ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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INFRARED SINGULARITIES OF MATRIX ELEMENTS IN SCALAR ELECTRODYNAMICS

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Инфракрасные особенности матричных элементов в скалярной электродинамике

Путем суммирования рядов теории возмущений с помощью ренормализационной группы получены выражения для инфракрасных (околопороговых) особенностей матричных элементов в скалярной электродинамике. Рассмотрены матричные элементы фотон-мезонного, электрон-мезонного и мезон-мезонного рассеяния. Сравнение результатов данной работы с соответствующими выражениями для спинорной электродинамики /1/ показывает, что вид главных инфракрасных особенностей не зависит от спина заряженных частиц.

Работа издается только на английском языке.

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Infrared Singularities of Matrix Elements in Scalar Electrodynamics

Expressions for the infrared (near threshold) singularities of the matrix elements in scalar electrodynamics have been obtained by summing up the perturbation theory series with the aid of the renormalization group. The matrix elements of the photon-meson, electron-meson, and meson-meson scattering have been treated. A comparison of the results of this paper with the corresponding expressions for spinor electrodynamics /1/ shows that the form of the main infrared singularities is independent of the charged particle spin.

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INFRARED SINGULARITIES OF MATRIX ELEMENTS IN SCALAR ELECTRODYNAMICS

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Introduction

The infrared singularities of the matrix elements in spinor electrodynamics were treated in /1/. Here we will be concerned with the case when charged particles are spinless.

Like in /1/, the index λ denotes the quantities calculated with the introduction of the mass $\sqrt{\lambda}$ into the photon propagation function. We write the S -matrix element for elastic scattering of two particles as

$$<p_2\, k_2 |\, S_\lambda - 1\, |\, p_1\, k_1> = -i(2\pi)^4\, (16\, p_1^o\, p_2^o\, k_1^o\, k_2^o)^{-1/2}\, \delta(p_1 + k_1 - p_2 - k_2)\; T_\lambda , \qquad (1)$$

where p_1 , k_1 and p_2 , k_2 are the particle momenta before and after scattering. We designate by $s=(p_1+k_1)^2$, $u=(p_1-k_2)^2$, $t=(p_1-p_2)^2$ the squares of the total energies of the direct, crossing and the third processes. The infrared divergences in T_λ are taken into account by resorting to formula /2/

$$T_\lambda = \exp\{ K_\lambda \}\, T , \qquad (2)$$

$$K_\lambda = -\sum_{i<j} z_i\, a_i\, z_j\, a_j\; F_\lambda\, ((p_i\, a_i + p_j\, a_j)^2,\; p_i^2 , p_j^2) , \qquad (3)$$

where the summing is being carried out over all the charged particles before and after the reaction, z_i is the sign of the charge , $a_i = 1$ or -1 for outgoing or ingoing particle with a momentum p_i ,

$$F_\lambda\, ((p_1 - p_2)^2,\; p_1^2 , p_2^2) = \frac{i a}{8\pi^3} \int \frac{dk}{k^2 - \lambda} \left(\frac{2p_1 - k}{2p_1 k - k^2} - \frac{2p_2 - k}{2p_2 k - k^2} \right)^2 \qquad (4)$$

(a is the fine structure constant, $\hbar = c = 1$, $ab = a^o b^o - \vec{a}\,\vec{b}$). For F_λ (with $\lambda \to 0$) the following representation holds

$$2F_\lambda\, (t, m^2, M^2) = \beta(x_t)\, \ln (mM/\lambda) - \epsilon (x_t , \nu) , \qquad (5)$$

$$4m\, M x_t = t - (m - M)^2 ; \qquad 4m\, M \nu = (m - M)^2 , \qquad (6)$$

$$\beta(x) = \frac{a x}{2\pi} \int_1^\infty \frac{(2z-1)\, dz}{\sqrt{z(z-1)}\;\, z\,(z - x - i\epsilon)} , \qquad (7)$$

$$\epsilon(x, \nu) = \frac{a x}{2\pi} \int_1^\infty \left[\frac{2z-1}{\sqrt{z(z-1)}} \ln \frac{z+\nu}{4z\,(z-1)} + \frac{\sqrt{z(z-1)}}{z+\nu} \right] \frac{dz}{z(z - x - i\epsilon)} \; ^* \qquad (8)$$

In what follows we are going to consider the infrared singularities of T , i.e, singularities for s , u , or t tending to the threshold values, for the elastic photon-meson, electron-meson and meson-meson scattering proces-

*Note, that this expression corresponds to the transverse gauge. If, on the other hand, use is made of the Coulomb gauge, then in this expression the second component should be omitted in the brackets under the integral.

ses. It turns out that the form of the main singularities remains the same as for the corresponding processes involving spinor charged particles treated in /1/.

In 4 the analytic properties of the matrix element of the meson-meson scattering were investigated in the fourth order of perturbation theory. Upon singling out the main infrared singularities this matrix element contains in the infrared region the terms tending to infinity like $y^{-1/2}\ln y$, $y^{-1/2}\ln t$, $y^{-1/2}$, $t^{-1/2}$, $\ln t$, where $y = s-(m+M)^2$ or $u-(m+M)^2$ i.e., it has integrable singularities. In the fourth order of perturbation theory the Mandelstam representation /3/ is valid for it.

2. Photon-Meson Scattering.

The kinematics of the photon-meson scattering was considered in paper /4/. Let p_1, p_2, and m be the meson momenta and mass, k_1, k_2 and e_1, e_2 are the photon momenta and polarization vectors. The matrix element can be put then in the form

$$T = A(s,u,t)H_A + B(s,u,t)H_B \tag{9}$$

where the structural expressions H_A, H_B are guage invariant and equal to

$$H_A = (e_1 e_2) - \frac{(e_1 k_2)(e_2 k_1)}{k_1 k_2}\ , \tag{10}$$

$$H_B = (e_1 q)(e_2 q) - \frac{(e_1 k_2)(e_2 q)(k_1 q)}{k_1 k_2} - \frac{(e_1 q)(e_2 k_1)(k_2 q)}{k_1 k_2} + \\ + \frac{(e_1 k_2)(e_2 k_1)(k_1 q)(k_2 q)}{(k_1 k_2)}; \qquad q = p_1 + p_2 \tag{11}$$

It is worth while to note that at $t = -2k_1k_2 = 0$ the momenta k_1 and k_2 become equal and $k_2 e_1 = k_1 e_2 = 0$. So, H_A and H_B are finite at $t=0$. In the c.m.s. of the direct process

$$H_A = -(\vec{e}_1\,\vec{e}_2) + \frac{2}{t}(\vec{e}_1\,\vec{p}_2)(\vec{e}_2\,\vec{p}_1)\ , \tag{12}$$

$$H_B = \frac{4}{t^2}(s-m^2)^2(\vec{e}_1\,\vec{p}_2)(\vec{e}_2\,\vec{p}_1)\ . \tag{13}$$

Now we consider the analytic properties of A and B in the lowest orders of perturbation theory. In the second order (Fig. 1)

$$A^{(2)} = -2e^2, \quad B = e^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\ . \tag{14}$$

In the fourth order diagrams 1-9 (Fig.2) contain the infrared divergences (at $\lambda\to 0$). Upon renormalizing the meson wave functions and singling out the infrared divergences by formula (2), where for the given case $K_\lambda = F_\lambda(t, m^2, m^2)$, we get that in the infrared region, i.e., at $s\to m^2$ or $u\to m^2$ the main singularities in $T^{(4)}$ yield, like in spinor electrodynamics /1/, diagrams 8-15 (Fig. 2).

The main singularities are of the form

$$T^{(4)} = [\,\beta(t/4m^2)\,\ell n\,[(m^2-s)/m^2] + \gamma(t/4m^2)\,]\; T^{(2)}_s + [\,\beta(t/4m^2)\ell n\,[(m^2-u)/m^2] + \gamma(t/4m^2)\,]\; T^{(2)}_u + \dots, \tag{15}$$

where the points indicate the terms less singular than the pole at $s \to m^2$ or $u \to m^2$, provided $\beta(x)$ is given by (7)

$$\gamma(x) = -\frac{a\,x}{4\pi}\int_1^\infty \left[\frac{(2z-1)\,\ell n 4z}{\sqrt{z(z-1)}} - \sqrt{\frac{z-1}{z}}\right]\frac{dz}{z(z-x-i\epsilon)}\;{}^{*}, \tag{16}$$

and $T^{(2)}_s$, $T^{(2)}_u$ are the contributions of diagrams 2 and 3 of Fig. 2, respectively.

Other diagrams of the fourth order are finite at $\lambda \to 0$ and at $s \to m^2\,(u \to m^2)$ have only integrable (i.e., weaker, than the pole) singularities.

Expression (15) may be expanded in structures (10),(11) up to the terms less singular at $s \to m^2$ or $u \to m^2$. As a result , we get that in the fourth order

$$A^{(4)} = A^{(4)}_a \tag{17}$$

$$B^{(4)} = e^2\,\{(s-m^2)^{-1}\,[\beta(t/4m^2)\,\ell n\,[(m^2-s)/m^2] + \gamma(t/4m^2)\,] + (u-m^2)^{-1}\,[\,\beta(t/4m^2)\,\ell n[(m^2-u)/m^2] + \gamma(t/4m^2)\,]\,\} + B^{(4)}_a, \tag{18}$$

where $A^{(4)}_a$ and $B^{(4)}_a$ have singularities weaker than the pole in the infrared region.

The equations of the renormalization group /5/ allow to sum up the main singularities in B. Let us consider, for example, the region $s \to m^2$. Representing B as $B = e^2(s-m^2)^{-1} M$ and considering M for different normalizations, the factors of external lines being fixed, we are able to write for M differential equations of the group. We choose the normalization momentum k_0 so that $Re\, d\,(1, m^2/k_0^2, e^2) = 1$ where $d(k^2/k_0^2, m^2/k_0^2, e^2)$ is the transverse Green function of the photon. By making k_0^2 tend to m^2 at $s \to m^2$ we get the relationship between $M((m^2-s)/m^2)$ and $M((m^2-s)/(m^2-k_0^2))$, for which the perturbation theory series is well convergent for all t in the region where the function $\beta(t/4m^2)$ is determined. As a result for M we obtain $M = \Psi(t)\exp\{e^2\beta(t/4m^2)\ell n\frac{m^2-s}{m^2}\}$, where e^2 is the observed charge. For $\Psi(t)$ it is again possible to write down the equation of the renormalization group which for all finite t (for $t \to 4m^2$ inclusive) yields $\Psi(t) = \exp\{\gamma(t/4m^2)\}$.

Considering in a similar manner the region $u \to m^2$ we get for B

$$B = -e^2\exp\{\gamma(t/4m^2)\}\,[(m^2-s)^{\delta(t)} + (m^2-u)^{\delta(t)}] + B_a, \tag{19}$$

$$\delta(t) = -1 + \beta(t/4m^2), \tag{20}$$

where β and γ are, generally speaking, series in a , whose first terms are represented in (7), (16), and B_a does not contain main infrared singularities.

The physical interpretation of the function $\delta(t)$ in (19) was treated in /1,6-8/ .

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* See a reference on page 8.

3. Electron-Meson Scattering.

Let p_1 , p_2 and m be the electron momenta and mass, k_1 , k_2 and M are the positive meson momenta and mass.

In the second order (diagram 1 of Fig. 3) the matrix element

$$T^{(2)} = e^2 \bar{u}(p_2)\,\gamma^n u(p_1)(k_1 + k_2)_n\, t^{-1} . \tag{21}$$

($\bar{u}\, u = 2m$) has a pole at $t = 0$. Unlike the previous case, the main infrared singularities in the fourth order arise at $t = 0$. In order to consider these singularities, we single out, first of all, the infrared divergences. In the fourth order the divergences are given by diagrams of the same type as in the previous case. However, as far as each charged particle gives the contribution to the divergences then the infrared factor K_λ in (2) is equal here to

$$K_\lambda = 2F_\lambda\,(s, m^2, M^2) - 2F_\lambda\,(u, m^2, M^2) + F_\lambda(t, m^2, m^2) + F_\lambda\,(t, M^2, M^2) \tag{22}$$

Unlike the previous case, diagrams 2 and 3 of Fig. 3 contain two virtual photons and give the infrared divergences and singularities when the momentum of each of them vanishes. Consider, e.g., diagram 2.

Representing its contribution as

$$\begin{aligned} T^{(4)}_{\lambda_2} = \frac{ie^4}{(2\pi)^4} \int \frac{dk}{(k^2 - 2p_1 k)(k^2 + 2k_1 k)} \Big\{ &\Big[\frac{N(0)}{k^2 - \lambda} + \frac{N(q)}{(k-q)^2 - \lambda} \Big] \frac{1}{q^2} + \\ &+ \Big[\frac{N(0)}{k^2-\lambda}\Big(\frac{1}{q^2 - 2qk} - \frac{1}{q^2} \Big) - \frac{N(q)}{(k-q)^2 - \lambda}\Big(\frac{1}{q^2 - 2qk} + \frac{1}{q^2} \Big) \Big] + \\ &+ \Big[\frac{N(k) - N(0)}{k^2 - \lambda} - \frac{N(k) - N(q)}{(k-q)^2 - \lambda} \Big] \frac{1}{q^2 - 2qk} \Big\}, \end{aligned} \tag{23}$$

where $q = p_1 - p_2$ and

$$N(k) = \bar{u}(p_2)\,\gamma^m d_{mn} (k_1 + k_2 + k)^n (\hat{p}_1 - \hat{k} + m)(2k_1 + k)^r d_{rs} \gamma^s u(p_1) \tag{24}$$

(d_{ij} is the factor of the photon Green function equal to g_{ij} for the Coulomb gauge or $g_{ij} - k_i k_j / k^2$ for the transverse gauge; in the latter case in $N(0)$ and $N(q)$ the momentum k in d_{ij} is not fixed) we can see that at $\lambda \to 0$ only the first component in curved brackets is divergent. It gives the contribution to $2F_\lambda\,(s, m^2, M^2)$ in (22). In the second and third components in (23) λ can be put to be 0. At the same time the second component at $t \to 0$ has at least the pole. It is possible to show that the third component has weaker singularities than the pole. Thus, the main infrared singularities in $T^{(4)}_2$ enter the second component in (23).

Just in a similar manner diagram 3 of Fig. 3. is considered. Diagrams 2 and 3 give the following contribution to the main singularities of the matrix element at $t \to 0$

$$T^{(4)}_{2,3} = \Phi\, T^{(2)} + \dots , \tag{25}$$

$$\Phi = [\beta(x_s) - \beta(x_u)]\,\ln[(-t)/mM] + \epsilon(x_s, \nu) - \epsilon(x_u, \nu), \tag{26}$$

where the functions β and ϵ are given by (6-8).

Besides diagrams 2 and 3 of Fig. 3 the pole singularities in T are given by diagram 4 of Fig.3 which has an electron-photon vertex function. Apart from the factor $F_\lambda(t, m^2, m^2)$ in (22), this function makes a contribution to the infrared singularities which depends on an additional magnetic moment of an electron μ':

$$T_4^{(4)} = \bar{u}(p_2)\, \mu'\, ½\, [\hat{q}, \gamma^n]_-\, u(p_1)\, e\, (k_1 + k_2)_n\, t^{-1} + \ldots \tag{27}$$

It is easy to see that in the sixth order (diagrams 5-8 of Fig. 3) a part of the electron-photon vertex function dependent on μ' leads to singularities of form (25), where one needs to substitute (27) for $T^{(2)}$. Thus, for the main infrared singularities of the matrix element T perturbation theory in the lowest orders yields a sum of expressions (21) and (25) where the matrices γ^n should be substituted by $\gamma^n + (\mu'/2e)\,[\hat{q}, \gamma^n]_-$.

Making use of the renormalization group, we get

$$T = e^2 \bar{u}(p_2)(\gamma^n + (\mu'/2e)[\hat{q}, \gamma^n]_-)\, u(p_1)(k_1 + k_2)_n\, t^{-1} \exp\{\Phi\} + T_a, \tag{28}$$

where Φ is a series in α, whose first term is represented in (26), and T_a has at $t \to 0$ singularities weaker than the pole.

If T is expressed in terms of the invariant functions

$$T = \bar{u}(p_2)\,(A(s,u,t) + (\hat{k}_1 + \hat{k}_2)\, B(s,u,t))\, u(p_1), \tag{29}$$

then

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e\mu'(s-u) \\ e^2 - 2me\mu' \end{pmatrix} \frac{e^{\Phi}}{t} + \begin{pmatrix} A_a \\ B_a \end{pmatrix} \tag{30}.$$

4. Meson-Meson Scattering.

We consider the matrix element of the scattering of the two oppositely charged mesons with the masses m and M Like in the previous case, we single out the infrared divergences and the main infrared singularities . Then we get for this matrix element the following representation

$$T = e^2 (s - u) \exp\{\Phi\}\, t^{-1} + T_a \tag{31}$$

where Φ is given by (26), and T_a has singularities weaker than the pole. We consider the singularities of T_a in the lowest (fourth) order of perturbation theory. The strongest of them are given by diagrams 2 and 3 of Fig. 3, as well as by diagrams 1-3 of Fig.4. They are of the form

$$T_a^{(4)} = -4\alpha^2 \{ (s - m^2 - M^2) \int_{(m+M)^2}^{\infty} \frac{1}{\sqrt{k(s')}} \ln \frac{-ts'}{k(s')} \frac{ds'}{s' - s - i\epsilon} + (s \to u) + \tag{32}$$

$$+ \frac{\pi^2 (m+M)}{\sqrt{-t}} + \ln \frac{-t}{mM} \} + \ldots,$$

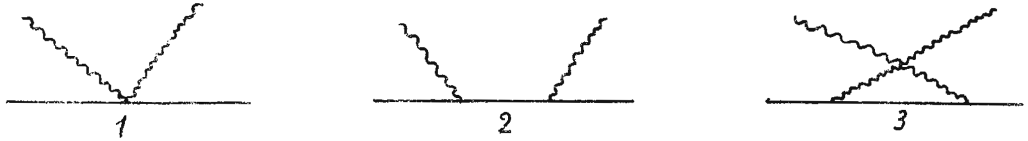

Fig.1.

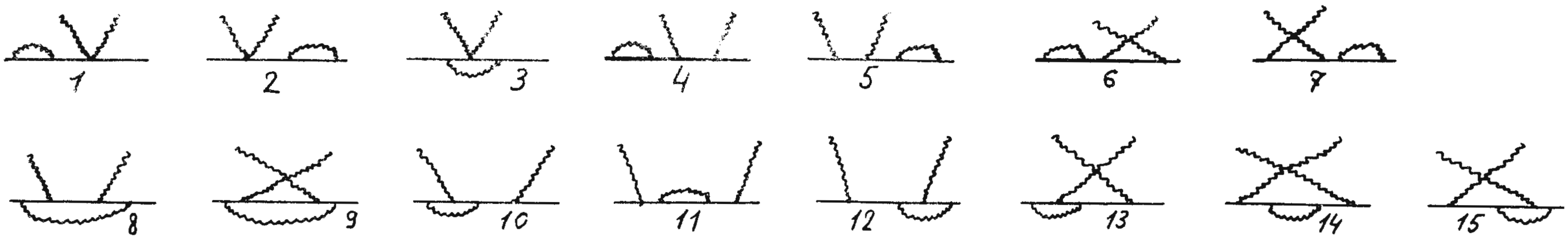

Fig. 2.

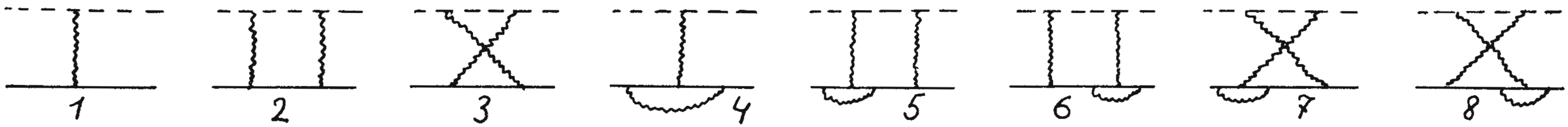

Fig.3.

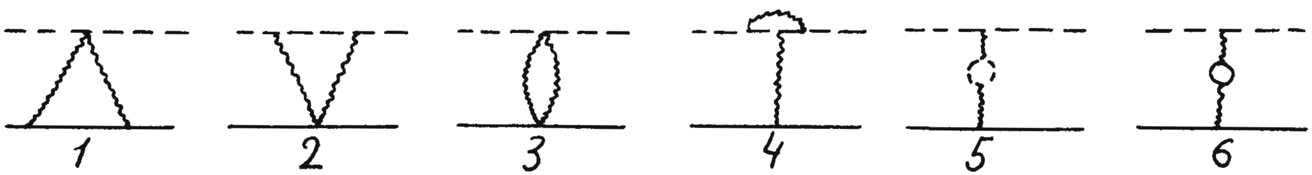

Fig.4.

where $k(s) = [\, s - (m+M)^2 \,][\, s - (m-M)^2]$, $(s \to u)$ denotes previous terms in which s is replaced by u and the points indicate the finite terms. We see that at $t \to 0$, $s \to (m+M)^2$ or $u \to (m+M)^2$ $T_a^{(4)}$ tends to infinity, these singularities being remained integrable. The consideration of the fourth order diagrams (2-4 of Fig. 3 and 1-6 of Fig. 4) shows that for $T_a^{(4)}$ the Mandelstam representation [3] of the following form holds true

$$T_a^{(4)} = (s - s_0) \int_{(M+m)^2}^{\infty} \frac{b_1(s')\, ds'}{(s' - s_0)(s' - s)} + (s - s_0)(t - t_0) \int_{(M+m)^2}^{\infty} ds' \int_0^{\infty} dt' \frac{b_2(s')}{(s' - s_0)(s' - s)(t' - t_0)(t' - t)} +$$

$$+ (s \to u) + (t - t_0) \int_0^{\infty} \frac{b_3(t')\, dt'}{(t' - t_0)(t' - t)} + (s - u)\left(\int_{4m^2}^{\infty} \frac{b_4(t')\, dt'}{t' - t} + \int_{4M^2}^{\infty} \frac{b_5(t')\, dt'}{t' - t} \right) \tag{33}$$

Note, that the spectral function $b_2\,(s')$ is independent of t'.

References

1. Л.Д. Соловьев. ЖЭТФ, 44, № 1 /1963/; Phys. Lett., 3, 172 /1963/.
2. D. R.Yennie, S.C.Frautschi, H.Suura. Ann. of Phys. , 13, 379 (1961).
3. S.Mandelstam. Phys.Rev., 112, 1341 (1958).
4. M.Gourdin, A.Martin. Nuovo Cim., 17, 224 (1960).
5. Н.Н. Боголюбов, Д.В. Ширков. Введение в теорию квантованных полей. Гостехиздат, гл. УШ, /1957/. В.З. Бланк. Диссертация. Препринт ОИЯИ Р-103, 1958.
6. Л.Д. Соловьев, О.А. Хрусталев. Препринт ОИЯИ Е-1116, ЖЭТФ, 44, № 2 /1963/.
7. M.Levy. Phys. Rev. Lett., 9, 235 (1962) .
8. H.Suura, D. Yennie. Phys. Rev.Lett., 10, 69 (1963).

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