ОБЪЕДИНЕННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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THREE-PARTICLE STATES IN THE RELATIVISTIC THEORY OF COMPLEX ANGULAR MOMENTUM

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1. Introduction

The analytic properties of partial wave amplitudes in the total angular momentum quantum number have been extensively investigated in a series of works. The attention devoted to this problem is due to the well-known fact that in a crossing-symmetric theory the asymptotic behaviour of scattering amplitudes at high energies is in a one-to-one correspondence with the location and nature of singularities in the angular momentum of a crossed channel. The investigation of analytic properties has reached a more or less satisfactory state, when one channel states containing more than two particles are neglected. In fact, for this case, it has been proved (see e.g.[1,2,3]) that the partial wave amplitude is meromorphic in a half-plane, so the asymptotic behaviour of the amplitude in a crossed channel is dominated by the extreme right pole.

In the present work we start a study of the role played by many-particle states, considering the first non trivial case, that of three spinless particles. The method applied is a direct extension of that used for the two-particle case: i.e. we expand the amplitudes in angular momentum eigen-states (Sec. 2), following the work of Wick[4] and Lee and Cook[5]. The continuation of the kinematic factors in j, the total angular momentum is effected according to the method, proposed in an earlier paper[6].

The amplitudes satisfying unitarity and having the necessary analytic properties are constructed by means of integral equations of the N/D type. (Sec. 4). The main difficulty here is in our incomplete knowledge of the analytic properties of production amplitudes. Therefore we adopt a model of the inelastic interaction by requiring that the dynamical singularities be given by some simple diagrams, their analytic properties being summarized in Sec.3. We then find that in spite of the presence of complex singularities, the N/D equations are of the Fredholm type, so the many particle amplitudes – at least in the frames of the model considered – are meromorphic functions of the total angular momentum. A new feature is brought in when one inverts the denominator matrix (Sec. 5) : due to the presence of the three-particle channel, the Regge-poles have condensation points, the positions of which depend on the relative signs and magnitudes of the coupling constants (Sec. 6). In Sec. 7 we sketch the physical implications of such a situation; some formal aspects of the Watson-Sommerfeld transform in the presence of complex singularities are summarized in the Appendix.

2. Partial wave expansion and unitarity

The partial wave expansion we start with is similar to that of Cook and Lee[5], i.e. the πN channel (2) is expanded into normal helicity amplitudes, while in the $\pi\pi N$ one (3) we expand two of the particles in their own CMS into partial waves (ℓ) ; the resulting states are treated as 'particles' with spin ℓ and mass $\sqrt{s_{12}}$, where s_{12} is the invariant energy squared of the pair[1,2]*. Thus, splitting off the kinematic singularities and disconnected diagrams from the amplitude, we obtain the unitarity condition, written in matrix form:

$$\frac{1}{2i}\left[M(j\ell m,\ s+io,\ s_{12}+io) - M(j\ell m,\ s-io,\ s_{12}-io) \right] = \qquad (2.1)$$

$$= M(j\ell m,\ s+io,\ s_{12}+io)\ \Sigma\ M(j\ell m,\ s-io,\ s_{12}-io)$$

*For a special case, such an expansion has been given independently by Ter-Martirosian. (Proc. Internat. Conf. on High Energy Physics, Geneva, 1962).

Here j stands for the total angular momentum, ℓ, m – the angular momentum and its projection of the pion pair; s – the total energy squared.

The elements of the 'phase space matrix' Σ are given by:

$$\Sigma_{22} = \frac{p_1^{2j+1}}{\sqrt{s}}$$

$$\Sigma_{33} = \frac{k_{12}}{\sqrt{s_{12}}} \; \frac{p_2^{2j+1}}{\sqrt{s}} \qquad (2.2)$$

$$\Sigma_{23} = \Sigma_{32} = 0 .$$

The matrix $M_{\alpha\beta}$ connected with the transition matrix $T_{\alpha\beta}$ by the relation:

$$T^{(j)}_{\alpha\beta} = p^j_\alpha \, M^{(j)}_{\alpha\beta} \, p^j_\beta \, , \qquad (\alpha, \beta = 2,3)$$

where:

$$p_2 = \tfrac{1}{2} s^{-\frac{1}{2}} \left[s - (M+\mu)^2 \right]^{\frac{1}{2}} \left[s - (M-\mu)^2 \right]^{\frac{1}{2}}$$

$$k_{12} = \tfrac{1}{2} \, s_{12}^{-\frac{1}{2}} \, (s_{12} - 4\mu^2)^{\frac{1}{2}}$$

$$p_3 = \tfrac{1}{2} s^{-\frac{1}{2}} \left[s - (M + \sqrt{s_{12}})^2 \right]^{\frac{1}{2}} \left[s - (M - \sqrt{s_{12}})^2 \right]^{\frac{1}{2}} .$$

The details of the derivation of these formulae are given in ref. /5/. In eq. (2.1) a summation over ℓ, m and integration over s_{12} is understood from $4\mu^2$ to $(\sqrt{s} - M)^2$.

In formula (2.2) factors like $\delta(s_{12} - s'_{12})$, $\delta_{jj'}$ etc are suppressed.

Sometimes it is convenient to go over to another representation; namely, instead of the quantum numbers $(j\ell m)$ we introduce the orbital momentum L of the nucleon with respect of the CM of the pion pair. The transformation matrix between both representations has been given by Jacob and Wick /7/, and for our case it reads:

$$\langle j M L \ell \mid j M \ell m \rangle = \left(\frac{2L+1}{2j+1} \right)^{\frac{1}{2}} \langle L\, 0\, \ell m \mid j m \rangle \qquad (2.3)$$

(M being the projection of j).

Inserting the expansion of $M_{\alpha 3}$ and $M_{3\beta}$ with the help of eq. (2.3) into the unitarity condition, we observe that the summation over the projection m can be carried out in a closed form. If j is an integer, then, according to the well-known sum rule of vector coupling coefficients, we obtain:

$$\sum_m \langle L 0 \ell m \mid j m \rangle \langle j m \mid L 0 \ell m \rangle = \begin{cases} \dfrac{2j+1}{2L+1} & \text{if } \; |j - L| \le L \le j + \ell \\ 0 & \text{otherwise} . \end{cases} \qquad (2.4)$$

If, however, we continue in the quantum number j to complex values, then the last formula is replaced by the expression[(6)]:

$$\frac{2L+1}{2j+1} \sum_m \langle L0\ell m \mid jm\rangle \langle jm \mid L0\ell m\rangle = \Delta(j; L\ell) \tag{2.5}$$

where
$$\Delta(j;L\ell) = \frac{\sin j\pi}{\pi}(-1)^{L-\ell}\Big\{ \frac{1}{j-|L-\ell|} + G(L+\ell+1-j) - G(|L-\ell|+1-j) + (j\to -j-1)\Big\}, \tag{2.6}$$

$G(z)$ being a polygamma function of order zero (Ref./8/, Ch. 1.).

In the new representation, we have to sum over L and ℓ , integrate over s_{12} as before, while the element Σ_{33} of the phase space matrix is of the form:

$$\Sigma_{33} = \frac{k_{12}}{s_{12}} \frac{p_3^{2j+1}}{\sqrt{s}} \Delta(j; L\ell), \tag{2.6}$$

the other elements being unchanged.

From our point of view, the main difference between eqs. (2.4) and (2.5) consists in the fact that for complex values of j , the series in L is not cut off for a finite value of the orbital momentum; therefore, for the analytic properties of the amplitudes in the total angular momentum, the convergence properties of this series are essential. The function $\Delta(j; L\ell)$ - as one can easily see - is an integer function in j ; if L or ℓ tends to infinity, $\Delta(j; L\ell)$ decreases as L^{-1} or ℓ^{-1} ,respectively.

Intuitively one ecpects that - due to the finite range of the interaction - the amplitudes $M_{\alpha\beta}$ decrease exponentially if L or ℓ tends towards infinity through real values; therefore the series $\sum_{L\ell}$ converge uniformly in j , so the presence of the inelastic channel does not introduce new kinds of singularities. At present we are unable to prove this conjecture in full generality; therefore , in what follows, we study the contribution of some simple diagrams, where the convergence properties can be investigated comparatively easily.

3. Model of the inelastic interaction

We study the inelastic processes, described by the diagrams Fig. 1a, b. In order to simplify the kinematics, in what follows we take all the masses equal (μ) and call them 'pions'. The contribution of the diagrams on Fig. 1 a, b is given by the following expression:

$$B^j_{23}(s,\sigma) = \frac{g f s}{\pi a(s,\sigma)} Q_j\Big(\frac{s(s-\mu^2-\sigma)}{a(s,\sigma)}\Big) + \frac{2^{*} g^2 f s}{\pi a(s,\sigma)} \int_{4\mu^2}^{\infty} dt\, Q_j\Big(\frac{\beta(s,t,\sigma)}{a(s,\sigma)}\Big) \times \times \{[t-(\mu+\sqrt{\sigma})^2][t-(\mu-\sqrt{\sigma})^2]\}^{-1/2} Q_0\Big(\frac{t(t-\mu^2-\sigma)}{a(t,\sigma)}\Big), \tag{3.1}$$

where

$$a(x,y) = \{ x[x-4\mu^2][x-(\mu+\sqrt{y})^2][x-(\mu-\sqrt{y})^2] \}^{1/2}$$

and

$$\beta(s,t,\sigma) = s[s+2t-\sigma-3\mu^2],$$

σ is the energy squared of the pion pair, emitted in a relative s -state. g and f are phenomenological coupling constants of the three- and four- particle vertices, respectively. Q_j - s are Legendre functions of the second kind. The reader will immediately notice that the first and second terms in the expression (3.1) arise from diagrams 1 a, and 1 b respectively.

Now we are going to study the singularities of the expression (3.1) in its variables. As one can immediately check, the amplitude is a meromorphic function in j, having simple poles at $j = -n$ $(n = 1, 2, \ldots)$

Singularities in the energy variables s, σ may arise, where

a) Some of the functions under the square root vanish.

b) the argument of - at least – one of the Legendre functions equals to ± 1.

The square - root singularities give rise to the kinematic cuts, while those coming from the Legendre functions to the left-hand and anomalous singularities.

As σ increases from below, at the value $\sigma = 3\mu^2$ the singularity of Q_0 reaches the point $t = 4\mu^2$, encircles it and goes backward down to the point $t = 3\mu^2$ (this is reached at $\sigma = 4\mu^2$). Afterwards the singularity moves out to the complex t -plane. In order to avoid this singularity, we have to deform the path of integration in t, as shown in Fig. 2.

A similar analysis shows that the singularity of the function Q_j reaches the point $t = 4\mu^2$ if s and σ are bound by the following equation:

$$\sigma = \frac{1}{2}[2\mu^2 + s + \sqrt{3s(4\mu^2 - s)}]. \qquad (3.2)$$

If $s > 4\mu^2$ or $s < 0$ an end point singularity in t cannot occur for real values of σ.

The partial wave amplitude is singular at

$$\left. \begin{array}{ll} \sigma = 4\mu^2 & \sigma = 0 \\ s = (\mu \pm \sqrt{\sigma})^2 & \end{array} \right\} \quad \text{(the kinematic branch points)}$$

and at the surface given by eq. (3.2). If we inserted a complete pion-pion partial wave amplitude in the place of the four-pion vertex in Fig. 1 a, b, the situation concerning the location of singularities would remain essentially the same. The convergence of a partial wave expansion in one of the angular momenta could be damaged, if a singularity in the momentum transfer reaches the end point of the integration contour as at $\sigma = 3\mu^2$ in the diagrams considered). However, according to a well-known theorem /9/, a Legendre expansion begins to diverge if the singularity reaching the end point is at least as strong as $(t-4)^{-3/4}$ (or correspondingly for the other end point). In the cases considered here, the singularities in question are logarithmic; therefore the Legendre expansions are very likely to continue to converge even if an 'anomalous' singularity is developed.

Bearing these considerations in mind, we retain the point vertices in our diagrams; this corresponds to setting $\ell = 0$ in all the previous formulas.

4. N/D equations for the matrix amplitude

We are now ready to construct the matrix amplitude $M_{\alpha\beta}$ (α , $\beta=2,3$), satisfying unitarity in two- and three particle channels. We follow the procedure of Cook and Lee[5], by assnming that the matrix M has the form:

$$M = N D^{-1}$$

the 'denominator matrix' D being chosen in such a way as to satisfy the unitarity condition for M while the nominator matrix contains the dynamical singularities. M satisfies the unitarity condition if the matrix elements of D are connected with those of N by means of the following relations:

$$D_{22}(s) = 1 - \int_{4\mu^2}^{\infty} \frac{ds'}{s'-s} \Sigma_{22}(s') N_{22}(s')$$

$$D_{32}(s,\sigma) = - \int_{9\mu^2}^{\infty} \frac{ds'}{s'-s} \Sigma_{33}(s,'\sigma) N_{32}(s',\sigma)$$

$$D_{23}(s,\sigma) = - \int_{4\mu^2}^{\infty} \frac{ds'}{s'-s} \Sigma_{22}(s') N_{23}(s',\sigma) \tag{4.1}$$

$$D_{33}(s,\sigma',\sigma) = \delta(\sigma'-\sigma) - \int_{9\mu^2}^{\infty} \frac{ds'}{s'-s} \Sigma_{33}(s',\sigma') \times$$
$$\times N_{33}(s',\sigma',\sigma),$$

The matrix elements of the phase space matrix $\Sigma_{\alpha\beta}$ are obtained from eqs. (2.2) and (3.6) by setting $M=\mu$ and $\ell=0$. In writing down the expressions (4.1) we suppressed the diagonal index j.

The equations for the nominator matrix N are constructed as in ref.[5], i.e. assuming that the dynamical singularities arise from the inelastic amplitude only, and equal to the dynamical cuts of eq. (3.1). The amplitude B_{23} can be represented by a contour integral of the form:

$$B_{23}(s,\sigma) = \int_C \frac{ds'}{s'-s} A_{23}(s',\sigma) \tag{4.2}$$

the contour of integration being shown in Fig. 3.

Because of the singularities of B_{23}, reaching the end points of the integrations, beginning at the elastic threshold $s=4\mu^2$, the contours should be correspondingly deformed. The continuation procedure necessary is described in the literature (e.g. in ref.[5]), we simply quote the resulting system of equations for the elements of the nominator matrix:

$$N_{22}(s) = \int_{4\mu^2}^{\infty} d\sigma \int_{C'+C''} \frac{ds'}{s'-s} A_{23}(s',\sigma+io)\, D_{32}(s',\sigma-io)$$

$$N_{23}(s,\sigma) = \int_{4\mu^2}^{\infty} d\sigma' \int_{C'+C''} \frac{ds'}{s'-s} A_{23}(s',\sigma'+io) \times D_{33}(s',\sigma'-io,\sigma) \tag{4.3}$$

$$N_{32}(s,\sigma) = \int_{C'+C''} \frac{ds'}{s'-s} A_{32}(s',\sigma)\, D_{22}(s') - 2\pi i \int_{C''} \frac{ds'}{s'-s} \Sigma_{22}(s') \times N_{22}(s') A_{32}(s',\sigma)$$

$$N_{33}(s,\sigma',\sigma) = \int_{C'+C''} \frac{ds'}{s'-s} A_{32}(s',\sigma)\, D_{23}(s',\sigma) - \tag{4.3}$$

$$- 2\pi i \int_{C''} \frac{ds'}{s'-s} \Sigma_{22}(s') N_{23}(s',\sigma)\, A_{32}(s',\sigma'),$$

the contour of integration being explained in Fig. 3.

By means of elementary manipulations from the expressions (4.1.) and (4.3) we can obtain a system of integral equations for the elements of the nominator matrix $N_{\alpha\beta}$. The resulting equations read as follows:

$$N_{22}(s) = -\int_{4\mu^2}^{\infty} ds \int_{(\mu+\sqrt{\sigma})^2}^{\infty} ds' \Sigma_{33}(s',\sigma-io) \frac{B_{23}(s,\sigma+io) - B_{23}(s',\sigma+io)}{s-s'} \times$$
$$\times N_{32}(s',\sigma-io)$$

$$N_{23}(s,\sigma) = B_{23}(s,\sigma) - \int_{4\mu^2}^{\infty} d\sigma' \int_{(\mu+\sqrt{\sigma'})^2}^{\infty} ds' \Sigma_{33}(s',\sigma'-io) \times$$
$$\times \frac{B_{23}(s,\sigma'+io) - B_{23}(s',\sigma'+io)}{s-s'} N_{33}(s',\sigma'-io,\sigma)$$

$$N_{32}(s,\sigma) = B_{32}(s,\sigma) - \int_{4\mu^2}^{\infty} ds' \Sigma_{22}(s') \frac{B_{32}(s,\sigma) - B_{32}(s',\sigma)}{s'-s} N_{22}(s') \tag{4.4}$$
$$- 2\pi i \int_{4\mu^2}^{\infty} d\sigma' \int_{C''} ds' \Sigma_{22}(s') A_{23}(s',\sigma'+io) \frac{B_{32}(s,\sigma) - B_{32}(s',\sigma)}{s-s'} \times$$
$$\times \int_{(\mu+\sqrt{\sigma'})^2}^{\infty} \frac{ds''}{s''-s'} \Sigma_{33}(s'',\sigma) N_{32}(s'',\sigma)$$

$$N_{33}(s,\sigma',\sigma) = 2\pi i \int_{C''} ds' \Sigma_{22}(s') A_{23}(s',\sigma+io) \times$$
$$\times \frac{B_{32}(s,\sigma') - B_{32}(s',\sigma')}{s-s'}$$

$$- \int_{4\mu^2}^{\infty} ds' \Sigma_{22}(s') \frac{B_{32}(s,\sigma') - B_{32}(s',\sigma')}{s-s'} N_{23}(s',\sigma)$$

$$- 2\pi i \int_{4\mu^2}^{\infty} d\sigma'' \int_{C''} ds' \Sigma_{22}(s') A_{23}(s',\sigma''+io) \frac{B_{32}(s,\sigma') - B_{32}(s',\sigma')}{s-s'} \times$$

$$\times \int_{(\mu+\sqrt{\sigma'})^2}^{\infty} \frac{ds''}{s''-s'} \Sigma_{33}(s'',\sigma''-io) N_{33}(s'',\sigma''-io,\sigma)$$

(Notice that in the system (4.4) all the integrations beginning at $s = 4\mu^2$ are to be undestood in the sense, ex - plained in Fig. 3).

Introducing for a moment the notation:

$$\phi_1 = N_{22}, \quad \phi_2 = N_{23}, \quad \phi_3 = N_{32}, \quad \phi_4 = N_{33}$$

and

$$f_1 = 0 \quad , \qquad f_2 = B_{23}(s,\sigma) \, , \qquad \text{etc,}$$

we see that our system of equations (4.4) can be written in the symbolic form:

$$\phi_i = f_i + \sum_{\ell} K_{i\ell} \, \phi_\ell \quad , \tag{4.5}$$

where the matrix K_{ij} has the following structure:

$$\| K_{ij} \| = \begin{pmatrix} 0 & 0 & K_{13} & 0 \\ 0 & 0 & 0 & K_{24} \\ K_{31} & 0 & K_{33} & 0 \\ 0 & K_{42} & 0 & K_{44} \end{pmatrix} . \tag{4.6}$$

and its elements K_{ij} can be read off by comparing eqs. (4.5) and (4.4), e.g.:

$$K_{13}(s, s', \sigma + io) = - \frac{B_{23}(s, \sigma + io) - B_{23}(s', \sigma + io)}{s - s'} \; \Sigma_{33}(s', \sigma - io)$$

etc.

In order to investigate the analytic properties of the amplitude $M = ND^{-1}$ in j , the total angular momentum quantum number, one has to apply essentially the same procedure as in the one-channel problem/1,2/ .

The method described in refs./1,2/ consists in establishing the conditions, under which the integral equations for N (or D) are of the Fredholm type.

Taking these conditions for granted, we can immediately apply the theory of parametric integral equations (cf. e.g. Iglisch/10/).

So we turn to the investigation of the kernels of the system (3.6) (or (3.7) respectively).

As is well known the condition for a system of integral equations to be of the Fredholm type is:

$$\int dx \; dy \sum_{ij} | K_{ij}(x,y) |^2 < \infty \quad , \tag{4.7}$$

where x and y stand for the set of variables correspondingly for each element of the matrix K_{ij} .

For our system (4.4) we are given the exact expression of the kernel matrix; a straightforward estimation of the integrals shows that (4.7) is fulfilled down to $Re\, j = -1$. Hence it follows that the matrix $N_{\alpha\beta}$ is a meromorphic function of j in the right half-plane $Re\, j > -1$. $N_{\alpha\beta}$ will have poles, where the Fredholm determinant

$$K = \exp \, Tr \, (\log K)_{ij} \tag{4.8}$$

vanishes; the operation 'Tr' means integration over the variables s, σ and summation over the index i of the diagonal elements of the matrix $(\log K)_{ij}$.

Besides that, at the poles of the elements of K_{ij} (i.e.at $j - n$), the solution $N_{\alpha\beta}$ will have in general an essential singularity.

As can be seen from the structure of eq. (4.8), the positions of the singularities of $N_{\alpha\beta}$ are independent of the energy variables s, σ.

If we had to do with a one-channel problem, by this analysis we could essentially be satisfied. In fact, when forming D , the same ('standing') poles would appear both in N and D , so the amplitude N/D would remain finite except at the roots of D . In our case, however, the inversion problem of the matrix $D_{\alpha\beta}$ leads once again to an integral equation, the kernel of which may be singular in the parameter; so we have to investigate this problem separately.

5. The inversion problem of D

The amplitude M in matrix form looks like $M = N D^{-1}$ so, after having determined N and D , we have to find the matrix D^{-1} . This is equivalent to solving the integral equation (cf. /5/):

$$f(s,\sigma) = \int_{4\mu^2}^{\infty} D_{33}(s,\sigma,\sigma')\ \phi(s,\sigma')\ d\sigma' \ , \qquad (5.1)$$

or taking into account the structure of D_{33} (eq. (4.1)) :

$$\phi(s,\sigma) = f(s,\sigma) + \lambda \int_{4\mu^2}^{\infty} K_{33}(s,\sigma,\sigma',\lambda)\ \phi(s,\sigma')\ d\sigma' \ , \qquad (5.2)$$

where

$$D_{33}(s,\sigma,\sigma') \equiv \delta(\sigma - \sigma') - \lambda\ K_{33}\ (s,\sigma,\sigma',\lambda,q)$$

and $\lambda = f g$.

Now, K_{33} has the following structure:

$$\lambda K_{33} = \int \frac{d\ s'}{s' - s}\ \Sigma_{33}(s',\sigma')\ N_{33}(s',\sigma',\sigma) \qquad (5.3)$$

and N_{33} is to be taken from the solution of eq. (4.5). The latter being a system of Fredholm equations, N_{33} can be written symbolically:

$$N_{33} \equiv \phi_4 = f_4 + \sum_k \frac{1}{N}\ \Gamma_{4k}\ f_k \quad , \qquad (5.4)$$

where $\frac{1}{N}\Gamma_{ik}$ is the resolvent matrix of the system (4.5).

So, N_{33} and by eq. (5.3) λK_{33} will have poles in the parameter λ , where the Fredholm determinant $N(\lambda, j, g)$ vanishes . The solutions of the eqnations

$$N(\lambda, j_o, g) = 0 \qquad (5.5)$$

can be written as :

$$j_o = j_o(\lambda, g) \quad , \qquad (5.6)$$

so equivalently, we can state that the kernel of eq. (5.2) is a meromorphic function of j, having poles at the solutions of eq. (5. 5).

According to the results of ref./10/, the solution of (5.2) (i.e. the inverse of D_{33}) will have – in general – essential singularities (condensation points of the eigenvalues) at the points given by (5.6), except when in the neighbourhood of $j = j_0$ the singular part of the kernel is a degenerate one.

We can see the effect of this phenomenon on the amplitude as follows.

Having found the resolvent of eq. (5.2), in the form:

$$H(s,\sigma',\sigma) = \frac{G(s,\sigma',\sigma)}{D(s)} \tag{5.7}$$

we find the matrix $D^{-1}_{\alpha\beta}(s,\sigma',\sigma)$ as in ref./5/ :

$$D^{-1}_{\alpha\beta} = \frac{d_{\alpha\beta}}{Det\ D_{\alpha\beta}} , \tag{5.8}$$

where the elements of the matrix are given by the following relations:

$$\begin{aligned}
d_{22}(s) &= D(s)\\
d_{23}(s,\sigma') &= -\int d\sigma_1\, D_{23}(s,\sigma_1)\, G(s,\sigma_1,\sigma') - D_{23}(s,\sigma')\\
d_{32}(s,\sigma) &= -\int d\sigma_1\, G(s,\sigma,\sigma_1)\, D_{32}(s,\sigma_1) - D_{32}(s,\sigma)\\
d_{33}(s,\sigma,\sigma') &= \Delta(s)\, G(s,\sigma,\sigma') + [\, D_{32}(s,\sigma) + \int d\sigma_1\, G(s,\sigma,\sigma_1)\times\\
&\times D_{32}(s,\sigma_1)\,]\,[\, \int d\sigma_2\, D_{23}(s,\sigma_2)\, G(s,\sigma_2,\sigma') + D_{23}(s,\sigma')\,]
\end{aligned} \tag{5.9}$$

Here $\Delta(s)$ is the following expression:

$$\begin{aligned}
\Delta(s) = D_{22}(s) - \int d\sigma_1\, d\sigma_2\, D_{23}(s,\sigma_1)\, G(s,\sigma_1\,\sigma_2)\times\\
\times D_{32}(s,\sigma_2) - \int d\sigma_1\, D_{23}(s,\sigma_1)\, D_{32}(s,\sigma_1) .
\end{aligned} \tag{5.10}$$

Notice the relation:

$$Det\ D_{\alpha\beta} = \Delta(s)\, D(s) . \tag{5.11}$$

Hence, e.g. the elastic scattering amplitude is given by the equation:

$$\begin{aligned}
M_{22}(s) = [\,\Delta(s)\, D(s)\,]^{-1} \{\, N_{22}(s)\, D(s) -\\
- \int d\sigma_1\, d\sigma_2\, N_{23}(s,\sigma_1)\, G(s,\sigma_1\,\sigma_2)\, D_{32}(s,\sigma_2) -\\
- \int d\sigma_1\, N_{23}(s,\sigma_1)\, D_{32}(s,\sigma_1)\, \} .
\end{aligned} \tag{5.12}$$

The poles of M_{22} (and every other amplitude) in the angular momentum plane are given by the equation:

$$\Delta(s)\, D(s) = 0 . \qquad (5.13)$$

Where there no coupling between channels '2' and '3', (5.13) would reduce to the condition, known from the one-channel problem, i.e. $D_{22} = 0$. In the present case, however, the presence of other channels 'induce' new poles even in the elastic one (we call them for the sake of brevity 'inelastic poles' although, as one can see from the structure of $\Delta(s)$ and $D(s)$, both are influenced by all the channels). The 'inelastic' poles are given by the equation:

$$D(s) = 0 .$$

They condense at the points where the Fredholm determinant, $N(\lambda, j, g) = 0$. The condensation point of the poles obviously does not depend on s, but on the strength of the coupling characterized by the parameters λ and g only.

This phenomenon, as the reader can immediately see, is characteristic of the presence of a many particle channel. In order to get some physical insight into the problem, we are going to study now this behaviour in an oversimplified model of a three-particle channel, not coupled to any other one. We are interested in the position of the poles of the nominator fuuction N_{33}, giving rise to the condensation of Regge poles of the amplitude.

6. Poles in the nominator function of a three particle channel

Consider a system of scalar particles of mass μ. We study the amplitude M_{33}, under the assumption that the three particle channel is completely decoupled from all the others. In order to get a nontrivial equation in this case, we have to generate a dynamical singularity of the amplitude by prescribing the function which according to our previous terminology is called B_{33} (the 'Born term' for the three particle scattering).

For B_{33} we take a simple pole diagram, shown in Fig.4. The expression for the partial wave projection of B_{33} can be found in a straightforward way; it reads as follows:

$$B^{j}_{33}(s,\sigma',\sigma) = \frac{2 f^{3} g\, s}{a(s,\sigma)\; a(s,\sigma')}\; \frac{1}{\mu^{2}-\sigma}\; \frac{1}{\mu^{2}-\sigma'} \times$$
$$\times\; Q_{j}\left(\frac{(s+\mu^{2}-\sigma)(s+\mu^{2}-\sigma') - 2\mu^{2} s}{a(s,\sigma)\; a(s,\sigma')}\right) , \qquad (6.1)$$

where the kinematic factors a are defined as in Sec.3. (In spite of the fact that we take particles of equal mass, we choose two kinds of vertices at the upper and lower parts of the diagram in Fig.4., with coupling constants f and g respectively). In a standard way we obtain the integral equations for N_{33} and D_{33} :

$$M_{33} = N_{33}\, D^{-1}_{33} ,$$

$$N_{33}(s,\sigma',\sigma) = B_{33}(s,\sigma',\sigma) -$$

$$- \int_{4\mu^{2}}^{\infty} d\sigma'' \int_{(\mu+\sqrt{\sigma''})^{2}}^{\infty} ds' \,\Sigma_{33}(s',\sigma'')\; \frac{B_{33}(s,\sigma,\sigma'') - B_{33}(s',\sigma',\sigma'')}{s - s'} \times \qquad (6.2)$$

$$\times\ N_{33}(s', \sigma'', \sigma) . \qquad (6.2)$$

(We suppressed the index j); the denominator function D_{33} is given by eq. (4.1).

As we have shown in the foregoing sections, the quantity we are interested in, is $N(f, g, j)$, , the Fredholm determinant of eq. (6.2). We want to work in a weak coupling approximation, i.e. we put:

$$N(f, g, j) \approx 1 - Tr\left[\frac{B_{33}(s, \sigma', \sigma) - B_{33}(s', \sigma', \sigma)}{s - s'} \quad \Sigma_{33}(s', \sigma')\right].$$

Taking into account eqs. (6.1) and (6.2), we find in this approximation:

$$N(f, g, j) = 1 - \frac{f^3 g}{128\ \pi^3} \int_{4\mu^2}^{\infty} d\sigma \int_0^{\infty} dp \frac{\partial}{\partial p} \left\{ p^{-2} Q_j\left(1+\frac{\mu^2}{2p^2}\right)\right\} \times$$

$$\times \frac{p}{\sqrt{s}} \left(\frac{\sigma - 4\mu^2}{\sigma}\right)^{1/2} (\mu^2 - \sigma)^{-2} , \qquad (6.3)$$

where

$$p = \frac{a(s,\sigma)}{2\sqrt{s}} .$$

After an integration by parts, we observe that for small values of j , the main contribution to the integral comes from low values of p^2 ; therefore we replace $Q_j\left(1+\frac{\mu^2}{2p^2}\right)$ by its asymptotic form (Ref. /8/, ch. III), thus arriving at the expression:

$$N(f, g, j) \approx 1 + \frac{f^3\ g\ 2^{-j}}{256\ \pi^{5/2}} \frac{\Gamma(j+1)}{\Gamma(j+3/2)} \int_0^{\infty} \frac{dp}{p^2}\ 3\left(1+\frac{\mu^2}{2\ p^2}\right)^{-j-1} \times$$

$$\times\ F(p) , \qquad (6.4)$$

where

$$F(p) = \int_{4\mu^2}^{\infty} d\sigma \left(\frac{\sigma - 4\mu^2}{\sigma}\right)^{1/2} (\mu^2 - \sigma)^{-2} \frac{\partial}{\partial p} \left(\frac{p}{\sqrt{p^2+\mu^2} + \sqrt{p^2+\sigma}}\right) ; \qquad (6.5)$$

$F(p) = 0(1)$ if $p \to 0$; $F(p) = 0(p^{-3})$ if $p \to \infty$ and $F(p) > 0$ everywhere.

The integral in (6.4) is singular for $j = -1/2$ (the singularity coming from the lower limit of integration), the behaviour coming from this singularity being described by a factor $\Gamma(j+1/2)$.

Therefore $N(f, g, j)$ is of the form:

$$N(f, g, j) = 1 + f^3 g\ 2^{-j} \frac{\Gamma(j+1)}{j+1/2}\ C + \dots , \qquad (6.6)$$

where $C > 0$. (Had we taken the exact integral (6.3) we would have obtained a similar expression, with C a slowly varying function of j in the neighbourhood of $j = -\frac{1}{2}$).

Let us remark that the higher order terms of the determinant N contain singularities beginning at $j = -3/2, -5/2, \ldots$ and the common factor $\Gamma(j+1)$; therefore, if we are near to $j = -\frac{1}{2}$, the first term in (6.6) is the dominant one, independently of the magnitude of the coupling constants.

It is easily seen that the roots of N cannot go out to infinity in the right half of the j-plane. In fact, for $j \to \infty$ $(-\pi/2 < \arg j < \pi/2)$, by making use of the asymptotic expression for the Legendre functions, one can show that $N = 1 + O(j^m \exp\{-a(+j+\frac{1}{2})\})$ where m and a are some constants. To study the behaviour of the first roots of (6.6) we replace $2^{-j}\,\Gamma(j+1)$ by $\frac{2}{j+1}$, thus obtaining that $N(f, g, j_0) = 0$ at

$$j_0 = \tfrac{1}{4}\,\{-3 \pm \sqrt{1 - 32\,C\,f^3 g}\,\}. \tag{6.6}$$

Now, these roots, as functions of the coupling constants, show a different behaviour, when $f^3 g \gtrless 0$. For $f^3 g > 0$ as the coupling constant increases from zero, the roots move towards each other afterwards going out to complex values; they, however, always remain in the left-hand part of the j-plane. For the other sign of the first root, starting from $j = -\frac{1}{2}$, moves to the right (the other one to the left) and can pass over to the right half plane if the coupling constants are sufficiently large.

7. Discussion

We hope that the qualitative picture arising from our previous considerations, will essentially be the same in more realistic models as well; therefore let us look for the physical implications of the results we have found.

The novel feature the presence of inelastic channels brings in is that the poles in the angular momentum plane can condense at points, the positions of which depends on the magnitudes and signs of the coupling constants. If such a condensation point can move out sufficiently far to the right half of the angular momentum plane, it can completely change the asymptotic behaviour of the amplitude in a crossed channel, originally determined by the dominant Regge pole. (Besides the coupling dependent condensation points of Regge-poles, there are of course other ones, at the poles of the Legendre functions Q_j, i.e. at $j = -1, -2, \ldots$ They are of the type found by Gribov and Pomeranchuk/11/). If a condensation point has come out sufficiently far to the right half of the j-plane, then by constructing the complete elastic scattering amplitude by means of a Watson-Sommerfeld transformation, one can go with the residual integral to the left down to the first condensation point. Thus, if t is the total energy, we obtain roughly for $t \to \infty$ (apart from trivial factors):

$$M_{22}(t,s) \approx \sum_{\nu} \zeta_{\nu}\, t^{a_{\nu}(s)} + \frac{1}{2i}\int \frac{(2j+1)\,dj}{\sin j\pi}\, M_{22}^{j}(s)\, t^{j}$$

the integration contour being $j = j_0 + i\tau$ $(-\infty < \tau < \infty)$ and j_0 is the position of the first condensation point of poles. If s, the momentum transfer becomes so large that $a_0(s) \approx j_0$ ($a_0(s)$ being the trajectory of the leading pole), the behaviour of the amplitude is no more characterized by one Regge-pole, but by an infinite number of them plus by the contribution of the residual integral. This change in the character of scattering reminds the semiclassical picture of diffraction scattering of elementary particles/12/. According to this picture, at small momentum transfers the scattering is essentially determined by the 'cloud of virtual pions', for increasing momentum transfers the dominant role being taken over by the 'core' - i.e. essentially by the multiparticles states. (As one can see from eq. (5.12), as s tends to the critical value, the amplitude M_{22} is mainly determined by the contribution of many-particle states ($D(s)$ – has a 'root of infinite order') .

We want to point out that we have found no energy dependent cuts in the angular momentum plane; this is connected with the fact that our partial wave expansions in the three particle unitarity are very likely to converge. In fact, making use of the apparatus, developed in Sec. 2., one can see that if one expansions – e.g. that over ℓ were replaced by a Watson-Sommerfeld integral and at least one of the elements of the matrix $M_{\alpha\beta}$ contained a Regge-pole in ℓ this would generate an energy - dependent cut in the angular momentum j .

Whether such cuts really arise or not can be answered therefore only after a detailed study of the singularities of the inelastic amplitudes in the momentum transfers. There is one further problem concerning the inelastic amplitudes. They surely contain complex singularities in the momentum transfer, therefore it may happen that apart from any condensation phenomenon of the Regge poles or the presence of singularities other than poles, the asymptotic behaviour in the total energy may not be of the Regge type. Some formal aspects of this problem are discussed in the Appendix, a final answer depending again on the structure of the amplitudes in question. Being aware of the incompleteness of our results, we nevertheless hope that they may serve as a basis for future investigations and that the qualitative features we have found will not undergo a substantial change.

In conclusion we would like to express our sincere thanks to Mr. J.Kwiecinski for having kindly checked some of our calculations and to Prof. A.A.Logunov for valuable critical remarks.

Appendix

We want to investigate the Watson-Sommerfeld transform in the presence of complex singularities of the scattering amplitude*. Assume that the partial wave amplitude is meromorphic for $Re\ \ell > \ell_0$ its poles being located at $\ell = a_\nu(s)$, where the a_ν satisfy the inequalities:

$$0 < \arg a_\nu < \pi / 2$$
$$0 < Re\ a_i < A, \qquad \text{(A.1)}$$

A is some fixed positive number.

For $|\ell| \to \infty$, $A(\vec{\ell}, s)$ has the following asymptotic behaviour:

$$A(\ell, s) = O\left(\ell^m \exp\left\{-\gamma(s)\ (\ell + 1/2)\right\}\right) \qquad \text{(A.2)}$$

inside the sector: $\ell = \ell_0 + Re^{i\psi}$, $|\psi| < \pi/r - \epsilon$;

m is some fixed number, $\gamma(s)$ – a complex – valued function of s .

Consider the Watson-Sommerfeld transform of $A(\ell, s)$:

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* At least part of these results – if not all – are surely known to Challifour and Eden [13], and to Zu.A.Simonov and K.A.Ter.Martyrosian.

$$A(z,s) = i \int_C \frac{\lambda\, d\lambda}{\cos \lambda \pi} A(\lambda, s) P_{\lambda-1/2}(-z) \tag{A.3}$$

where for the sake of convenience we write $\ell + 1/2 = \lambda$.

The contour C , originally taken around the positive roots of $\cos \lambda\pi$ (for $\lambda > \ell_0 + 1/2$) can be deformed into the vertical line $C^1: \lambda = \ell_0 + 1/2 + i\pi$ and the semicircle of infinite radius, R .

Then we have:

$$A(z,s) = i \int_{\ell_0+1/2-i\infty}^{\ell_0+1/2+i\infty} \frac{\lambda\, d\lambda}{\cos \lambda \pi} A(\lambda, s) P_{\lambda-1/2}(-z) + \\ + 2\pi \sum_\nu r_\nu(s) \frac{a_\nu(s)+1/2}{\sin \pi a_\nu(s)} P_{a_\nu(s)}(-z) , \tag{A.4}$$

provided the integral along R vanishes. In (A.4), $r_\nu(s)$ stand for the residues of the poles at $\ell = a_\nu$. Put $z = \cos\theta$, $\theta = \theta_0 + i\theta_1$ and $\gamma = \gamma_0 - i\gamma_1$ with $\gamma_1 > 0$ (the other case can be treated analogously). Then making use of the asymptotic formula: (Ref./8/, Ch. III),

$$P_{\lambda-1/2}(-\cos\theta) \approx 2(\pi\lambda \sin\theta)^{-1/2} \cos[\lambda(\theta - \pi) - \pi/4] \quad (\lambda \to \infty)$$

and of eq. (A.2) we find that the integral along the part of R in the lower half-plane vanishes if

$$\theta_1 < \gamma_0 ; \quad \theta_0 > \gamma_1 , \tag{A.5}$$

while the integral along the part of R in the upper half-plane vanishes if

$$\theta_0 > 0 , \quad \theta_1 < 2 \frac{\gamma_0}{\gamma_1} [\gamma_1 + \pi - 1/2\, \theta_0] . \tag{A.6}$$

The intersection of the domains determined by (A.5) and (A.6) has always a common part with the 'physical' interval $0 < \theta_0 < \pi_1, \theta_1 = 0$ provided $\gamma_1 < \pi - \epsilon$. (By considering the diagrams in Sec. 3, one can show that for them $|\gamma_1| < \pi/2$). If the last condition is satisfied, one can always continue the Legendre expansion into the representation (A.4) and if (A.5) and (A.6) are satisfied the series in (A.4) converges. If $\gamma_1 > \eta > 0$ the function $A(z,s)$ has complex singularities in z on the boundary of the intersection of the domains (A.5), (A.6).

If one is going to investigate the asymptotic behaviour of $A(z,s)$ for $z \to -\infty$ (physically this corresponds to the behaviour at large energies in a crossed channel), one has to consider complex values of θ given by $\theta = \pi + i\theta_1$. If there are no complex singularities in the amplitude (correspondingly $\gamma_1 < \epsilon$ where ϵ is an arbitrarily small positive number), any finite portion of the line $\theta = \pi + i\theta_1$ lies within the domain (A.6). However, for $\gamma > \eta > 0$ this is not the case, so the series in (A.4) will in general diverge, and so does – formally – the integral along the vertical line. By a suitable analytic continuation the residual integral can be given a sense even in this case, and the series exists at any rate in the asymptotic sense.

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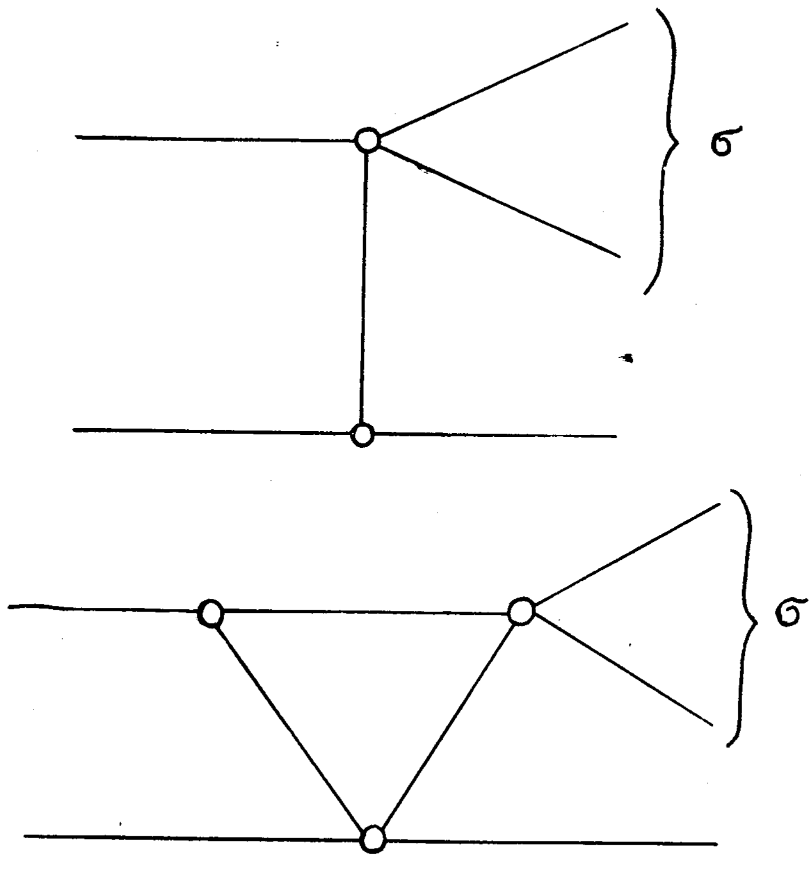

Fig. 1. Diagrams determining the dynamical singularities of the inelastic amplitudes ('Born terms').

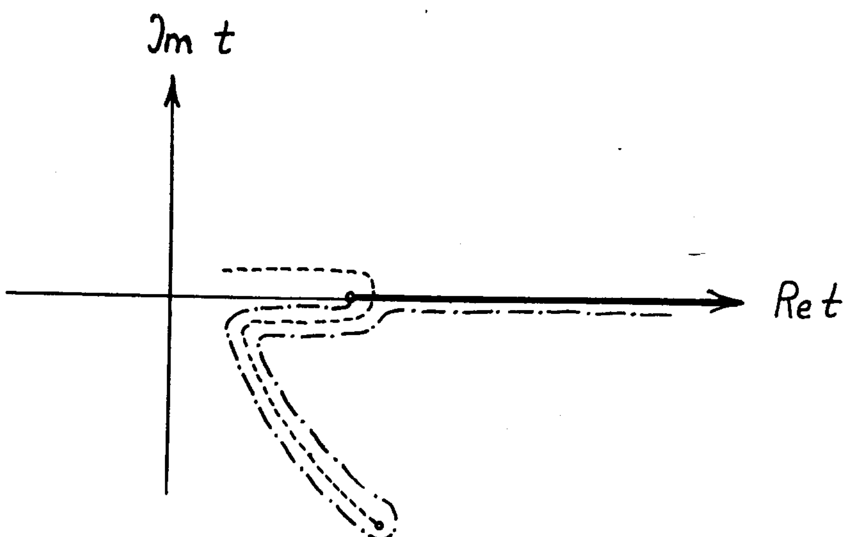

Fig. 2. Path of integration in the t-plane for diagram Fig. 1b. The kinematic cut runs along the real axis from $4\mu^2$ to infinity. The dashed line (– – – – – –) corresponds to the left-hand cut of. The line (—.—.—.) indicates the path of integration to be chosen.

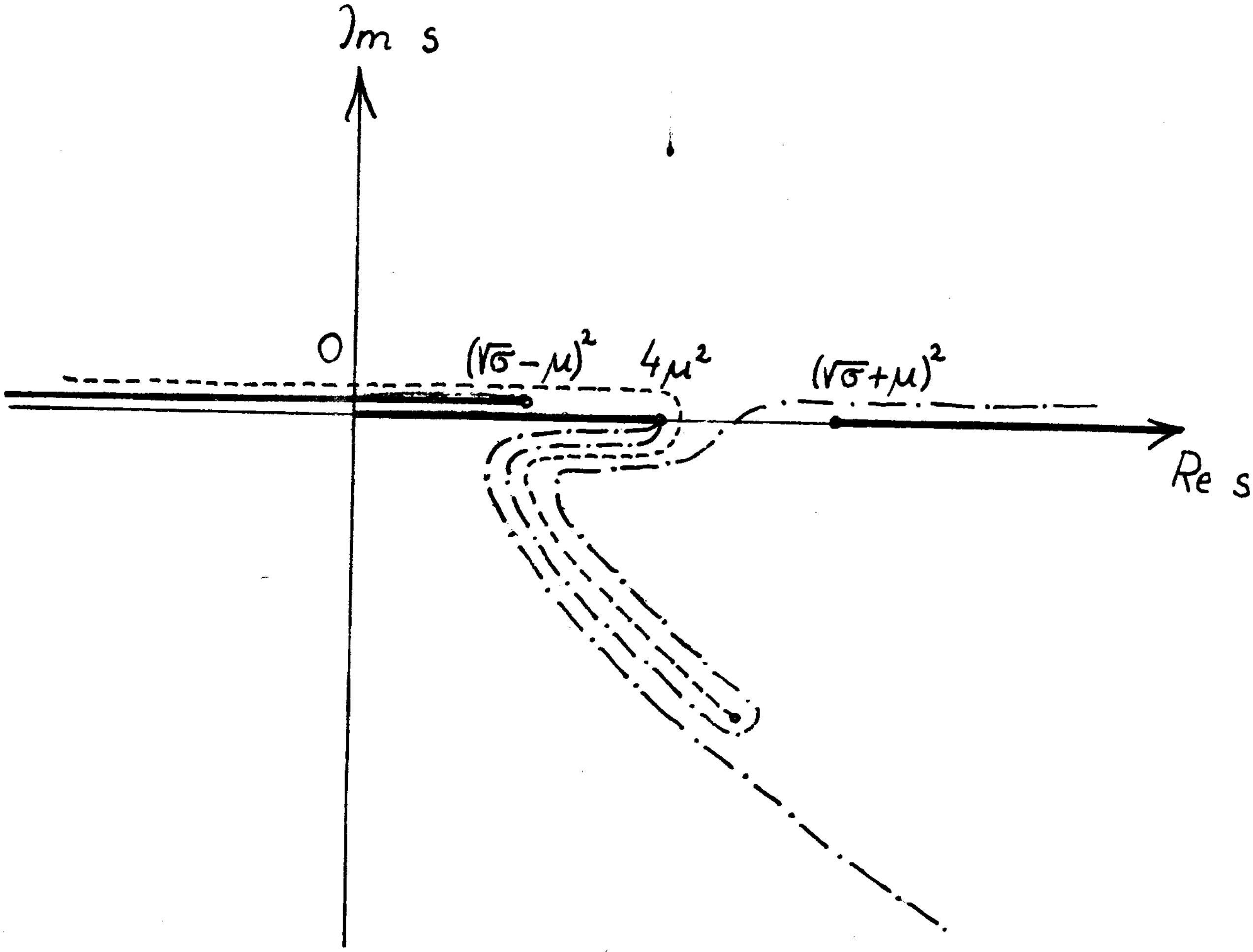

Fig. 3. Singularities and path of integration in the N/D equations. The solid lines(_______) along the real axis show the kinematic cuts; the dashed line the dynamical cut, while the line (–.–.–.–.–.) the deformed integration path in the N/D equations.

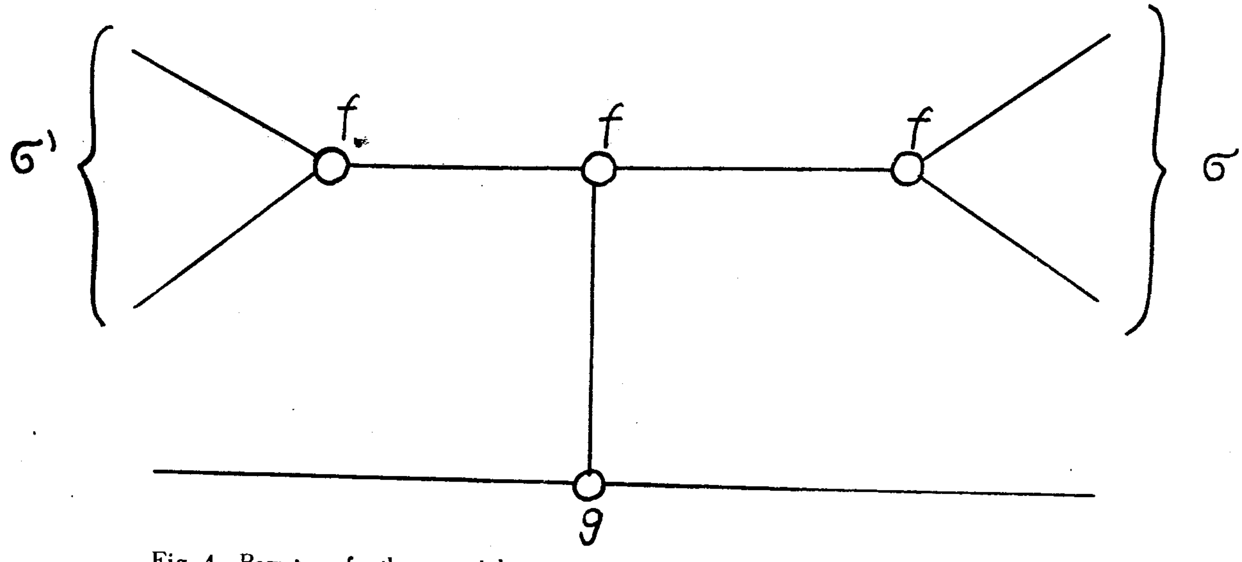

Fig. 4. Born term for three particle scattering.