ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

A.A.Logunov, Nguyen Van Hieu, O.A.Khrustalev

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QUASIOPTICAL METHOD AND ASYMPTOTIC BEHAVIOUR OF MANY CHANNEL AMPLITUDES

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1. Introduction

In previous papers[1-5] a relativistic equation of the Schrödinger type was obtained for a system of two identical particles, and on its basis the properties of the scattering amplitude and its asymptotic behaviour were investigated. In this paper we deduce equations for the systems of two different scalar particles. A procedure for deducing equations and the scattering amplitudes is examplified by the reactions (in the annihilation channel)

$$
\begin{aligned}
&a+\bar{a}\to a+\bar{a}\,,\\
&a+\bar{a}\leftrightarrows b+\bar{b}\,,\\
&b+\bar{b}\to b+\bar{b}\,.
\end{aligned}
\qquad (1)
$$

Other channels are taken into account by introducing the complex potentials. The equations obtained allow to investigate the asymptotic behaviour of the scattering amplitudes for the reactions

$$
\begin{aligned}
&a+a\to a+a\,,\\
&a+b\to a+b\,,\\
&b+b\to b+b
\end{aligned}
$$

in the s- channel. In the case of potentials without subtraction the scattering amplitudes of all these reactions are of the Regge type with one and the same trajectory $a(s)$. Equations for determining $a(s)$ are the analytical continuation in ℓ of the equations determining the energy spectrum of the system. As a result the well-known relationships are obtained both between the differential elastic scattering cross sections and the total cross sections. The subtraction terms in the potentials may lead to the non-Regge-type behaviour of the amplitudes. It is shown, however, that the relationships between the total cross sections for the processes under consideration remain valid.

2. Two-time Green Functions and Lippman-Schwinger Equation

Consider the Green functions for the systems of particles in reactions (1)

$$
\begin{aligned}
&G_{aa}\,(1,2;1',2')=\langle 0|\,T\{\,\phi_a(1)\;\phi_{\bar a}(2)\;\bar\phi_a(1')\;\bar\phi_{\bar a}(2')\,\}\,|0\rangle\,,\\
&G_{ba}\,(1,2;1',2')=\langle 0|\,T\{\,\phi_b(1)\;\phi_{\bar b}(2)\;\bar\phi_a(1')\;\bar\phi_{\bar a}(2')\,\}\,|0\rangle\,,\\
&G_{ab}\,(1,2;1',2')=\langle 0|\,T\{\,\phi_a(1)\;\phi_{\bar a}(2)\;\bar\phi_b(1')\;\bar\phi_{\bar b}(2')\,\}\,|0\rangle\,,\\
&G_{bb}\,(1,2;1',2')=\langle 0|\,T\{\,\phi_b(1)\;\phi_{\bar b}(2)\;\bar\phi_b(1')\;\bar\phi_{\bar b}(2')\,\}\,|0\rangle\,,
\end{aligned}
\qquad (2)
$$

where $\phi_a, \phi_b, \phi_{\bar a}, \phi_{\bar b}$ are the annihilation operators of particles a, b and antiparticles $\bar a$, $\bar b$, respectively, $\phi_{\bar a}=\bar\phi_a$, $\phi_{\bar b}=\bar\phi_b$.

These functions satisfy the equations written down in a symbolic form[6,7]:

$$G_{aa} = G^{\circ}_{aa} + G^{\circ}_{aa} K_{aa} G_{aa} + G^{\circ}_{aa} K_{ab} G_{ba},$$

$$G_{ba} = G^{\circ}_{bb} K_{ba} G_{aa} + G^{\circ}_{bb} K_{bb} G_{ba},$$

$$G_{ab} = G^{\circ}_{aa} K_{aa} G_{ab} + G^{\circ}_{aa} K_{ab} G_{bb},$$

$$G_{bb} = G^{\circ}_{bb} + G^{\circ}_{bb} K_{ba} G_{ab} + G^{\circ}_{bb} K_{bb} G_{bb}. \qquad (3)$$

Here G°_{aa} and G°_{bb} are the Green functions of free particles. The kernels K_{ij}, $i,j = a,b$ are real, if only the two-particle intermediate states of the systems ($a\bar{a}$) and ($b\bar{b}$) i.e. only channels (1) are taken into account.

In the opposite case these kernels are complex.

If we rule out, for instance, G_{ba} from the first two equations (2), then we obtain an equation for G_{aa} and, hence, we get an equation for the amplitude of the first elastic scattering process (1). The kernel of this equation is complex even when the kernels K_{ij} in system (3) are real. This equation has been investigated in /1-5/.

By introducing the matrices:

$$G = \begin{pmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{pmatrix}, \qquad G^{\circ} = \begin{pmatrix} G^{\circ}_{aa} & 0 \\ 0 & G^{\circ}_{bb} \end{pmatrix}, \qquad K = \begin{pmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{pmatrix} \qquad (4)$$

we are able to write system (3) as one matrix equation

$$G = G^{\circ} + G^{\circ} K G, \qquad (5)$$

whose solution can be found by iterations

$$G = G^{\circ} + G^{\circ} K G^{\circ} + \dots \qquad (6)$$

Following the procedure of /1-5/ we introduce the two-time (matrix) function $\approx\atop G$. It satisfies an equation of the form:

$$\Sigma \overset{\approx}{G} = 1 \qquad (7)$$

where

$$\Sigma = [\overset{\approx}{G^{\circ}}]^{-1} - [\overset{\approx}{G^{\circ}}]^{-1} \widetilde{\widetilde{G^{\circ} K G^{\circ}}} [\overset{\approx}{G^{\circ}}]^{-1} - \dots \qquad (8)$$

Up to a normalization factor we have:

$$[\overset{\approx}{G^{\circ}}]^{-1} = F(p^2, E^2) = \begin{pmatrix} \mathcal{F}_a(p^2, E^2) & 0 \\ 0 & \mathcal{F}_b(p^2, E^2) \end{pmatrix}, \qquad (9)$$

$$\mathcal{F}_{a,b} = (p^2 + m_{a,b}^2 - E^2) \sqrt{p^2 + m_{a,b}^2} \ , \tag{10}$$

where m_a and m_b are the masses of particles a and b, respectively. Therefore, Eq. (7) reads in an explicit form:

$$F(p'^2, E^2) \approx G(\vec{p}\,', \vec{p}, E) - \frac{1}{(2\pi)^3} \int d^3q \ V(\vec{p}\,', \vec{q}, E) \approx G(\vec{q}, \vec{p}, E) = \delta(\vec{p} - \vec{p}\,') \ , \tag{11}$$

where $V(\vec{p}\,', \vec{p}, E)$ is the potential matrix

$$V(\vec{p}\,', \vec{p}, E) = \begin{pmatrix} V_{aa}(\vec{p}\,', \vec{p}, E), & V_{ab}(\vec{p}\,', \vec{p}, E) \\ V_{ba}(\vec{p}\,', \vec{p}, E), & V_{bb}(\vec{p}\,', \vec{p}, E) \end{pmatrix} . \tag{12}$$

For the wave function of the systems

$$\Psi = \begin{pmatrix} \Psi_{a\bar{a}} \\ \Psi_{b\bar{b}} \end{pmatrix} \tag{13}$$

we have the corresponding homogeneous equation

$$F(p^2, E^2)\, \Psi(\vec{p}, E) - \frac{1}{(2\pi)^3} \int d^3q \ V(\vec{p}, \vec{q}, E) \ \Psi(\vec{q}, E) = 0 \ . \tag{14}$$

From this equation it is possible to get, with the aid of the well-known method [1], the Lippman-Schwinger equation for the transition amplitudes

$$T(\vec{p}\,', \vec{p}, E) = V(\vec{p}\,', \vec{p}, E) + \frac{1}{(2\pi)^3} \int d^3q \ V(\vec{p}\,', \vec{q}, E) \frac{1}{F(q^2, (E+i\epsilon)^2)} T(\vec{q}, \vec{p}, E) \tag{15}$$

$$T(\vec{p}\,', \vec{p}, E) = \begin{pmatrix} \mathcal{G}_{aa}(\vec{p}\,', \vec{p}, E) & \mathcal{G}_{ab}(\vec{p}\,', \vec{p}, E) \\ \mathcal{G}_{ba}(\vec{p}\,', \vec{p}, E) & \mathcal{G}_{bb}(\vec{p}\,', \vec{p}, E) \end{pmatrix} \tag{16}$$

On the mass shell the amplitudes $\mathcal{G}_{ij}(\vec{p}\,', \vec{p}, E)$ coincide with the scattering amplitudes: $\mathfrak{M}_{ij}(\vec{p}\,', \vec{p}, E)$

$$\mathcal{G}_{ij}(\vec{p}\,', \vec{p}, E) = \mathfrak{M}_{ij}(\vec{p}\,', \vec{p}, E) \ , \qquad p'^2 + m_i^2 = p^2 + m_j^2 = E^2 .$$

3. Local Potentials .

In this section we shall deduce Lippman-Schwinger equation with local potentials (according to the nomenclature, employed in [4]). We introduce the notations: p_1, p_2 and p_1', p_2' are the 4-momenta of particles in the initial and final states, respectively, θ is the angle between $\vec{p}_1$ and $\vec{p}_1'$ in the c.m.s. ,

$$s = -(p_1+p_2)^2, \quad t = -(p_1'-p_1)^2, \quad \bar{t} = -(p_2'-p_1)^2.$$

As here we consider all the processes in the annihilation channel, t and $\bar{t}$ are the momentum transfers, and s is the energy squared. The substitution $t \leftrightarrows \bar{t}$ corresponds to the change in the sign of $\cos\theta$. In other channels s is the momentum transfer, and t or $\bar{t}$ is the energy squared. It is possible to prove, by resorting to perturbation theory, that for

$$-4\mu^2 < s < 4\mu^2$$

the scattering amplitudes have a spectral representation*

$$\mathfrak{M}_{ij}(t, \bar{t}, s) = \frac{1}{\pi}\int_{\mu^2}^{\infty} dt' \, \frac{\rho_{ij}(t', s)}{t' - t} + \frac{1}{\pi}\int_{\mu^2}^{\infty} dt' \, \frac{\sigma_{ij}(t', s)}{t' - \bar{t}} \, ,$$

or in a matrix form:

$$M(t, \bar{t}, s) = \frac{1}{\pi}\int_{\mu^2}^{\infty} dt' \, \frac{\rho(t', s)}{t' - t} + \frac{1}{\pi}\int_{\mu^2}^{\infty} dt' \, \frac{\sigma(t', s)}{t' - \bar{t}} \, . \qquad (17)$$

By analogy with the one-channel amplitudes treated in /1-5/ we introduce the functions

$$M^{\pm}(t, s) = \frac{1}{\pi}\int_{\mu^2}^{\infty} dt' \, \frac{\rho(t', s) \pm \sigma(t', s)}{t' - t} \, . \qquad (18)$$

The total amplitude is related to them by

$$M(t, \bar{t}, s) = \tfrac{1}{2}\{ M^{+}(t, s) + M^{+}(\bar{t}, s) + M^{-}(t, s) - M^{-}(\bar{t}, s) \} \, . \qquad (19)$$

Now we define the local potentials $V^{\pm}((\vec{p}\,'-\vec{p})^2, E)$ as potentials of the Lippman-Schwinger equation for the amplitudes $T^{\pm}(\vec{p}\,', \vec{p}, E)$ coinciding with amplitudes (18) $M^{\pm}((\vec{p}\,'-\vec{p})^2, E^2)$ on the mass shell

$$T^{\pm}(\vec{p}\,', \vec{p}, E) = V^{\pm}(\vec{p}\,', \vec{p}, E) + \frac{1}{(2\pi)^3}\int d^3q \; V^{\pm}(\vec{p}\,', \vec{q}, E) \frac{1}{F(q^2, (E+i\epsilon)^2)} \; T(\vec{q}, \vec{p}, E) \qquad (20)$$

or in a symbolic form

$$T^{\pm} = V^{\pm} + V^{\pm} \times \frac{1}{F} \times T^{\pm}. \qquad (21)$$

If we expand the amplitudes and potentials in series in the coupling constant

$$T^{\pm} = \sum_n T^{(n)\pm}, \qquad V^{\pm} = \sum_n V^{(n)\pm},$$

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* Here μ^2 is the least (by the absolute magnitude) limit for all integrations. If the limit of a certain integral lies above μ^2 , then the spectral function is multiplied by the corresponding θ function.

then from (21) the following equation is obtained

$$T^{(n)\pm} - V^{(n)\pm} = \sum_{m=1}^{n-1} V^{(m)\pm} \times \frac{1}{F} \times T^{(n-m)\pm} \quad . \tag{22}$$

This equation leads to the fact that if the potentials in some region of energy E have a spectral representation of the form:

$$V^{\pm}((\vec{p}\,' - \vec{p}\,)^2, E) = \frac{1}{\pi} \int_{\mu^2}^{\infty} dt_0 \frac{v^{\pm}(t_0, E)}{t_0 + (\vec{p}\,' - \vec{p}\,)^2} , \tag{23}$$

then the amplitudes in Eq. (20) in this region have also a similar form

$$T^{\pm}(\vec{p}\,', \vec{p}, E) = \frac{1}{\pi} \int_{\mu^2}^{\infty} dt_0 \frac{\tau^{\pm}(t, p'^2, p^2, E)}{t_0 + (\vec{p}\,' - \vec{p}\,)^2} . \tag{24}$$

Analogously in the region where the scattering amplitudes may be presented in a form of (24), the potentials of Eq. (20) have the form of (23). In proving these assertions we follow the arguments of /3/ *. However, it should be noted that the amplitudes $T^{\pm}$ and the potentials $V^{\pm}$ are the matrices 2x2 and in all the equations the multiplication is being done according to the rules of matrix multiplication.

4. Asymptotic Behaviour of the Scattering Amplitude

Now we investigate the asymptotic behaviour of the solutions of Eq. (20) with the potentials of the form of (23). These solutions have the form of (24), and we shall consider them on the mass shell of one variable, viz., for $p^2 = E^2 - m_j^2 = s_0^j$, $j = a, b$. We put $p'^2 = s$. When $s = E^2 - m_i^2$, $i = a, b$ we have the scattering amplitudes.

Note that both the potentials $V^{\pm}((\vec{p}\,' - \vec{p}\,)^2, E)$ and the solutions $T^{\pm}(\vec{p}\,', \vec{p}, E)$ of Eq. (2) depend on E as a parameter. For the sake of simplicity in the following we suppress both this parameter and the signs $\pm$.

In an explicit form Eq. (20) means a system of equations for the amplitudes $\mathcal{G}_{aa}$, $\mathcal{G}_{ab}$, $\mathcal{G}_{ba}$, $\mathcal{G}_{bb}$ of different processes (1), viz.,

$$\mathcal{G}_{aa}(\vec{p}\,', \vec{p}) = V_{aa}((\vec{p}\,' - \vec{p}\,)^2) + \frac{1}{(2\pi)^3} \int d^3q \, V_{aa}((\vec{p}\,' - \vec{q}\,)^2) \frac{1}{\mathcal{F}(q^2)} \mathcal{G}_{aa}(\vec{q}, \vec{p}) \tag{25}$$

$$+ \frac{1}{(2\pi)^3} \int d^3q \, V_{ab}((\vec{p}\,' - \vec{q}\,)^2) \frac{1}{\mathcal{F}_b(q^2)} \mathcal{G}_{ba}(\vec{q}. \vec{p})$$

or in a symbolic form

$$\mathcal{G}_{aa} = V_{aa} + V_{aa} \times \frac{1}{\mathcal{F}_a} \times \mathcal{G}_{aa} + V_{ab} \times \frac{1}{\mathcal{F}_b} \times \mathcal{G}_{ba} , \tag{26}$$

$$\mathcal{G}_{ba} = V_{ba} + V_{ba} \times \frac{1}{\mathcal{F}_a} \times \mathcal{G}_{aa} + V_{bb} \times \frac{1}{\mathcal{F}_b} \times \mathcal{G}_{ba} , \tag{27}$$

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* Here we do not concern ourselves with the problem as to whether the boundaries of vanishing the spectral functions of the scattering amplitudes and potentials are consistent. To tell the truth, this problem is not so important, for in what follows it is supposed to investigate the behaviour of the amplitude starting from the given properties of the potential.

$$\mathcal{G}_{ab} = V_{ab} + V_{aa} \times \frac{1}{\mathcal{F}_a} \times \mathcal{G}_{ab} + V_{ab} \times \frac{1}{\mathcal{F}_b} \times \mathcal{G}_{bb}, \tag{28}$$

$$\mathcal{G}_{bb} = V_{bb} + V_{ba} \times \frac{1}{\mathcal{F}_a} \times \mathcal{G}_{ab} + V_{bb} \times \frac{1}{\mathcal{F}_b} \times \mathcal{G}_{bb} \tag{29}$$

If account is taken of only the two-particle intermediate states of the systems ($a\bar{a}$) and ($b\bar{b}$), i.e. only of channels (1) then the potentials V_{ij} , $i, j = a, b$ are real. In the opposite case they are complex. Note that by using iterations we can rule the non-diagonal amplitudes $\mathcal{G}_{ab}$ and $\mathcal{G}_{ba}$ out in Eqs. (26) - (29) and get independent equations for $\mathcal{G}_{aa}$ and $\mathcal{G}_{bb}$. These latter equations will be alike Lippman-Schwinger equation with the complex potential even for real V_{ij} . This approach enables us to investigate the asymptotic behaviour of the amplitude for each scattering process separately. However, in order to find the relationship between the asymptotic behaviour of the processes in question, it is necessary to investigate their amplitudes as a solution of a system of equations, viz. a system of Eq. (26)-(29).

We represent the potentials and the amplitudes as (23) and (24).

$$V_{ij}\ ((\vec{p}\,' - \vec{p})^2) = \frac{1}{\pi} \int dt_0 \frac{v_{ij}(t_0)}{t_0 + (\vec{p}\,' - \vec{p})^2}, \tag{30}$$

$$\mathcal{G}_{ij}\ (\vec{p}\,', \vec{p}) = \frac{1}{\pi} \int dt_0 \frac{\tau_{ij}(s, t_0)}{t_0 + (\vec{p}\,' - \vec{p})^2}, \quad i, j = a, b. \tag{31}$$

Substituting these expressions into Eqs. (26)-(29) we get a system of equations for the imaginary parts

$$\tau_{aa}(s,t) = v_{aa}(t) + \int ds'\,dt' \frac{Q_{aa}(s,t;s',t')\tau_{aa}(s',t')}{s' - s_0^a - i\epsilon} + \int ds'\,dt' \frac{Q_{ab}(s,t;s',t')\tau_{ba}(s',t')}{s' - s_0^b - i\epsilon}, \tag{32}$$

$$\tau_{ba}(s,t) = v_{ba}(t) + \int ds'\,dt' \frac{Q_{ba}(s,t;s',t')\tau_{aa}(s',t')}{s' - s_0^a - i\epsilon} + \int ds'\,dt' \frac{Q_{bb}(s,t;s',t')\tau_{bb}(s',t')}{s' - s_0^b - i\epsilon}, \tag{33}$$

$$\tau_{ab}(s,t) = v_{ab}(t) + \int ds'\,dt \frac{Q_{aa}(s,t;s',t')\tau_{ab}(s',t')}{s' - s_0^a - i\epsilon} + \int ds'\,dt' \frac{Q_{ab}(s,t;s',t')\tau_{bb}(s',t')}{s' - s_0^b - i\epsilon}, \tag{34}$$

$$\tau_{bb}(s,t) = v_{bb}(t) + \int ds'\,dt' \frac{Q_{ba}(s,t;s't')\tau_{ab}(s',t')}{s' - s_0^a - i\epsilon} + \int ds'\,dt' \frac{Q_{bb}(s,t;s',t')\tau_{bb}(s',t')}{s - s_0^b - i\epsilon}. \tag{35}$$

Here

$$Q_{ij}(s,t;s',t') = \frac{1}{(2\pi)^3} \int dt_0\, v_{ij}(t_0) \frac{K(t, t', t_0, s, s', s_0)}{\sqrt{s' + m_j^2}}, \tag{36}$$

$$K(t, t', t_0, s, s', s_0^j) = \frac{\theta(\sqrt{t} - \sqrt{t'} - \sqrt{t_0})\,\theta(\Delta)}{\sqrt{\Delta}}, \tag{37}$$

and Δ is the well-known determinant /8/ .

For $t \to \infty$ the system of Eqs. (32)-(35) has an asymptotic form

$$\tau_{aa}(s,t) = \int ds' d\frac{t'}{t} \frac{P_{aa}(s, s', \frac{t'}{t})\, \tau_{aa}(s',t')}{s - s_0^a - i\epsilon} + \int ds' d\frac{t'}{t} \frac{P_{ab}(s,s', \frac{t'}{t})\, \tau_{ba}(s',t')}{s - s_0^b - i\epsilon}, \tag{38}$$

$$\tau_{ba}(s,t) = \int ds' d\frac{t'}{t} \frac{P_{ba}(s,s', \frac{t'}{t})\, \tau_{aa}(s',t')}{s - s_0^a - i\epsilon} + \int ds' d\frac{t'}{t} \frac{P_{bb}(s,s', \frac{t'}{t})\, \tau_{bb}(s', t4)}{s - s_0^b - i\epsilon}, \tag{39}$$

$$\tau_{ab}(s,t) = \int ds' d\frac{t'}{t} \frac{P_{aa}(s,s', \frac{t'}{t})\, \tau_{ab}(s',t')}{s - s_0^a - i\epsilon} + \int ds' d\frac{t'}{t} \frac{P_{ab}(s,s', \frac{t'}{t})\, \tau_{bb}(s',t')}{s - s_0^b - i\epsilon}, \tag{40}$$

$$\tau_{bb}(s,t) = \int ds' d\frac{t'}{t} \frac{P_{ba}(s,s', \frac{t'}{t})\, \tau_{ab}(s, t')}{s - s_0^a - i\epsilon} + \int ds' d\frac{t'}{t} \frac{P_{bb}(s,s', \frac{t'}{t})\, \tau_{bb}(s', t')}{s - s_0^b - i\epsilon}, \tag{41}$$

where

$$P_{ij}(s,s',x) = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{s' + m_j^2}} \int dt_0\, v_{ij}(t_0) \frac{\theta(s' - sx - t\frac{x}{1-x})}{(1-x)^{1/2}[s' - sx - t_0\frac{x}{1-x}]^{1/2}} \tag{42}$$

We emphasize that the system of two Eqs. (38), (39) coincides exactly with that of Eqs. (40), (41). Let us consider the first system. It follows from the invariance of this system of homogeneous equations with respect to the substitution $t \to nt$, $t' \to nt'$ that the system admits a solution of the form:

$$\begin{aligned} \tau_{aa}(s,t) &= f_{aa}^{(a)}(s)\, t^a, \\ \tau_{ba}(s,t) &= f_{ba}^{(a)}(s)\, t^a. \end{aligned} \tag{43}$$

Similarly the solution of the second system has the form

$$\begin{aligned} \tau_{ab}(s,t) &= f_{ab}^{(a)}(s)\, t^a, \\ \tau_{bb}(s,t) &= f_{bb}^{(a)}(s)\, t^a. \end{aligned} \tag{44}$$

Substituting (43) and (44) into (39)-(41) we get a new system of equations

$$f_{aa}^{(a)}(s) = \int ds' \frac{R_{aa}^{(a)}(s,s')\, f_{aa}^{(a)}(s')}{s' - s_0^a - i\epsilon} + \int ds' \frac{R_{ab}^{(a)}(s,s')\, f_{ba}^{(a)}(s')}{s' - s_0^b - i\epsilon}, \tag{45}$$

$$f_{ba}^{(a)}(s) = \int ds' \frac{R_{ba}^{(a)}(s,s')\, f_{aa}^{(a)}(s')}{s' - s_0^a - i\epsilon} + \int ds' \frac{R_{bb}^{(a)}(s,s')\, f_{ba}^{(a)}(s')}{s' - s_0^b - i\epsilon}, \tag{46}$$

$$f_{ab}^{(a)}(s) = \int ds' \frac{R_{aa}^{(a)}(s,s')\, f_{ab}^{(a)}(s')}{s' - s_0^a - i\epsilon} + \int ds' \frac{R_{ab}^{(a)}(s,s')\, f_{bb}^{(a)}(s')}{s' - s_0^b - i\epsilon}, \tag{47}$$

$$f_{bb}^{(a)}(s) = \int ds' \frac{R_{ba}^{(a)}(s,s')\, f_{ab}^{(a)}(s')}{s' - s_0^a - i\epsilon} + \int ds' \frac{R_{bb}^{(a)}(s,s')\, f_{bb}^{(a)}(s')}{s' - s_0^b - i\epsilon}, \tag{48}$$

where

$$R_{ij}^{(a)} = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{s'+m_j^2}} \int dt_0\, v_{ij}(t_0) \int_0^1 dx \frac{x^{a}}{(1-x)^{1/2}} \frac{\theta(s'-sx-t_0\frac{x}{1-x})}{[s'-sx-t_0\frac{x}{1-x}]^{1/2}} . \qquad (49)$$

Let us emphasize once more that the system of Eqs. (45)-(46) coincides exactly with that of (47)-(48). Therefore, the trajectories $a(s)$ in (43) and (44) are identical. So, in the case we are considering the imaginary parts of all processes have the same asymptotic behaviour at $t \to \infty$.

Now we consider equations for bound states. Let us show that systems of equations (45), (46) or (47), (48) are analytical continuation in ℓ of the equations for the wave functions of bound states.

These equations are of the form:

$$\Psi_a(\vec p', \vec p) = \frac{1}{(2\pi)^3} \int d^3q\, V_{aa}((\vec p' - \vec q)^2) \frac{1}{\mathfrak{F}_a(q^2)} \Phi_a(\vec q, \vec p) + \frac{1}{(2\pi)^3} \int d^3q\, V_{ab}((\vec p' - \vec q)^2) \frac{1}{\mathfrak{F}_b(q^2)} \Psi_b(\vec q, \vec p), \qquad (50)$$

$$\Psi_b(\vec p', \vec p) = \frac{1}{(2\pi)^3} \int d^3q\, V_{ba}((p'-q)^2) \frac{1}{\mathfrak{F}_a(q^2)} \Psi_a(\vec q, \vec p) + \frac{1}{(2\pi)^3} \int d^3q\, V_{bb}((\vec p - \vec q)^2) \frac{1}{\mathfrak{F}_b(q^2)} \Psi_b(\vec q, \vec p). \qquad (51)$$

The wave function of the state with the angular momentum ℓ can be presented as

$$\Psi_{a,b}(\vec p', \vec p) = f_{a,b}^{(\ell)}(p'^2)\, \mathcal{Y}_\ell^m(\vec p', \vec p) . \qquad (52)$$

Substituting this expression into equations (50), (51) with the potentials of form (30) and making use of the equality[9]

$$\int \frac{d^3q\,\delta(q^2-s')}{t_0+(\vec p'-\vec q)^2} Y_\ell^m(\vec q \vec p) = \pi Y_\ell^m(\vec p \vec p) \int_0^1 \frac{x\, dx}{(1-x)^{1/2}} \frac{\Theta[s'-sx-t_0\frac{x}{1-x}]}{[s'-sx-t_0\frac{x}{1-x}]^{1/2}}$$

we get equations for the radial wave functions, which coincide exactly with Eqs. (45), (46) or (47), (48) with the substitution of a by ℓ . If there exist bound or resonance states, then these equations have a solution.

Let us remind that we are considering equations for the amplitudes $T^{\pm}$ and suppressed the signs $\pm$. It follows from (31) that the total amplitudes (19) have the asymptotic behaviour

$$\mathfrak{M}_{ij}(t, \bar t, s) = \tfrac{1}{2} \frac{1+e^{-i\pi a^+}}{\sin \pi a^+} t^{a+} f_{ij}^{(a+)}(s) + \tfrac{1}{2} \frac{1-e^{-i\pi a^-}}{\sin \pi a^-} t^{a^-} f_{ij}^{(a^-)}(s) . \qquad (53)$$

As has been already remarked, the systems of homogeneous equations (45), (46) and (47), (48) coincide, and, therefore,

$$f_{aa} = c\, f_{ab}$$

$$f_{ba} = c\, f_{bb} ,$$

where c is the constant, and

$$f_{aa} f_{bb} = f_{ab} f_{ba}. \tag{54}$$

This expression yields relationships between the cross sections /10-12/, both between the differential cross sections for elastic scattering and the total cross sections

$$\sigma_{aa} \sigma_{bb} = \sigma_{ab}^2 . \tag{55}$$

5. The Role of Subtraction Terms.

We have investigated the case of potentials without subtraction of the form (30). Here the amplitudes of the processes in question have the asymptotic behaviour (53) with one and the same trajectory. Moreover, there is a relation between the differential cross sections for elastic scattering processes and a similar one between the total cross sections.

Now we shall be concerned with the case of the potentials with the subtraction

$$V_{ij}(t) = \sum_{n=0}^{\ell_0} a_{ij}^{(n)} t^n + \frac{1}{\pi} \int dt_0 \frac{v_{ij}(t_0)}{t_0 - t} . \tag{56}$$

We expand the amplitudes $\mathcal{G}_{ij}(\vec{p}\,', \vec{p})$ in partial waves and write down equations for the partial amplitudes. They have the form:

$$\mathcal{G}_{aa}^{(\ell)}(s) = V_{aa}^{(\ell)}(s, s_0^a) + \frac{1}{(2\pi)^2} \int \frac{ds'}{s' - s_0^a - i\epsilon} V_{aa}^{(\ell)}(s, s') \mathcal{G}_{aa}^{(\ell)}(s') \frac{1}{\sqrt{s' + m_a^2}} \tag{57}$$

$$+ \frac{1}{(2\pi)^2} \int \frac{ds'}{s' - s_0^b - i\epsilon} V_{ab}^{(\ell)}(s, s') \mathcal{G}_{ba}^{(\ell)}(s') \frac{1}{\sqrt{s' + m_b^2}}$$

$$\mathcal{G}_{ba}^{(\ell)}(s) = V_{ba}^{(\ell)}(s, s_0^a) + \frac{1}{(2\pi)^2} \int \frac{ds'}{s' - s_0^a - i\epsilon} V_{ba}^{(\ell)}(s, s') \mathcal{G}_{aa}^{(\ell)}(s') \frac{1}{\sqrt{s' + m_a^2}} \tag{58}$$

$$+ \frac{1}{(2\pi)^2} \int \frac{ds'}{s' - s_0^b - i\epsilon} V_{bb}^{(\ell)}(s, s') \mathcal{G}_{ba}^{(\ell)}(s') \frac{1}{\sqrt{s' + m_b^2}}$$

and similarly for $\mathcal{G}_{ab}^{(\ell)}$ and $\mathcal{G}_{bb}^{(\ell)}$, where $V_{ij}^{(\ell)}$ are the coefficients in the expansion

$$V_{ij}(\vec{p}\,', \vec{p}) = \frac{1}{p} \sum_{\ell} (2\ell + 1) P_{\ell}(\vec{p}\,' \vec{p}) V_{ij}^{(\ell)}(p'^2, p^2). \tag{59}$$

We denote by $\tilde{\tilde{V}}_{ij}^{(\ell)}$ the expansion coefficients of the Yukawa part of the potential - the second term in (56), and put:

$$V_{ij}^{(\ell)} = \tilde{\tilde{V}}_{ij}^{(\ell)} + \delta V_{ij}^{(\ell)} . \tag{60}$$

It is obvious, that

$$\delta V_{ij}^{(\ell)} = 0 \qquad \text{for} \qquad \ell > \ell_0 . \tag{61}$$

Let $\tilde{\tilde{\mathcal{G}}}_{ij}^{(\ell)}$ be a solution of the system of equations (57)-(58) with the potentials $\tilde{\tilde{V}}_{ij}^{(\ell)}$. This solution can be continued analytically in the ℓ -plane.

Let us denote

$$\mathcal{G}_{ij}^{(\ell)} = \tilde{\tilde{\mathcal{G}}}_{ij}^{(\ell)} + \delta\, \mathcal{G}_{ij}^{(\ell)} , \tag{62}$$

$$\delta\, \mathcal{G}_{ij}^{(\ell)} = 0 \qquad \text{for} \quad \ell > \ell_0 . \tag{63}$$

The additions $\delta\, \mathcal{G}_{ij}^{(\ell)}$ satisfy some equations which can be obtained from equations (57), (58) and from definitions (60) and (62). The kernels of the equations for $\delta\, \mathcal{G}_{ij}^{(\ell)}$ depend on the potentials and on the Yukawa parts $\mathcal{G}_{ij}^{(\ell)}$ of the amplitudes. It is likely that for some potentials this system of equations has a solution of the form : $\delta\, \mathcal{G}_{ij}^{(\ell)} = 0$ for one process and $\delta\, \mathcal{G}_{ij}^{(\ell)} \neq 0$ for other processes. The total scattering amplitudes are as follows

$$\mathcal{G}_{ij}\,(t,s) = \frac{1}{p} \sum_{\ell=0}^{\infty} (2\ell+1)\, \tilde{\tilde{\mathcal{G}}}_{ij}^{(\ell)}(s)\, P_\ell(z) + \frac{1}{p} \sum_{\ell=0}^{\ell_0} (2\ell+1)\, \delta\, \mathcal{G}_{ij}^{(\ell)}(s)\, P_\ell(z) . \tag{64}$$

In the analytical continuation of $\mathcal{G}_{ij}\,(t,s)$, $t \to \infty$ the first part of (64) is complex and has a Regge behaviour, while the second part is real and has a power behaviour. In the case $\ell_0 = 1$ the scattering amplitudes have the following asymptotic behaviour

$$\mathfrak{M}_{ij}\,(t, \bar{t}, s) = A_{ij}(s) + B_{ij}(s)\, t^{\alpha(s)} \tag{65}$$

This behaviour does not contradict unitarity if $\alpha(0) = 1$. The first term in (65) determines the asymptotic behaviour of the elastic scattering cross section at $s \neq 0$. In this case relation (55) between the differential elastic scattering cross sections is not fulfilled .However, this term is real, and the total cross sections are completely determined by the second complex term in (65). In contrast to the elastic scattering cross sections the relation between the total cross sections exists even if there are subtraction terms.

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