{2pp'}\right) \tag{4}$$

where the weight function $\rho(y) = \sqrt{\frac{y-4\mu^2}{y}}$

and

$$F(p,\omega,s) = \left[p^2+m^2-i\epsilon-\left(\frac{\sqrt{s}}{2}-\omega\right)^2\right]\left[p^2+m^2-i\epsilon-\left(\frac{\sqrt{s}}{2}+\omega\right)^2\right].$$

After a counter-clockwise rotation of the ω integration contour we may see, that K_ℓ^S defines essentially an equation of Fredholm type, unlike kernels K_ℓ^V and K_ℓ^P . Nevertheless, the iterated kernels $(K_\ell^V)^n$, $(K_\ell^P)^n$ exist, only their traces are infinite. As we mentioned the integral, equation with kernel K_ℓ^S is solved /3/ so we shall examine only the other cases.

We define the 'asymptotic kernels' $\bar{K}_\ell^V$, $\bar{K}_\ell^P$ so that all the masses and s would be equal to zero in the expression of kernels K_ℓ^V and K_ℓ^P (3), (4).

Introducing new variables:

$$p = a \sin \alpha , \quad p' = b \sin \beta , \quad p'' = c \sin \gamma ,$$
$$\omega = a \cos \alpha , \quad \omega' = b \cos \beta \quad \omega'' = c \cos \gamma , \tag{5}$$
$$y = b^2 . z$$

and symmetrizing, we obtain new integral equation. (We write here for sake of simplicity the equations for the resolvents $\bar{\Gamma}_\ell^V(a,\alpha;b,\beta)$ and $\bar{\Gamma}_\ell^P(a,\alpha;b,\beta)$:

$$\bar{\Gamma}_\ell^V (a,\alpha;\ c,\gamma) = \bar{K}_\ell^V (a,\alpha;\ c,\gamma) + \int_0^\infty db \int_0^\pi d\beta\ b\ \bar{K}_\ell^V(a,\alpha;\ b,\beta)\ \bar{\Gamma}_\ell^V(b,\beta;\ c,\gamma) \tag{6}$$

and a similar equation for $\bar{\Gamma}_\ell^P (a,\alpha;\ b,\beta)$.

Here the kernels are defined in the following way:

$$\bar{K}_\ell^V(a,\alpha;\ b,\beta) = \frac{2G^2}{(2\pi)^3}\ \frac{1}{b^2}\left[1+\frac{1}{(\frac{a}{b})^2}\right] \times Q_\ell\left(\frac{(\frac{a}{b})^2+1-2\frac{a}{b}\cos\alpha\ \cos\beta}{2\frac{a}{b}\sin\alpha\ \sin\beta}\right), \tag{7}$$

$$\bar{K}_\ell^P (a,\alpha;\ b,\beta) = \frac{f^2}{8(2\pi)^5}\ \frac{1}{b^2}\ \frac{1}{(\frac{a}{b})^2} \times \int_0^\infty dz\ Q_\ell\left(\frac{(\frac{a}{b})^2+1+z-2\frac{a}{b}\cos\alpha\ \cos\beta}{2\frac{a}{b}\sin\alpha\ \sin\beta}\right). \tag{8}$$

From this point we shall work only with kernel (7), but all the following results are true for kernel (8) too.

Kernel $\bar{K}_\ell^V$ may be easily diagonalized in variable b by means of a Mellin transformation.

We define

$$\bar{\Gamma}_\ell (x;\ \alpha,\beta) = b^{2-x} \int^\infty da\ a^{x-1}\ \bar{\Gamma}_\ell^V (a,\alpha;\ b,\beta)$$

and

$$\bar{K}_\ell (x;\ \alpha,\beta) = \frac{2G^2}{(2\pi)^3} \int_0^\infty dz\ .\ z^{x-1}\left[1+\frac{1}{z^2}\right] \times Q_\ell\left(\frac{z^2+1-2z\ \cos\alpha\cos\beta}{2z\sin\alpha\ \sin\beta}\right) = \frac{2G^2}{(2\pi)^3}\int_1^\infty dz\,(z^{x-1}+z^{1-x}) \times \tag{9}$$

$$\times [1 + \frac{1}{z^2}]\ \mathcal{Q}_\ell \left(\frac{z^2+1-2z\ \cos a \cos \beta}{2z \sin a \sin \beta} \right). \qquad (9)$$

The resolvent $\bar{\Gamma}_\ell(x; a, \beta)$ satisfies the integral equation of Fredholm type:

$$\bar{\Gamma}_\ell(x; a, \gamma) = \bar{K}_\ell(x; a, \gamma) + \int_0^\pi d\beta\ \bar{K}_\ell(x; a, \beta)\,\bar{\Gamma}_\ell(x; \beta, \gamma). \qquad (10)$$

The definition (9) shows that $\bar{K}_\ell(x; a, \beta)$ exists if

$$1 - \mathrm{Re}\,\ell < \mathrm{Re}\,x < 1 + \mathrm{Re}\,\ell\ . \qquad (11)$$

On the real axis (11) the positive kernel $\bar{K}_\ell(x; a, \beta)$ and the iterated kernels $\bar{K}^n_\ell(x; a, \beta)$ satisfy the following conditions:

$$\mathrm{sgn}\left\{ \frac{\partial \bar{K}^n_\ell(x; a, a\gamma)}{\partial x} \right\} = \mathrm{sgn}\{x - 1\},$$

$$\frac{\partial^2 \bar{K}^n_\ell(x; a, \gamma)}{\partial x^2} > 0, \qquad (12)$$

$$\frac{\partial \bar{K}^n_\ell(x; a, \gamma)}{\partial \ell} < 0,$$

$$\bar{K}^n_\ell(x; a, \beta) \to 0 \quad \text{if} \quad \ell \to \infty,$$

$$\bar{K}^n_\ell(x; a, \beta) \to \infty \quad \text{if} \quad \begin{array}{ll} \ell \to x - 1, & \text{for } x > 1, \\ \ell \to 1 - x, & \text{for } x < 1, \end{array}$$

$$n = 1, 2, \ldots\ .$$

The conditions (12) define the behaviour of the Fredholm determinant of equation (10), which may be seen on figure 1. The equation

$$\mathrm{Det}\,(1 - \bar{K}_\ell(x; a, \beta)) = 0 \qquad (13)$$

is satisfied at two points

$$x = 1 \pm x_1(\ell).$$

The iterative solution exists in the interval

$$1 - x_1(\ell) < \mathrm{Re}\,x < 1 + x_1(\ell). \qquad (14)$$

The resolvent $\bar{\Gamma}^v_\ell(a, a; b, \beta)$ is given by the following integral:

$$\bar{\Gamma}^v_\ell(a, a; b, \beta) = \frac{1}{b^2} \int_{-i\infty + x_0}^{i\infty + x_0} dx \left(\frac{a}{b} \right)^{-x} \frac{D_\ell(x; a, \beta)}{\mathrm{Det}\,(1 - \bar{K}_\ell(x; a, \beta))} \qquad (15)$$

Here x_0 satisfies inequality (14), D_ℓ is the Fredholm numerator of equation (10).

The leading singularity of $\bar{\Gamma}_\ell^v(a, a; b, \beta)$ determined by the point when the two zeros of $Det(1 - \bar{K}_\ell(x; a, \beta))$ pinch the integration countour in integral (15). This ℓ_0 value is defined by equation (13) if we put $x = 1$ (ℓ_0 depend only on G!). The behaviour of $x_1(\ell)$ near $\ell = \ell_0$ may be obtained by developing $Det(1 - \bar{K}_\ell(x; a, \beta))$ into Taylor's series around the point $\ell = \ell_0$, $x = 1$:

$$x_1(\ell) \approx k\sqrt{\ell - \ell_0}, \tag{16}$$

where $k > 0$, $\ell_0 > 0$.

From here we obtain the behaviour of $\bar{\Gamma}_\ell^v(a, a; b, \beta)$ near $\ell = \ell_0$:

$$\bar{\Gamma}_\ell^v(a, a; b, \beta) \approx \frac{1}{b^2}\left(\frac{a}{b}\right)^{-1} \frac{D_\ell(1, a, \beta)}{\frac{\partial}{\partial x}(Det(1 - \bar{K}_\ell(x; a, \beta)))\,|_{x = 1 + x_1(\ell)}} \approx$$

$$\approx \frac{1}{b^2}\left(\frac{a}{b}\right)^{-1} \frac{D_{\ell_0}(1, a, \beta)}{k\, x_1(\ell)} \approx \frac{1}{b^2}\left(\frac{a}{b}\right)^{-1} \frac{D_{\ell_0}(1, a, \beta)}{k\sqrt{\ell - \ell_0}}. \tag{17}$$

It is simple to prove that the solution of the original integral equation (1) has the same leading singulatity (17) as the solution of equation (6).

Introducing variables $a, b, c, \alpha, \beta, \gamma$ (5) we diagonalize equation (1) by some integral transformation . The generating function of S satisfies the following homogeneous integral equation:

$$S_\ell^{-1}(s, x, b, a, \beta) \cdot \int_0^\infty da\; S_\ell(s, x, a, \alpha, \beta) \times$$

$$\times K_\ell(s, a, \alpha, b, \beta) = k_\ell(s, x, \alpha, \beta). \tag{18}$$

The kernel $K_\ell(s, a, \alpha; b, \beta)$ has a continous spectrum[5] with eigenvalues $k_\ell(s, x, \alpha, \beta)$ In the limiting case $b \to \infty$, $a \to \infty$ the kernel $K_\ell(s, a, \alpha; b, \beta) \to \bar{K}_\ell(a, \alpha; b, \beta)$; so asymptotically the Mellin transformation diagonalizes equation (1). Having $k_\ell(s, x, \alpha, \beta)$ independent of b, $k_\ell(s, x, \alpha, \beta) = K_\ell(x, \alpha, \beta)$. From here we may easily obtain the wanted results. Still we remark that the singularity described by formula (17) leads to an asymptotic behaviour in the crossed channel:

$$A(t, s) \approx f(s)\,\frac{t^{\ell_0}}{\sqrt{\log t}}, \tag{19}$$

similar to that recommanded by Gribov[6] earlier.

There remains an interesting question: what is the essential difference between equations with kernels K_ℓ^S and K_ℓ^V or K_ℓ^P ?

To explain the situation we assume that we have an integral equation for the partial wave amplitude:

$$A_\ell(\lambda, \dots) = A_\ell^0(\lambda, \dots) + \int_0^\infty d\lambda'\, K_\ell(\lambda', \lambda, \dots)\, A_\ell(\lambda', \dots) \tag{20}$$

where λ is some quantity with the dimensions of momentum or momentum squared, K_ℓ contains some power of the coupling constant. We are interested in the asymptotic behaviour of $K_\ell(\lambda', \lambda, ...)$. We define the asymptotic kernel $\overline{K}_\ell(\lambda', \lambda)$ in the same way as we did in formulas (7) and (8). We define $\overline{K}^1_\ell(\lambda', \lambda)$ so that $\frac{1}{\lambda'}\overline{K}^1_\ell(\lambda', \lambda) = \overline{K}_\ell(\lambda', \lambda)$ where $\overline{K}^1_\ell(\lambda', \lambda)$ is dimensionless. If the coupling constant is dimensionless $\overline{K}^1_\ell(\lambda', \lambda)$ may depend only on $\frac{\lambda}{\lambda'}$. It results that our kernel will not be of the Fredholm type, but it may be diagonalized by means of a Mellin transformation. At the inverse transformation there appear the same type of leading singularities as in eq.(17).

On the other side, it the coupling constant has the dimensions of length on some negative power, then our kernel may be written like $\overline{K}_\ell(\lambda', \lambda) = \frac{1}{\lambda'^k}\overline{K}^1_\ell(\lambda', \lambda)$ where $k > 1$ It results equation (20) will be of the Fredholm type, the leading singularity being a Regge pole.

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5. I am grateful to Dr. G.Domokos for calling my attention to his proof, that if the kernel has a discrete spectrum we obtain Regge poles, if it has a continous one there appear cuts as leading singularities (G.Domokos: Theses). The possible discreet spectrum of the kernel $K_\ell(s, a, \alpha, b, \beta)$ may lead to poles in the ℓ plane, but they are on the left from ℓ_0 for s values near to $s = 0$, because $K_\ell(0, a, \alpha, b, \beta) < \overline{K}_\ell(a, \alpha, b, \beta)$ and so $T_\ell(0, a, \alpha, b, \beta) < \overline{T}_\ell(a, \alpha, b, \beta)$.
6. V.N.Gribov. Nucl. Phys., 22, 249 (1961). If $S^{-1}_\ell(s, 1, b, a, \beta) = 0$ the type of the singularity in ℓ may be $\approx\sqrt{\ell - \ell_0}$, which results an asymptotics in the crossed channel $A \approx f(s)\frac{t^{\ell_0}}{(\log t)^{3/2}}$

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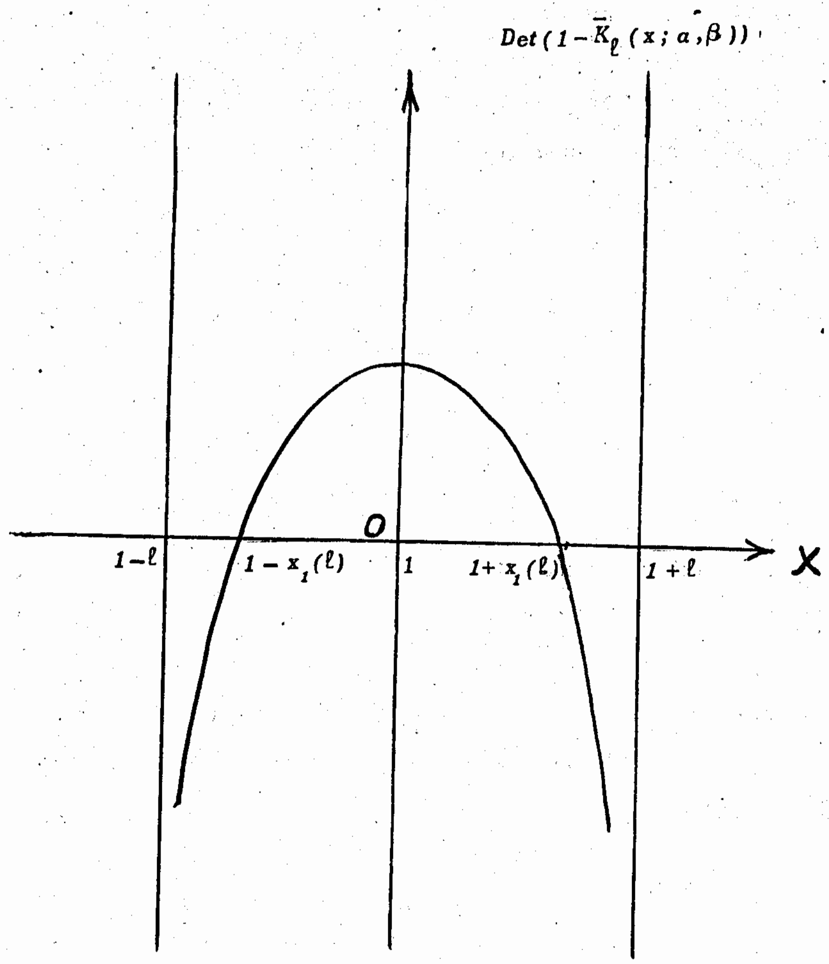

Fig. 1. The behaviour of $Det(1-\bar{K}_\ell(x;a,\beta))$ as the function of x at a given $\ell > \ell_0$ value.